The propositional logic of *Principia Mathematica* and some of its forerunners

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1. Introduction

Principia Mathematica, a monumental work in the foundations of mathematics and the development of logic, has been the subject of much discussion and the genesis of many valuable mathematical results. This paper describes an examination of some of the demonstrations in Part I, Section A, the part dealing with propositional logic. This part of *Principia Mathematica* can be traced to works of Peano and Frege through some earlier work of Russell.

I am interested in this part of *Principia Mathematica* for a number of reasons. The preface states: "The proofs of Part I, Section A, however, are necessary [for the reader's understanding], since in the course of them the manner of stating proofs is explained. The proofs of the earliest propositions are given without the omission of any step, but as the work proceeds the proofs are gradually compressed, retaining however sufficient detail to enable the reader by the help of the references to reconstruct proofs in which no step is omitted." [33, p. vi] The claim of reconstructing the proofs from the compressed versions is interesting. This paper will outline my attempts to complete the reconstruction and will examine some of the difficulties encountered.

A second reason for looking at this part of *Principia Mathematica* lies in the fundamental changes in the subject of logic. This aspect of logic has diminished in importance in recent years as new advances are made in logic. Many results of propositional logic are still exciting and beautiful. We may now look back upon some of this work from an interesting position; we have answers to questions which vexed Russell, Peano, Frege, and others. For example Russell wrote to Frege on 12 December 1904 and offered a set of axioms. He says, "I chose these instead of others because it seemed to me that their number would be the smallest adequate number." [4] The problem of the smallest adequate number of primitive propositions was later solved by Paul Bernays [1] in the context of *Principia Mathematica*. He also showed that one of the axioms of *Principia Mathematica* is redundant.

2. Before and after Principia Mathematica

One source which would have answered all questions is, of course, Russell. I imagine that he worked out all the demonstrations in detail and then condensed them to the current abbreviated form. At the least he must have had a sketch for the complete demonstration. Presumably the next step would be to send them to Whitehead for comments. For a discussion of this process see Douglas Lackey's "The Whitehead Correspondence" [10]. In any case neither Russell's working papers nor the letters to and responses from Whitehead concerning Part I, Section A are in the Russell Archives. There is however a letter from Whitehead to Russell which indicates that, in an early draft of Part I, Section A, the demonstrations had not been written out in sufficient detail. Whitehead says, "Everything, even the object of the book, has been sacrificed to making the proofs look short and neat. It is *essential*, especially in the early parts that the proofs be written out fully—" (emphasis in original). This letter [31] had been later dated by Russell and a typescript [32] prepared. On the bottom of the letter Russell has written, "Whitehead's criticism of $*1$–$*5$ of *Principia Mathematica*". On the bottom of the typescript Russell wrote, "A criticism of my first draft of the Logic of Propositions for the beginning of *Principia Mathematica*. Whitehead was entirely right." Russell assigned a date of 1902. The manuscript to which this letter refers is not extant.

In his autobiography Russell mentions this letter in connection with his work on symbolic logic. He says ([26], I: 227):

> [I] devoted myself to the mathematical elaboration which was to become *Principia Mathematica*. By this time I had secured Whitehead's co-operation in this task, but the unreal, insecure, and sentimental frame of mind into which I had allowed myself to fall affected even my mathematical work. I remember sending Whitehead a draft of the beginning, and his reply: "Everything, even the object of the book, has been sacrificed to making proofs look short and neat." This defect in my work was due to a moral defect in my state of mind.

The troubled mental state here refers to Russell's sudden realization that he no longer loves his first wife, Alys. They continued to live together for many years, but Russell indicates that it was a bad experience for both of them.

This early work was also mentioned in a letter to Frege which Russell wrote on 24 May 1903, a year and a day after completing *The Principles of Mathematics* [4]. This is an excited letter announcing that "I believe I have discovered that classes are entirely superfluous." Russell concludes the letter by stating:

> In the first two parts of my book [*Principles of Mathematics*] there are many things which I did not discuss thoroughly, and many opinions which no longer seem to me correct. But in the later parts most things seem to me correct, provided classes are replaced by functions. In the second volume I hope to work out everything in symbols.

In 1906 Russell's paper "The Theory of Implication" appears [20]. This work, a forerunner to the symbolic logic of *Principia Mathematica*, contains a different logical system but all the essential features of *Principia Mathematica*. The letter from Whitehead to Russell must have been written before the "Theory of Implications".

Russell sent the manuscript to Frank Morley, the editor of *The American Journal of Mathematics*, on 23 July 1905 [19].

