Textual studies

Part II of *The Principles of Mathematics*

by Michael Byrd

THE RECENT TEXTUAL study in *Russell*, of *The Principles of Mathematics*, Part I, revealed substantial alterations between the copy Russell sent to the printer and the published text.[1] I was drawn to undertake a similar collation of Part II by some striking philosophical incongruities in the published text. Initially, my interest involved Russell's views about the concept "one". In Part I of *Principles*, in the early portions of Part II, and in several unpublished manuscripts written prior to his discovery of Peano, Russell held, prominently and consistently, that the concept "one" applies to every entity whatever.[2] Consequently, each person, number, and spatial point is one. However, in Chapter XV of Part II at section 128, Russell suddenly advocates, without accompanying argument, a more Fregean treatment of the concept. On this view, "one" is to be applied exclusively to classes. The sense of "one" in which every object is one is said to be a "very shadowy" sense which is "not relevant to Arithmetic". The change of view here is sufficiently important and precipitous that I guessed that the later view had been peremptorily inserted as Russell became familiar with Frege's work in 1902.

I examined the printer's copy of the manuscript of Part II on a visit to the Russell Archives during the summer of 1985 and found that §128, in its published form, is entirely new. Indeed, the subject-matter of the two versions is different. In the printer's copy, Russell considers,

[1] Kenneth Blackwell, "Part I of *The Principles of Mathematics*", *Russell*, n.s. 4 (Winter 1984–85): 271–88.

[2] See *The Principles of Mathematics* (Cambridge: at the University Press, 1903), pp. 43, 132: 12–17. Among unpublished manuscripts, the idea is found in "An Analysis of Mathematical Reasoning" (RA 230.030300), fol. 8, and "The Fundamental Ideas and Axioms of Mathematics" (RA 230.030310), fol. 3.

and fundamentally accepts, the objection that logicism, as regards the cardinal numbers, fails because the equivalence classes, which are to be identified with the numbers, cannot exist in a theory which adequately handles the Contradictions. Having discovered these significant divergences between printer's copy and published text, I decided it would be well worthwhile to do a complete collation of Part II.

I twice read a photocopy of the printer's copy of the manuscript line by line against the text of the first impression of *The Principles of Mathematics*. The first impression was chosen since the purpose of the collation was to bring to light how Russell's views developed in the crucial months when he first studied Frege carefully, and the extent of correction and even revision in the subsequent eight impressions until 1972 is unknown. There are substantial alterations in Part II, amounting to approximately 1,100 words of text.[3] By comparison, the list of variants for Part I, which is over twice as long as Part II, comprises 1,900 words in passages not appearing in the published text. The major changes are found in Chapter XV, "The Addition of Terms and the Addition of Classes". This is the first, and longest, of three chapters in Part II which consider the philosophical questions raised by the mathematical theory of cardinal numbers set out in Chapters XI to XIV. Section 128 of Chapter XV is, as I said earlier, essentially new, and §132, the final and summary section, overlaps the printer's copy only in its initial two sentences.

A list of substantive, or verbal, variants is given below. I have followed Blackwell's model in its construction. It is read as follows. At the left is a number such as 111: 8. This means page 111, line 8 from the top. To the right there is first the reading from the first impression of the published text, followed by a square bracket. The words after the bracket are the corresponding final manuscript reading. Editorial brackets enclose my comments. There are many additions and deletions on the manuscript itself, but they fall outside the scope of the textual series to which my study belongs.

The initial leaf of the printer's copy is dated June 1901 by Russell. This places its composition a month after the composition of what Blackwell calls the penultimate version of Part I, whose first leaf is dated May 1901. At the upper left-hand corner of each leaf is the notation "Nc", Russell's way of indicating the Part to which these leaves belong. The leaves are numbered consecutively, 1 to 86. Several leaves have two numbers, indicating that Russell had removed them from ear-

[3] I was concerned to locate only what are called substantive variants; see Blackwell, pp. 44–5.

lier work.[4]

