Recent researches on the mathematical Russell

by I. Grattan-Guinness

F. A. Medvedev. *Rannyaya istoriya askiomi vibora* [Early history of the axiom of choice]. Moscow: Naukal, 1982. Pp. 303. 3r, 60k.

Gregory H. Moore. *Zermelo's Axiom of Choice: Its Origins, Development, and Influence.* New York: Springer-Verlag, 1982. Pp. xiv + 410. U.S. $38.00.

Jean Cassinet and Michael Guillemot. *L'axiome du choix dans les mathématiques de Cauchy (1821) à Gödel (1940).* Toulouse: Université Paul Sabatier, 1983. 2 vols. Pp. xx + 504, 618. Obtainable while supplies last from the authors at U.E.R. Recherches, 118, route de Narbonne, 31062 Toulouse, France.

THE LEAST STUDIED aspect of Russell is the mathematical and logical research which dominated his twenties and thirties and culminated in *Principia Mathematica*, that massive work which shares with its Newtonian predecessor the dual properties of wide fame of title and minimal knowledge of content. Recently, however, a remarkable change has taken place in part of this story, for three extensive scholarly studies of the axioms of choice, prepared independently, have appeared within a few months.

The axioms—their forms, need, mathematical power, and philosophical acceptability—emerged into consciousness during the 1900s after Zermelo published his paper of 1904 in which a form of axiom was consciously employed to prove Cantor's well-ordering theorem. The polemics of the immediately succeeding years are well remembered to have occurred, although the fascinating and rich details have only been recalled and surveyed in three works cited above. Among the features which rapidly fell into oblivion was the importance of Russell.

Independently of Zermelo and before the publication of the famous paper, Russell came to see the need for a form of the axiom in a context different from Zermelo's—the definition of the infinite product of cardinals (Grattan-Guinness *1977*, 80; Russell *1911*, 171). He paid careful attention to the ensuing discussion, although his perspective was rather different; while most of the active discussants were mathematicians,

Russell felt governed by more general philosophical considerations. He wished rather to find the axioms proven false by means of a counter-example, or else to absorb them into his logicist programme. The first option seemed most unlikely, although Russell hoped for its success as late as his *1911*, 172; the second option remained difficult, for the seeming unlikelihood of expressing the action of the axioms *within the limits of his formal logical language*. For, in some form or another, the infinite selection has to be made *without a rule*, by definition of the need for the axioms; but then a finite number of symbols would be inadequate to tell the full story.

Russell did not explain this difficulty as clearly as he might in his writings; one of the best sources of which I am aware is in a letter to Jourdain in which he said that one assumes the existence of the multiplicative class without accepting its abstractability via a propositional function (Grattan-Guinness *1977*, 68–9). As a result, the matter does not find adequate discussion in any of these volumes, which also pass over, in more silence than I would have liked, the related question of whether the axioms were mathematical or logical (as these notions were understood at the time, not only by Russell but others).

However, overall each volume gives an excellent survey of the material that it treats, beginning with some measure or other of nineteenth-century anticipations and stopping its detailed account at different places after the main discussions; around 1920 for Medvedev, basically in the late 1920s for our French authors, and with Gödel's proof of consistency with axiomatized set theory in 1940 for Moore. Each volume gives due attention to Russell's writings of the 1900s, and the 1911 paper mentioned above, but only the French authors give proper due to *Principia Mathematica*, which contains perhaps the first systematic statement of the forms and uses of the axioms in the areas of (logico-) mathematics which fell within its purview (Cassinet and Guillemot, 345–69). Further, in their second volume, they provide translations into French of several of Russell's and Whitehead's writings in this area from the period 1902–06 (pp. 165–236, 323–36). English-speaking readers will, of course, not need the translation from that tongue, although some may welcome the translation into French that Cassinet and Guillemot have made of entirely symbolic passages that Russell (and others) wrote at this time. It is to be hoped that some full form of publication of their work, a double *docteur d'état* sustained at the University of Toulouse, will be possible.

Were I forced to apply an axiom of choice to these three publications, then I might just decide for Moore for the history as a whole; he has the best chronological spread, and perhaps the largest measure of historical appraisal. Russellians, however, might feel most rewarded by Cassinet and Guillemot. All should rejoice in the combined value of these three efforts.

Middlesex Polytechnic

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Grattan-Guinness, I. *1977*. *Dear Russell—Dear Jourdain*.... London: Duckworth.