After the appearance of the first edition Russell and Wittgenstein discussed *Principia Mathematica* and Wittgenstein criticized it. Kenneth Blackwell [2] reports that in late summer of 1913 there was a plan to rewrite the first volume with Wittgenstein reworking the first eleven chapters. I presume that would be Part I, Sections A and B, propositional logic and quantifiers. In a letter to Lady Ottoline Morrell, dated 23 February 1913 and quoted by Blackwell, Russell told her "Wittgenstein has persuaded me that the early proofs of *Principia Mathematica* are very inexact."

The second edition of *Principia Mathematica* did appear but Part I, Section A was not influenced by any criticisms offered by Wittgenstein. This is easily seen by a proposition by proposition comparison of the two editions. The type for the second edition was reset and a few typographical errors were corrected. By the time of the second edition, however, Russell would have abandoned this part in favour of the new one-axiom method of Nicod [14] using the one primitive idea of Sheffer [28]. These new ideas were not incorporated into the second edition since that would have meant rewriting many of the later proofs to incorporate new references.

3. Construction of the proofs

The analysis presented here is based on the second edition because of its availability. The first and second editions are mathematically the same. Part I, Section A is titled "The Theory of Deduction" and is divided into five chapters numbered ∗1 to ∗5. The first of these contains primitive ideas. As a starting-point negation and disjunction are taken as primitive propositions and implication is defined in terms of them. Next comes a list of ten primitive propositions. A modern view would divide these into three groups: those which describe well formed formulas (∗1·7, ∗1·71, and ∗1·72), those which provide axioms for the system (∗1·2 to ∗1·6), and those which act as inference rules (∗1·1 and ∗1·11). I wish to treat Part I, Section A as a formal system starting with a set of axioms and inference rules and proceeding with a series of proofs in which each line is a consequence of one or more preceding lines.

To help in constructing the proofs I used a digital computer at the University of Maine. I wrote a computer programme, in the language SNOBOL4 [7], which accepts an outline of a proof, applies the indicated inference rules, and writes out the proof in full. This is a particularly valuable technique since it reduces the drudgery and increases the accuracy of the work. The most valuable assistance comes with the substitution rule (described below) for some of the resultant expressions may have fifty to sixty symbols.

The demonstrations of *Principia Mathematica* were translated to a form based upon that in *Elements of Mathematical Logic* by Jan Lukasiewicz [13]. The propositions were rewritten in Polish notation. This has the advantage of not requiring special characters on the keyboard, all the symbols necessary being found on a standard typewriter keyboard. The demonstrations were also rewritten in a form which is not unlike the proof format used by Lukasiewicz. The structure of the demonstrations seems to be particularly amenable to this format.

The digital computer method has another advantage. It does not make assumptions. One must tell the programme exactly what to do. The symbols must match character for character or the programme indicates an error. For example, a conjunction must have the conjuncts in the proper order if it is to be used as the antecedent in an application of modus ponens. This method has a curious feature for which I must apologize to Russell. In *My Philosophical Development* ([25], p. 110), he criticizes the formalists, led by Hilbert, for maintaining that "arithmetical symbols are merely marks on paper, devoid of meaning, and that arithmetic consists of certain arbitrary rules, like the rules of chess, by which the marks can be manipulated." This is the "point of view" taken by my computer programme; the symbols are merely electronic marks to be manipulated by certain rules built into the programme and invoked by the outline of the proof provided. My computer programme does not find a proof but constructs one by following an explicit set of directions. In mitigation I quote Warren Goldfarb [5]:

> The upshot of formalization, for Frege and Russell, is thus somewhat more limited than we might suppose. They wish to render mathematics in such a way that all principles used in mathematical argumentation are explicit. Formalization enables us to do this: by precise transformation rules we see exactly what each step depends upon. The precision, and the possibility of checking formal proofs without knowing what the formulas mean, is essential only for the purpose of insuring that we have not let any principle be used without due notice. But this is not to say that in the end we take the precise rules as uninterpreted.

4. Part I, Section A as a formal system

To view Part I, Section A of *Principia Mathematica* as a formal system one must present (at least) a set of axioms and a set of inference rules. In this case it is best to precede these by an enumeration of the primitive ideas and definitions. Section A is based upon two primitive ideas, negation and disjunction. (The second edition would have these reduced to one, incompatibility.) From these two connectives the other three connectives used, implication, conjunction, and equivalence, are defined. There is also a set of rules which tell us how to build (or to check) propositions from constituents. These are given only in terms of the two primitive connectives, since any proposition may, by application of the definitions, be rewritten using just these two primitive connectives. The definitions used are:

∗1·01 $p \supset q . = . \sim p \vee q$ Df,
∗3·01 $p . q . = . \sim(\sim p \vee \sim q)$ Df,
∗4·01 $p \equiv q . = . p \supset q . q \supset p$ Df.