The May 1901 version of Part I differs considerably in structure from the copy sent to the printer a year later. For example, there are no independent chapters on Propositional Functions or the Contradictions.[5] There are several reasons, however, for supposing that Russell did not engage in such extensive alterations of Part II after June 1901. First, in the file of rejected leaves for Part I of *Principles*, there is a Table of Contents for the entire book which corresponds exactly, in its entries for Part I, with the May 1901 version of that Part. But the entries for Part II are just the same as those of the printer's copy and the published text.[6] Second, we know that Russell rewrote Part I in May 1902. The account he gave of his work at that time, which is found in correspondence, leaves little time for extensive changes in Part II. As Blackwell points out, Russell wrote his wife Alys that he finished the revision of Part I on 13 May, and that he finished the manuscript of *Principles* on 23 May. Russell's work is interrupted by visitors on the 18th and 19th, and in a letter to Alys on the 21st, he says that he is working on matter and motion, Part VII (Blackwell, pp. 278–80). Thus, only six or seven days remain for work on Parts II to VI. Russell's major revision of Part I involved the writing of 168 leaves in just eleven days. Extrapolating this pace, he might have written as much as a hundred new leaves in the remaining week. The printer's copy of Part II is 86 leaves long. So, had Russell completely rewritten Part II, little time, if any, would have been left for consideration of Parts III to VI, over 300 pages of published text. Finally, the subject-matter of Part II seems largely unaffected by the major stumbling-block of Part I—namely, the Contradictions. This seems clearly true of all of Part II except §§111, 122, 123, and Chapter XV.[7]

There are indications, however, of some revision subsequent to June 1901. As noted, Russell had not recognized fully the problems posed by the Contradictions in May 1901. This is manifested in the fact that the penultimate version of Part I contains no chapter on that subject. It is doubtful that the version of Part II written one month later would attribute great importance to them. However, the printer's copy of §§128 and 132 claims that the Contradictions pose grave problems for the logicist view that numbers are classes of similar classes (RA 230.030350–F5, fos. 44, 53). Also at several places, Russell makes interlinear insertions that involve the Contradictions (RA 230.030350–F6, fos. 57, 60, 64, 67, 85). These insertions are in pencil where the remainder of the manuscript is written in ink.

I conclude by briefly discussing the character of the most significant alterations. The collation brings out that all references to Frege in Part II were introduced *after* May 1902. This includes both issues of philosophical substance (§128, §132) and matters of acknowledgement (pp. 111: 8, 142: 10). Outside the Appendices, there are just five other references to Frege in *Principles*. Two footnotes in Part I (pp. 68, 76) do not occur in the printer's copy and so postdate May 1902. Another footnote occurs in Part VI at page 451. The portion of the printer's copy which includes this section is missing. The footnote acknowledges

[4] John King's report on pre-*Principles* manuscripts reveals seven such pages. Folio 49 of the printer's copy is from Part I (Numbers), Chapter I (Collections) of the unpublished manuscript "Principles of Mathematics" (1899–1900). Fos. 60, 61, and 63 are from Part II (Whole and Part), Chapter I (Meaning of Whole and Part). Fos. 72, 73, and 75 are from the same Part, Chapter IV (Infinite Wholes). See John King, "A Report on the Manuscripts for 'An Analysis of Mathematical Reasoning', 'The Fundamental Ideas and Axioms of Mathematics', and 'Principles of Mathematics (1899–1900)'" (unpublished typescript, Russell Editorial Project, 1984), pp. 9–10.

[5] Blackwell offers a reconstruction of this version at p. 276 of his study. It should be noted that the reconstruction of Chapter VI (Implication) is erroneous. On the account given, five leaves of this chapter, numbered 44 to 49, can be located in the final manuscript at fos. 135 to 138 of Part I. Examination of the final manuscript reveals that fos. 135 and 136 have double numbers: 44 and 45, respectively. They bear the marking "Nc" in their upper left-hand corner. Fos. 137 and 138 have triple numbers: One set is 48 and 49, respectively; the other set is 4 and 5, respectively. They bear the marking "N" in the upper left-hand corner. So, the origin of the first pair of leaves is the June 1901 version of Part II. The second pair originated in the "Principles of Mathematics" (1899–1900) manuscript. See King, p. 9. The second pair of folio numbers (48 and 49) indicates, in all likelihood, their use at that position in the June 1901 version of Part II. The claim that these four leaves did not form part of the penultimate version of Part I also derives from the inappropriateness of their subject-matter to the projected folio position. If these leaves had been in Part I, they would have been in the chapter on implication. But these leaves are devoted exclusively to the topics of term conjunction (e.g. "Brown and Jones") and the corresponding extensional view of classes. However, had these leaves been in the penultimate version of Part II, they might well have fallen in Chapter XV on the addition of terms and the addition of classes, to which their subject-matter is germane. I conclude that these leaves did not belong in Chapter VI of the penultimate version of Part I, and hence that all leaves of this chapter are missing.