Note that ∗4·01 is not defined in terms of the primitive connectives, but rather in terms of connectives which had already been defined. The four connectives negation, disjunction, conjunction, and implication hold a special place as fundamental functions in *Principia Mathematica*, a place not shared by equivalence. In the introduction to *Principia Mathematica* ([33], I: 7) one finds that:

> These four functions of propositions are the fundamental constant (i.e. definite) propositional functions with *propositions as arguments*, and all other constant propositional functions with propositions as arguments, so far as they are required in the present work, are formed out of them by successive steps.... The simplest example of the formation of a more complex

function of propositions by the use of these four fundamental forms is furnished by "equivalence."

Russell holds that definitions are volitional; they may be eliminated in favour of the *definiens*. Most of us would be reluctant to proceed without the definitions noted above. There is another set of definitions for which most of us would not feel quite a strong need. These have more the flavour of a short notation than the "seemingly" more fundamental definitions given above. This class of definitions serve "only for the avoidance of brackets." These definitions are:

∗2·33 $p \vee q \vee r . = . (p \vee q) \vee r$ Df,
∗3·02 $p \supset q \supset r . = . p \supset q . q \supset r$ Df,
∗4·02 $p \equiv q \equiv r . = . p \equiv q . q \equiv r$ Df,
∗4·34 $p . q . r . = . (p . q) . r$ Df.

The next item to present in the formal system is the set of axioms. These are given as:

∗1·2 $\vdash : p \vee p . \supset . p$ Pp,
∗1·3 $\vdash : q . \supset . p \vee q$ Pp,
∗1·4 $\vdash : p \vee q . \supset . q \vee p$ Pp,
∗1·5 $\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$ Pp,
∗1·6 $\vdash :. q \supset r . \supset : p \vee q . \supset . p \vee r$ Pp.

As noted above Bernays showed that ∗1·5 may be derived from the others. Emil Post [15] showed that this axiom set is sufficient (with an appropriate set of inference rules) to derive all tautologies.

We must now turn our attention to the inference rules. This is the area which has caused the most difficulty in my analysis. Russell is clear about what constitutes an inference. In *Principles of Mathematics* ([18], p. 17) he says that "all our axioms are principles of deduction...." Since Russell is here arguing from a logicism position he cannot go outside the system of logic; logic is all that there is. Thus the axioms serve both as starting places in derivations and as potential rules of passage from one line in a demonstration to another. This view is confirmed in *Introduction to Mathematical Philosophy* ([23], p. 149):

> these [axioms] are the *formal* principles of deduction employed in *Principia Mathematica*. A formal principle of deduction has a double use.... It has the use as the premiss of an inference, and a use as establishing the fact that the premiss implies the conclusion. In the schema of an inference we have a proposition *p*, and a proposition "*p* implies *q*," from which we infer *q*. Now when we are concerned with the principles of deduction, our apparatus of primitive propositions has to yield both the *p* and the "*p* implies *q*" of our inferences. That is to say, our rules of deduction are to be used, not *only* as *rules* which is their use in establishing "*p* implies *q*," but *also* as substantive premisses, i.e. as the *p* of our schema.

In my work I have found eight inference rules in use in this part of *Principia Mathematica*. Some of the rules are given explicitly and some must be developed

by a study of the demonstrations. The rules given here are those used in the passage from one proposition to another. The rules which describe well-formed propositions are not included here.

A distinction must be made between primitive inference rules and derived inference rules. It is a situation not unlike that involving axioms and propositions in a formalist view. Certain inference rules are taken as primitive; other inference rules may be used, but one must show that these rules are derivable in reference to the system under consideration.

Some of the rules given here have more than one form. These different forms haven't been counted separately in the eight rules. Table 4.1 summarizes the rules and indicates the demonstration which contains the first use of each rule.

4.1 The first rule which I shall describe is *modus ponens*. In doing so I wish to skirt any discussion of formal and material implication. I found three forms of *modus ponens*: the conventional form with implication and two involving equivalence as the major binary connective.

The conventional form is established by ∗1·11. I will paraphrase it here as "From $\vdash . P$ and $\vdash . P \supset Q$ infer $\vdash . Q$." Here and below *P*, *Q*, etc. will refer to arbitrary (well-formed) propositions.

The second form involves equivalence. It may be paraphrased as "From $\vdash . P$ and $\vdash . P \equiv Q$ infer $\vdash . Q$."

The third form also involves equivalence, but goes the other way. This may be paraphrased as "From $\vdash . Q$ and $\vdash . P \equiv Q$ infer $\vdash . P$." These two forms are curious in that they are used in their own right. One could have reduced an equivalence to a conjunction of implications (by definition ∗4·01) and then written each conjunct on a proof line. This method is not used. In fact it appears that reducing a conjunction to its conjuncts is not used at all in this section of *Principia Mathematica*.

The use of the second and third form is anticipated by the Introduction, but these forms are never given as formal rules. In the Introduction ([33], I: 7) we find, "Equivalence rises in the scale of importance when we come to 'formal implication' and thus to 'formal equivalence'.... Equivalence in its origin is merely mutual implication...."

4.2 The next rule to consider is substitution. Well-formed formulas may be substituted for the sentential variables in a proposition when the substitutions are uniform and simultaneous. By uniform I mean that each occurrence of a sentential variable must have the same well-formed formula substituted for it. By simultaneous I mean that all substitutions for sentential variables must be done at the same time.

In *Introduction to Mathematical Philosophy* ([23], p. 151n.) Russell points out the necessity for this rule. Alonzo Church, in *Introduction to Mathematical Logic* ([3], p. 32), attributes an explicit statement of the rule of substitution to C.I. Lewis in *A Survey of Symbolic Logic* [12].

Russell clearly understood the rule and even gives an example of its use on page 94 of *Principia Mathematica*. At various places in the demonstrations necessary substitutions are indicated. The indications are not always given; but when given they are complete and correct. In general more indications are given earlier than later, so this is part of the process of compressing the proofs.

4.3 The rule of replacement has two forms. Neither form is given explicitly, but both are used extensively in the work.

This rule allows the use of definitions. As actually used in practice it allows a little more. Stating both forms at once we get: The *definiens* (*definiendum*) of a definition or the result of a simultaneous and uniform substitution in a definition may be replaced by the *definiendum* (*definiens*). Thus, not only may a definition be used but any substitution instance of a definition is also allowed. This later form is not mentioned in any examples but is used throughout Section A.

4.4 The rule of replacement, above, dealt with definitions. There is a similar rule which deals with propositions in which the major connective is an equivalence. The rule has two forms, one dealing with equivalence and one dealing with mutual implication. The two forms are necessary because the rule is needed early in Section A, but equivalence is not defined until near the end (of Part I, Section A). The two forms of the rule are: (*a*) If some previously proved proposition is of the form $P \supset Q$ and some previously proved proposition or substitution instance of a previously proved proposition is of the form $Q \supset P$, then an occurrence of P in some proposition may be replaced by Q. (*b*) If some previously proved proposition or substitution instance of a previously proved proposition is of the form $P \equiv Q$, then an occurrence of P in some proposition may be replaced by Q and vice versa.

The demonstrations never explicitly mention the use of this rule, it must be inferred from the various lines as they are constructed. The primary purpose of this rule is in implementing associativity and commutativity and in removing double negations.

This is not a primitive rule. Hilbert and Ackermann ([8], p. 33) provide a proof which may be applied to *Principia Mathematica*. The proof requires proposition *2·16, but the first use of this rule is not made until *2·38.

This rule is anticipated in the Introduction to *Principia Mathematica* in a discussion of truth-values ([33], I: 8):

> [I]t is easy to see (though it cannot be formally proved except in each particular case) that if a proposition p occurs in any proposition $f(p)$ which we shall ever have occasion to deal with, the truth value of $f(p)$ will depend, not upon the particular proposition p, but only upon its truth value; i.e. if $p \equiv q$, we shall have $f(p) \equiv f(q)$. Thus whenever two propositions are known to be equivalent, either may be substituted for the other in any formula with which we shall have occasion to deal.

4.5 In *Principia Mathematica* some of the derived rules are proved. Two such rules are proved in Section I, Part A. Both proofs require an assumption, so we need a rule which allows us to place an assumption on a proof line. This rule is only used in the proof of inference rules, not in the demonstrations of propositions.

4.6 The first derived inference rule proved is *3·03. This rule allow one to assert the conjunction of two asserted propositions.

4.7 In some cases the demonstrations of *Principia Mathematica* require a proposition with implication as the major connective, but the form proved actually has equivalence as the major connective. For example the demonstration of *5·15 requires the proposition

$$\sim(p \supset q) . \supset . p . \sim q$$

be obtained from *4·61 which is

$$\sim(p \supset q) . \equiv . p . \sim q.$$

The rule here is similar to the form of modus ponens involving equivalence in that this rule is used in its own right. There is no mediating use of a rule which allows one to assert each conjunct of a conjunction.