[6] This Table of Contents coincides with that of the published text for all chapters in Parts III to VI, except for Chapter LI of Part VI. It is entitled "Absolute and Relative Position" in this table rather than "Logical Arguments Against Points". There are no section headings for Part VII. This table can be found at RA 230.030350 (rejected portions).

[7] Against my hypothesis that Part II was not significantly rewritten in May 1902, I should note that in 1910, Russell wrote to Jourdain that Parts I *and* II were "wholly later, May 1902". See I. Grattan-Guinness, *Dear Russell—Dear Jourdain* (London: Duckworth, 1977), p. 133.

Frege as a common proponent of the view that numbers are mind-independent entities and cites *Grundgesetze* I, p. XVII. There are two references to Frege in Part I which are found in the printer's copy. One occurs in the body of the text at page 19, where Russell groups Frege and Peano as logicians who clearly distinguished the membership relation from the subset relation. In a footnote, Russell cites *Grundgesetze* I, by page number (p. 2), and *Grundlagen* with no page citation. The remaining reference in the printer's copy occurs in a footnote on page 35 of the published text. There Russell writes, "Frege (*loc. cit.*) has a special symbol to denote assertion." It is reasonable to suppose that the relevant citation here is to *Grundgesetze* I, §5. On page 19, Russell has cited both *Grundgesetze* and *Grundlagen*, but no special symbol is introduced to denote assertion in *Grundlagen*. Such a symbol is introduced and discussed early in *Grundgesetze* I at §5 (pp. 9–10). Russell's copy of *Grundgesetze* I, which is to be found in the Russell Archives, contains extended marginal comments on this section, and these comments echo remarks made in the section of *Principles* where the footnote occurs.[8]

This pattern coheres well with Russell's own account of his knowledge of Frege's work. In his famous initial letter to Frege on 16 June 1902 Russell writes: "For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work." Later in the letter, he adds: "I already have your books or shall buy them soon, but I would be very grateful if you could send me reprints of your articles in various periodicals."[9] As we see, all citations in the body of *Principles* are to Frege's books, and all but three at most postdate May 1902.[10] Indeed the only citations by page number are to pages in the Preface and Introduction of *Grundgesetze* I. Again, this fits with Russell's recollection of the matter. In *Portraits from Memory*, he recalls being led to purchase the *Grundgesetze* by Peano's review which,

[8] There are two relevant marginal comments. The first is appended to the initial paragraph of §5. It reads: "In grammar, assertion is distinguished by the indicative verb, the mere proposition apart from its assertion, being best expressed by a verbal noun." The second attaches to the final paragraph of the section. It reads: "Assertion is thus something new over and above truth and falsehood. This is obviously correct: if P is a proposition, 'the truth of P' is not the same as 'P is true'."

[9] Gottlob Frege, *Philosophical and Mathematical Correspondence*, ed. G. Gabriel *et al.*, abridged by B. McGuinness, trans. H. Kaal (Oxford: Blackwell, 1980), pp. 130–1.

[10] Russell's correspondence gives no indication as to when the references to Frege were added. (See Blackwell, p. 282.) Appendix A on Frege was received by the printer on 15 November.

according to Russell, accuses Frege of "unnecessary subtlety".[11] The copy of *Grundgesetze* I in the personal library at the Russell Archives is inscribed "B. Russell, Oct. 1900". But, Russell admitted limited success in reading it; in *Portraits*, he writes: "I read the introduction with passionate admiration, but I was repelled by the crabbed symbolism which he had invented and it was only after I had done the same work for myself that I was able to read what he had written in the main text" (p. 22). The limited citations to the earliest parts of *Grundgesetze* are just what one would expect from such a reading.[12]