4.8 Section I, Part A has proofs of two derived rules. The first proof is in a numbered demonstration, *3·03, but the second is in a note following *4·3. This rule has only two uses, in the demonstrations of *4·3 and *4·31. It is used to show that if an implication can be proved showing that some two-place connective commutes, then the implication may be replaced by an equivalence.

Table 4.1
Summary of Inference Rules Used in Section I, Part A of *Principia Mathematica*

Rule	Type	First Use
Modus Ponens		
Conventional	Primitive	*2·04
Equivalence	Derived	*4·32
Equivalence	Derived	*5·15
Substitution	Primitive	*2·01
Replacement	Primitive	*2·01
Replacement of Equivalents	Derived	*2·38
Assumption	—	*3·03
Conjunction Introduction	Derived	*3·47
Changing Equivalence to Implication	Derived	*5·15
Equivalence of Commutative Connectives	Derived	*4·3

5. An examination of some demonstrations

In examining the demonstrations of Section A a number of problems were encountered. From most demonstrations one may easily reconstruct the proof, other demonstrations are difficult, some seem impossible.

To claim that a demonstration doesn't go through is not to claim that the associated proposition is false, but merely that the demonstration in question is not adequate to show the desired proposition. If required a proposition may be shown by showing that it is a tautology. This is, however, a modern view of logic, a view which states that truth lies in an interpretation. Interpretation is a metatheoretic concept (and so outside the formal system) and has no place in *Principia Mathematica*.

The concept of interpretation is, however, very important to my work. If a demonstration fails the proposition asserted is suspect. Suspicion may be removed by showing (by truth-tables methods) that the proposition is a tautology. It is now known that all tautologies, and only tautologies, may be derived in this formal system.

As mentioned above the second edition was used. There are a few typographical errors which I will indicate. In ∗2·32 the first and second lines of the demonstrations should each end with a dot. In ∗4·72 a horizontal bar is missing from the substitution.

5.1 Prior to the demonstration of ∗2·31 an abbreviation used in some of the subsequent demonstrations is introduced. Some understanding of this abbreviation is necessary to understand a problem which is discussed below. Moreover, this abbreviation creates some problems of its own. A series of propositions of the form $a \supset b$, $b \supset c$, $c \supset d$, all asserted, may be used to assert $a \supset d$. The detailed scheme for this is given as ([33], I: 104):

> This proposition and ∗2·32 together constitute the associative law for logical addition of propositions. In the proof, the following abbreviation (constantly used hereafter) will be employed*: When we have a series of propositions of the form $a \supset b$, $b \supset c$, $c \supset d$, all asserted, and "$a \supset d$" is the proposition to be proved, the proof in full is as follows:

[Syll]	$\vdash :. a \supset b . \supset : b \supset c . \supset . a \supset c$	(1)
	$\vdash : a . \supset . b$	(2)
[(1).(2).∗1·11]	$\vdash : b \supset c . \supset . a \supset c$	(3)
	$\vdash : b . \supset . c$	(4)
[(3).(4).∗1·11]	$\vdash : a . \supset c$	(5)
[Syll]	$\vdash :. a \supset c . \supset : c \supset d . \supset . a \supset d$	(6)
[(5).(6).∗1·11]	$\vdash : c \supset d . \supset . a \supset d$	(7)
	$\vdash : c . \supset . d$	(8)
[(7).(8).∗1·11]	$\vdash : a . \supset . d$	

where Syll means ∗2·06. In general the process is abbreviated as

$\vdash : a . \supset . b .$
[etc.] $\supset . c .$
[etc.] $\supset . d$

where $a \supset d$ is the desired result.

This provides an extension of the syllogism to prove a sorites of any length. This is the second, but more common, way in which this process is abbreviated. Following the demonstration of ∗2·15 there is a note which indicates that such proofs may also be indicated as

$$[\text{Syll}] \vdash . (a) . (b) . (c) . \supset \vdash . (d).$$

These two combine to raise the question of the status of such abbreviations as inference rules. Should one consider these as new inference rules or as compressions of the proofs? I have taken the later view, and present the proofs as the expanded version noted above.