The collation reveals Russell's attempts to bring Frege's views to the defence of logicism. This occurs at §§128 and 132 of the published text, where Russell invokes Frege's view that "one" is not a property of objects, but of concepts. This view is called upon in response to an objection set out three times in Chapter XV—namely, that logicism illicitly and circularly presupposes the concept "one" through its employment of the concept "a term" (see 130: 10–14, 132: 12–16, 135: 36–40).[13] The published text contains *two* replies of questionable compatibility. In §127, a section which contains only minor changes from the printer's copy, Russell attempts to meet the objection by distinguishing between what is *implied* by the concept "a term" and what is *presupposed* in it. He admits "one" is implied in the concept "a term", because every term is one. But, Russell claims, this does not yield vicious circularity, because we may consistently deny that the concept "a term" presupposes the concept "one". The distinction between implication and presupposition is one which Russell has already put to crucial use in *Principles*: it is an essential element in his reply to Bradley's relational regress (pp. 51, 99–100).

The material, incorporated in §§128 and 132 at the proofreading stage, offers a rather different reply. It, in effect, denies that there is a problem to be solved. The source of the circularity argument is alleged to be the acknowledgement that it is true to say of every term

[11] For Peano's review, see Victor Dudman, "Peano's Review of Frege's *Grundgesetze*", *Southern Journal of Philosophy*, 9 (1971): 25–37.

[12] Russell's copy of *Grundgesetze* contains marginal comments of the sort his recollections would suggest. In the Preface, the marginal notes include: "Hear Hear!", "Excellent!", "Good", and "Splendid". There are almost no marginal markings of any sort beyond p. 65. Many notes in the earlier technical portions of *Grundgesetze* are attempts to translate Frege's symbolism into Peano's. Russell's library also contains copies of *Begriffsschrift* and *Grundlagen*, apparently bound for Wittgenstein, but these contain no internal markings.

[13] The word "term" is used by Russell in *Principles* to indicate the concept "entity". See *Principles*, p. 43.

that it is one. On Frege's view, this is not so, if "one" is taken to be a numerical concept, as opposed to, say, self-identity. Indeed to say of an *object* that it is one is non-sense, a category mistake. The exact nature and seriousness of Russell's objection, as well as the adequacy of these replies, warrants extended discussion.[14]

Finally, passages omitted from the published text show clearly Russell's early realization that the Contradictions challenge the logicist identification of cardinal numbers with classes of similar classes. The tone of several omitted passages is quite pessimistic in this respect. The May 1902 version of §128 concludes: "Thus numbers, it would seem, philosophically though not formally, will have to be readmitted as indefinables." This claim is repeated in deleted portions of §§132 and 148. (See the list of variants, 132: 35–133: 11, 136: 3–19, 152: 27.) The reason for pessimism is that one of Russell's first attempts to resolve the Contradictions had the consequence that, for instance, the class of all trios could be regarded only as a class as many and not as a class as one. It is thus unsatisfactory to identify this class (as many) with the number 3, since the number 3 standardly functions in Arithmetic as a single entity.

Russell's early reaction to the Contradictions was that their source lay in permitting variation over such items as propositional functions. This diagnosis occurs in Russell's first letter to Frege: "There is just one point where I have encountered a difficulty. You state (p. 17) that a function, too, can act as an indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction."[15] But, of course, ban on such variation takes with it the classes with which Russell had identified the cardinal numbers. And this is the core of the objection set out in the deleted §128. (See the list of variants, 132: 35– 133: 11.)

In the published text of Part I, Russell argues that variation over propositional functions must be allowed on a variety of grounds

[14] Russell's notes on *Grundlagen* nicely reveal his ambivalence concerning Frege's views about those concepts which the medievals labelled "transcendental" (e.g. existence, number). He notes Frege's view that numbers are properties of concepts and the application of this idea to the concept "existence", construed by Frege as denying the application of 0. This, Frege says, is the source of the failure of the ontological argument. To these comments, Russell appends the editorial remark: Mistake. See RA 230.030420–F2.

I have here maligned Frege's view somewhat to make the contrast with Russell's early views striking. Frege held that ascriptions of number are ascriptions of concepts to concepts, but that the numbers themselves are objects.