While this facilitates the analysis which I wish to perform I wonder if it is true

to the intentions of Russell and Whitehead. *Principia Mathematica* makes a distinction between the use of a proposition as a rule of inference and as an ordinary line in a proof. The use as a rule of inference is indicated by enclosing the name or number of the proposition in square brackets. The use as an ordinary line in a proof is indicated by the assertion sign, ⊢. thus the notation for a sorites,

$$[\text{Syll}] \vdash . (a) . (b) . (c) . \supset \vdash . (d),$$

indicates the use of Syll as an inference rule. How should this be taken: to mean repeated applications of Syll or a "super-Syll" which allows a sorites of any length?

The convention of enclosing rules in square brackets is violated by the first abbreviation discussed above. In the full form the only rule of inference used is modus ponens. Syll is used as a line in the deduction not as an inference rule. When the abbreviation is introduced, the references to Syll are deleted. The propositions required in lines (2), (4), and (8) have their names or numbers enclosed in square brackets, but they too are just lines in the demonstration.

In any case, the distinction is blurred for all demonstrations from ∗2·4 onward. We find ([33], I: 106):

> [W]hen a general rule is adduced in drawing an inference, as when we write "[Syll] ⊢.(1).(2).⊃⊢.Prop," the mention of "Syll" is only required to remind the reader how the inference is drawn. The rule of inference may, however, also occur as one of the ordinary premisses, that is to say, in the case of "Syll" for example, the proposition "$p \supset q . \supset : q \supset r . \supset . p \supset r$" may be one of those which our rules of deduction are applied, and it is then an ordinary premiss. The distinction between the two uses of principles of deduction is of some philosophical importance, and in the above proofs we have indicated it by putting the rule of inference in square brackets. It is, however, practically inconvenient to continue to distinguish in the manner of reference. We shall therefore henceforth both adduce ordinary premisses in square brackets where convenient, and adduce rules of inference, along with other propositions, in asserted premisses, i.e. we shall write e.g. "⊢(1).(2).Syll.⊃⊢.Prop" rather than "[Syll] ⊢.(1).(2).⊃⊢.Prop"

5.2 The first demonstration to discuss in detail is ∗2·85 which is reproduced below. I will also use it as an example of how the demonstrations may be read. Recall that the demonstrations are compressed as the work progresses.

∗2·85 $\vdash :. p \vee q . \supset . p \vee r : \supset : p . \vee . q \supset r$

Dem.

[Add.Syll]	$\vdash :. p \vee q . \supset . r : \supset . q \supset r$	(1)
$\vdash$. ∗2·55 . $\supset \vdash :: \sim p . \supset :. p \vee r . \supset . r :.$		
[Syll]	$\supset :. p \vee q . \supset . p \vee r : \supset : p \vee q . \supset . r :.$	
[(1).∗2·83]	$\supset :. p \vee q . \supset . p \vee r : \supset : q \supset r$	(2)
$\vdash$.(2). Comm . $\supset \vdash :.$	$p \vee q . \supset . p \vee r : \supset : \sim p . \supset . q \supset r :$	
[∗2·54]	$\supset : p . \vee . q \supset r :. \supset \vdash$. Prop	

This demonstration contains three component parts. The first part is the first line, the second part is the next three lines, and the third part is the last two lines. Notice that each part ends with either a number or "Prop". Each one of these parts could have been a demonstration in its own right. They are combined here for the convenience of the authors, since the intermediate propositions are not required for use in any other demonstration.

Certain propositions have names, a practice carried over from Peano. Part I, Section A is the only place in *Principia Mathematica* where names are assigned. Thus the first part of this demonstration tells us to combine the propositions named Add and Syll in some way to yield the proposition

$$\vdash : p \lor q . \supset . r : \supset . q \supset r$$

and call it (1) for further use.

Generally such a notation calls for substitutions in the named propositions followed by modus ponens to get the desired result. *Principia Mathematica* contains a table entitled "Alphabetical List of Propositions Referred to by Names". In this table one may find the following entries:

Add $*1{\cdot}3$ $\vdash : q . \supset . p \lor q$
Syll $*2{\cdot}05$ $\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$
Syll $*2{\cdot}06$ $\vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r.$

Here no substitutions are required in $*1{\cdot}3$. One must use $*2{\cdot}06$ with q substituted for p and $p \supset q$ substituted for q. (The substitutions must be uniform and simultaneous.) The result from $*2{\cdot}06$ is

$$\vdash :: q . \supset . p \lor q : \supset :. p \lor q . \supset . r : \supset : q \supset r.$$

The antecedent here is (equiform with) $*1{\cdot}3$, modus ponens may be applied, and the desired result obtained. This is a common practice in the demonstrations.