[15] Frege, p. 130.

(104: 2–11). This admission is followed by two attempts at a more satisfactory resolution. One idea is to sort out a class of propositional functions, called quadratic forms, which cause problems because function and argument cannot vary independently (104: 13–30). The other is a primitive statement of the theory of types (104: 42–105: 15). As Blackwell's collation shows, neither of these proposals is found in the printer's copy.[16] The proposal about quadratic forms was sent to the printer on 25 June 1902 with the note: "To be added at the end of § 103" (Blackwell, p. 288). One assumes that the type-theoretic solution is later and was developed later in conjunction with Appendix B.[17]

The pessimism of the deleted sections is gone from the published version of Part II. Section 132, the summary section of Chapter XV, concludes: "Thus it appeared that no philosophical argument could overthrow the mathematical theory of cardinal numbers set forth in Chapters XI to XIV." (The use of "appeared" is admittedly disconcerting.) Other summary statements, in Part II and later, are similar in tone (152: 25–7, 497: 9–12). Nevertheless, Russell could scarcely have failed to raise the question as to how these new resolutions affect the identification of number with classes of similar classes. And indeed, in Appendix B, Russell admits that numbers, as standardly conceived, fit nowhere in the hierarchy of types and that the definitions offered in Part II of particular numbers (e.g. "0") will not suffice. He concedes there that he has no clear idea about how to carry out the required definitions within the new framework.[18]

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[16] Blackwell, p. 288. Instead of the published paragraph beginning at the bottom of fol. 103, the printer's copy contains an analysis along the lines suggested in Russell's first letter to Frege (*ibid.*, p. 287). Note that this view survives in the published text at §85.

[17] This assumption is based on the chronology of topics in the Russell–Frege correspondence. Russell's second letter to Frege, dated 24 June, sketches the idea about quadratic forms. It is later, in his fifth letter, on 8 August, that a type-theoretic approach is first discussed. Russell points out its similarity to Frege's hierarchy of functions and notes its application to the class and relation forms of the Contradictions. Russell's next letter, on 29 September, sets out the paradox about propositions which he leaves unresolved in the final section of Appendix B. See Frege, pp. 133, 144, 147.

[18] I have benefited greatly, in the preparation of this study, from the expert assistance of Kenneth Blackwell.

VARIANTS BETWEEN *THE PRINCIPLES OF MATHEMATICS*, PART II, "NUMBER", AND ITS MS.

CHAPTER XI. DEFINITION OF CARDINAL NUMBERS.

111: 4 of cardinal integers] of integers
111: 5 Logic*.] Logic <*added fn. indicators are not noted any further*>
111: 8 Cantor, Frege, and] Cantor and
111: 26–9 <*fn. added*>
111: 30–1 †See ... 314 ff.] *See Peano F_3, p. 6 ff. and Padoa, Theorie Algébrique des Nombres Entiers, Bibliothèque du Congrès Internationale de Philosophie, T. III (Paris 1901), p. 314 ff. This work will be quoted hereafter as *Congrès*.
113: 13 it (§ 59).] it.
113: 13–14 men conjoined in] men in
113: 16 conjoined in a] in a
113: 16 as thus conjoined] as thus denoted
113: 20 one to one] one by one
113: 28–30 when ... identical.] when, if *x* has the relation in question to *y*, and *y* differs from *y'*, *y'* differs from any term to which *x* has the relation; while if *x* differs from *x'*, *x'* differs from any term which has the said relation to *y*.
113: 30–1 Thus it is possible] Thus diversity suffices
113: 35 one to one] one by one
115: 15 Membership of this class] This class
115: 26 [a class] *a*] a class *a*
115: 34 "logical product of class of men and couple"] "class of men and couple"
116: 21 but the class] but either some other predicate, or the class
116: 22 such classes] such predicates or classes
116: 23 numbers.] numbers. I shall return to this point in connection with whole and part. For the present, it will be well to return to more arithmetical topics.

CHAPTER XII. ADDITION AND MULTIPLICATION.

117: 18–19 this and logical multiplication.] this one
118: 11 another exclusive] another
118: 43 one] only one
119: 40 stated. If] stated: If
119: 43 Oct. 1902] Vol.

CHAPTER XIII. FINITE AND INFINITE.

122: 5–9 of *u*. It ... *u*."] of *u*.
122: 24 two different] two

CHAPTER XIV. THEORY OF FINITE NUMBERS.

124: 18 five fundamental] fundamental <*both "five" and "six" deleted in MS.*>
124: 28 <*fn. added*>
126: 23 the class] a class
128: 4 classes which are not null and] classes which

CHAPTER XV. ADDITION OF TERMS AND ADDITION OF CLASSES.

130: 10 we shall examine] we examine
130: 32 identity] diversity
130: 33–5 when ... *y'*.] when *x* has the relation R to *y*, and *x'* differs from *x* and *y'* from *y*, then any relation which *x'* has to *y* or *x* has to *y'* differs from R.
130: 37 a new inquiry] an inquiry
131: 26 reconsider] consider
131: 36–132: 3 It might ... sense.] Now in this view of classes, as we saw in Part I, a class of only one term must be identified with that one term; hence, if formalism requires a distinction to be made, it is essential to substitute either the class-concept or the concept of the class for the class itself. But in this there is a difficulty, since we must choose, as the class-concepts which are to be numbers, concepts which are determinate when the classes are given. Such concepts will be "member of the class of couples" or "of trios" or etc. That is, having defined, by any predicate, a certain class *u*, we can form the class-concept "member of u", and this is determinate when *u* is given. It must be class-concepts of this nature, and not classes themselves, that are numbers; unless, indeed, other class-concepts, defining the same classes, can be found, which are also uniquely determined when the class is given.
132: 23 said] saw <*misprint, surely*>
132: 23 in simple cases] usually
132: 27 but ... many] but these assertions have many subjects
132: 28 as in] like
132: 30–1 belongs, in this view, to] belongs to
132: 35–133: 11 **128.** It ... term.] § 128. An objection to our definition of cardinal number may be based on the conclusion which we found necessary in Chapter XII of Part I for the solution of the contradiction. For we have to vary classes and relations in order to obtain our definition, and thus we do not necessarily obtain as a number an entity which can be treated as a single logical subject. Philosophically, this is a serious objection, for it shows that we must find a class-concept corresponding to the class of classes similar to a given class, and that this class-concept cannot be derived from the actual definition of the class of classes concerned, because this definition has the formal characteristics which prevent it from insuring the existence of a corresponding class-concept. Thus numbers, it would seem, philosophically though not formally, will have to be re-admitted as indefinables.

134: 1–2 notion of a numerical conjunction, or more shortly, a *collection*.] notion of a *collection*.
134: 3 to begin with,] at least to begin with,
134: 20 not one.] not one. ¶We have to consider, then, the collection A and B and C and etc., where A, B, C, etc. are each one. Since we do not want to presuppose number in this discussion, it may be well to express what is equivalent to the condition that each should be one in form free from reference to the number one. This can be done by a consideration of the nature of the terms which are one. Now in the first place, all indefinable simple terms are each one: points, instants, numbers, particular shades of colour, etc. But many other terms are one. A series, a planet, a man, a society, are each one in some sense. To go into this sense at length would take us too far from our subject. But the following general statement seems irrefutable: If A is one without being simple, then A is other than all its constituents together: it is a whole, which, in virtue of the relations contained in it, is different from all the parts of which it consists. All wholes are of this nature: every whole is one, and every collection of terms, except certain infinite collections, composes a whole. I shall return to this subject later; for the present I remark merely the following property of whatever is one: If A is one, then either A is simple, or A is falsified by analysis, *i.e.* it is other than all its constituents together. In all cases where A is one, A is, in a certain fundamental sense (though not quite the usual sense) indefinable.
134: 22 gives] seems scarcely distinguishable, if at all, from
134: 22–3 *A* and *B* are] *A and B* is <*Also at 134: 28 and elsewhere.*>
134: 38 applies practically] applies
134: 43–5 classes." The ... case.] classes."
135: 26–7 "if ... terms."] "*A is one* and *B is one* are together evuivalent to *A and B are two*."
135: 31–2 the three propositions, but their logical product.] the two propositions, but the whole composed of this conjunction.
135: 38–136: 2 This ... *term*.] I think, however, that some account can be given of 1, namely the above account that what is one is either incapable of analysis, or is different from the numerical conjunction of its unanalyzable constituents. However this may be, it seems that we can either take 1 or *a term* as indefinable, but that it is impossible to define both. For when we say that anything is unanalyzable, it is impossible to explain what we mean without introducing the notion of a term.
136: 3–19 To sum up: Numbers ... XIV.] To sum up: numbers are properties of classes taken as many, or numerical conjunctions, but they do not apply to such classes as are only many, for to these classes the notion of *all* is not applicable. For formal purposes, numbers may be taken to be classes of similar classes, but philosophically they must be defined as certain properties of these classes, because these classes are of the kind that are not each a single term as well as many. Numbers, it would seem, are thus philosophically, not formally, indefinable; formal definability results from the assumption made by the symbolism that a definable class can always be taken as a single term. But philosophically numbers are not predicates and not class-concepts; for predicates and class-concepts apply to single terms. But numbers are closely allied to predicates, for they are asserted of classes in the same kind of way in which predicates are asserted of terms: they are concepts occurring otherwise than as terms in propositions which are not in the ordinary sense relational: "A and B are two" does not express a relation of *A and B* to 2. Thus the majority of the remarks which we made concerning adjectives in Part I will apply also to what philosophy and common sense recognize as numbers; and these indefinable entities are different from the classes of classes which it is convenient to call classes <*sic; "classes" is underlined lightly in pencil and should, I think, be "numbers" here*> in mathematics.