The next part of the demonstration is meant to yield (2) which is

$$\vdash :: \sim p . \supset :. p \lor q . \supset . p \lor r : \supset : q \supset r.$$

The use of this type of demonstration was discussed above. Notice here the call for the intermediate step [(1) . $*2{\cdot}83$]. As explained previously the demonstration requires

$$\vdash :: p \lor q . \supset p \lor r : \supset : p \lor q . \supset . r :. \supset :. p \lor q . p \lor r : \supset : q \supset r$$

at this point. This proposition is a tautology, so is provable, but it cannot be proved by (1) and $*2{\cdot}83$. In this context one does not expect to make substitutions in (1). The substitution of $p \lor q . \supset r$ for p in $*2{\cdot}83$ will provide the correct antecedent, i.e (1), but the wrong consequent. The substitution of $p \lor q . \supset . p \lor r$ for p, $p \lor q$ for r, and r for s in $*2{\cdot}83$ will produce the correct consequent but not the correct antecedent.

It is not clear how the demonstration is to go, short of recognizing that this is simply an error. If the required line is supplied (2) is produced and the rest of the demonstration proceeds without difficulty. The required line is a tautology, so may be proved, thus the demonstration could be made to succeed. I suspect that the error lies in (1), that the wrong proposition was proved but there does not seem to be a way to know. Neither $*2{\cdot}85$ nor (1) appear in any of the forerunners to be discussed below.

5.3 The next point of difficulty is in $*3{\cdot}47$. The demonstration does not work as written. This is particularly interesting because it is the first place where $*3{\cdot}03$, the rule which allows conjunctions of asserted propositions, is called for.

The demonstration, reproduced below, is in three parts.

$*3{\cdot}47$ $\vdash :. p \supset r . q \supset s . \supset : p . q . \supset . r . s$

This proposition, or rather its analogue for classes, was proved by Leibniz, and evidently pleased him, since he calls it "praeclarum theorema."

Dem.

$\vdash . *3{\cdot}26 . \supset \vdash :. p \supset r . q \supset s . \supset : p \supset r :$
[Fact] $\supset : p . q . \supset . r . q :$
[$*3{\cdot}22$] $\supset : p . q . \supset . q . r.$ (1)
$\vdash . *3{\cdot}27 . \supset \vdash :. p \supset r . q \supset s . \supset : q \supset s :$
[Fact] $\supset : q . r . \supset . s . r :$
[$*3{\cdot}22$] $\supset : q . r . \supset . r . s$ (2)
$\vdash . (1) . (2) . *3{\cdot}03 . *2{\cdot}83 . \supset$
$\vdash :. p \supset r . q \supset s . \supset : p . q . \supset . r . s :. \supset \vdash .$ Prop

The first and second parts prove

$$\vdash :. p \supset r . q \supset s . \supset : p . q . \supset . q . r$$

and

$$\vdash :. p \supset r . q \supset s . \supset : q . r . \supset . r . s$$

respectively. The third part, $\vdash (1) . (2) . *3{\cdot}03 . *2{\cdot}83$, indicates that one should form the conjunction of (1) and (2) using the rule $*3{\cdot}03$, make a substitution in $*2{\cdot}83$ for which the conjunction is the antecedent, apply modus ponens, and get the proposition. No such substitution will work.

Moreover, on page 110 of the second edition (page 115 of the first edition), this demonstration is cited as an example of the use of the conjunction rule. The description has a different third part for the demonstration, one which uses $*3{\cdot}43$ and $*3{\cdot}33$, but never mentions $*2{\cdot}83$:

> As an example of the use of this proposition, take the proof of $*3{\cdot}47$. We there prove
> $\vdash :. p \supset r . q \supset s . \supset : p . q . \supset . q . r$ (1)
> and $\vdash :. p \supset r . q \supset s . \supset : q . r . \supset . r . s$ (2)
> and what we wish to prove is
> $$p \supset r . q . \supset s . \supset : p . q . \supset . r . s,$$
> which is $*3{\cdot}47$. Now in (1) and (2), p, q, r, s are elementary propositions (as everywhere in Section A); hence by $*1{\cdot}7{\cdot}71$, applied repeatedly "$p \supset r . q \supset s . \supset : p . q . \supset . q . r$" and "$p \supset r . q \supset s . \supset : q . r . \supset . r . s$" are elementary propositional functions. Hence by $*3{\cdot}03$, we have

$$\vdash :: p \supset r . q \supset s . \supset : p . q . \supset . q . r :. p \supset r . q \supset s . \supset : q . r . \supset . r . s,$$

whence the result follows by $*3{\cdot}43$ and $*3{\cdot}33$.