CHAPTER XVI. WHOLE AND PART.

137: 12 not classes] one

137: 14 characterized, though not defined,] characterized

137: 15 presuppositions.] presuppositions. (A proposition is said to have no presuppositions when there is no proposition formally implied by the proposition in question with its subject made variable, but not implying it.)

137: 16 not classes] one

137: 18 not a class] one

137: 25–6 urged ... what] urged, what

138: 44 XXVII] IV

139: 1 some non-quadratic] some

139: 3 the class] the class, provided it is not a member of itself,

139: 4–5 terms ... whole.] terms.

139: 12–14 the relation ... aggregate,] the relation of aggregates, containing some but not all the terms of our aggregate, to our aggregate,

139: 41–4 The ... terms.] It is to be observed that this symbol requires that the relatum should be one: hence the relatum must be either the class-concept or the whole formed of the terms of the class, but cannot be all the terms of the class.

140: 5–6 a non-quadratic propositional function] a propositional function

140: 44 Chapter IV, esp. § 54] Chapter III

141: 1–2 a class ... unit.)] one is a unit.)

141: 17 a single term.] one.

141: 41 what we call classes as one] the classes which appear in Symbolic Logic

141: 43 quadratic] variable

141: 44 <*fn. added*>

142: 1 classes as one] classes

142: 10 Peano and Frege] Peano

141: 12–13 predicate or propositional function] predicate

142: 15 class-concepts or propositional functions] class-concepts

CHAPTER XVII. INFINITE WHOLES.

144: 27 paradox.] paradox.* <*fn. added in MS.*> *Though, as we have seen, it is still necessary to maintain this paradox as regards classes defined by variable propositional functions.

144: 27–8 where ... function,] where we can speak of *all* of a collection,

144: 33 *141*] § 142. <*misnumbered in MS.*>

145: 44 <*fn. added*>

147: 13 term] sum <*misprint, surely, in text*>

CHAPTER XVIII. RATIOS AND FRACTIONS.

149: 25–6 Also we may define mn as $0+mn$.] Also we have $0+mn = mn$.

149: 30–1 B', where ... b.] B′.

150: 40 occur in daily life.] be of any importance.

150: 41 occur] are important

151: 22 ordered wholes.] wholes.

151: 24–5 two similar] two

152: 7 Chapter XI] Chapter I <*In this paragraph and the next three, the MS. chapter references result from subtracting 10 (or X) from those in the text.*>

152: 27 others).] others), though we admitted that the resulting definitions did not give us the entities which should philosophically be called numbers. <*This clause is added in pencil in the MS.*>