This version can be made to work if a syllogism is applied. This is basically the question discussed as to whether or not a syllogism should count as a derived rule.

The more interesting aspect of this problem is an attempt to reconstruct the demonstration as given. This can also be made to work, but requires that the reference to $*3{\cdot}03$ be deleted.

One commentator on *Principia Mathematica* is Jørgen Jørgensen in *Introduction to Mathematical Logic*. He expands some of the demonstrations and examines this one ([9], I: 237). He follows the demonstration (not the note on page 110 of *Principia Mathematica*) and indicates that the conjunction of (1) and (2) is to formed, but he never indicates a use for the conjunction in the demonstration.

The demonstration of $*3{\cdot}48$ is analogous to the version of $*3{\cdot}47$ which does not use the conjunction rule.

5.4 The statement of $*4{\cdot}87$ is not well formed. This proposition combines the principles of exportation, importation, and the commutative principle by showing that they are equivalent. This is indicated by using three equivalence signs of equal status. Recall that $*4{\cdot}02$ ($p \equiv q \equiv r . = . p \equiv q . q \equiv r$ Df) is a definition which shows how two equivalence signs may be combined by the use of conjunction, but it is not clear how three equivalence signs are to be combined.

I should remark here that I haven't been able to reconstruct all the proofs. Some still remain a mystery to me. I haven't mentioned all the problems encountered in Part I, Section A, just some of the more interesting ones.

6. *A troublesome "rule" of inference*

The most serious difficulty is the seeming use of a false rule of inference. The problem is easily shown by an example. Consider the demonstration of $*4{\cdot}32$:

$*4{\cdot}32 \ \vdash : (p . q) . r . \equiv . p . (q . r)$

Dem.

$\vdash . *4{\cdot}15 . \quad \supset \vdash :. p . q . \supset . \sim r : \quad \equiv : q . r . \supset . \sim p :$

$[*4{\cdot}12] \qquad \equiv : p . \supset . \sim (q . r) \qquad (1)$

$\vdash . (1) . *4{\cdot}11 . \quad \supset \vdash : \sim (p . q . \supset . \sim r) . \equiv . \sim \{p . \supset . \sim (q . r)\} :$

$[(*1{\cdot}01 . *3{\cdot}01)] \supset \vdash . \text{Prop}$

Note. Here (1) stands for "$\vdash :. p . q . \supset . \sim r : \equiv : p . \supset . \sim (q . r)$," which is obtained from the above steps by $*4{\cdot}22$. The use of $*4{\cdot}22$ will often be tacit, as above. The principle is the same as that explained in respect of implication in $*2{\cdot}31$.

The first part, (1), requires

$$\vdash :. q . r . \supset . \sim p : \equiv : p . \supset . \sim (q . r)$$

be obtained from $*4{\cdot}12$. By substituting $q . r$ for p and p for q in $*4{\cdot}12$ one gets

$$\vdash :. q . r . \equiv . \sim p : \equiv : p . \equiv . \sim (q . r)$$

from which the desired result seems to follow by changing the symbols for equivalence to symbols for implication. Such a rule holds in case the equivalence is the primary connective (as explained above), but not generally for other instances of equivalence. Consider, for example,

$$*4{\cdot}21 \ \vdash : p \equiv q . \equiv . q \equiv p.$$

An application of the "rule" provides

$$p \supset q . \equiv . q \supset p$$

which fails to be a tautology when p is true and q is false.

7. *A rule which was not used*

It is well known that Russell was greatly influenced by Peano as a result of their meeting at the International Congress of Philosophy in Paris, 1900. Russell reports that he read much of the material which Peano and other members of his school had written. In *The Collected Papers of Bertrand Russell*, Volume 1, one finds ([27], p. 345), "From February 1891 to March 1902 Russell kept a monthly list of books, partial books and some individual papers which he read during that time. This list, which totals 785 entries, bears the printed title 'What Shall I Read?' and is subtitled 'A Record of Books Read and to Be Read'." From this record we know that Russell read Peano's *Formulaire de mathématiques*, I, II.

Russell adopted many ideas from Peano including the basic techniques of proof in symbolic logic. The structure of the demonstrations in *Principia Mathematica* is similar to those presented by Peano in his early work.

An idea which Resell did not adopt from Peano is the deduction theorem which was used extensively by Peano. For example in *Formulaire de mathématiques*, I, Section I [16] a demonstration is given as

$$21.\ a = b . b \supset c . \supset . a \supset c \qquad [\text{Hp} . \supset . a \supset b . b \supset c . \supset a \supset c].$$

Peano used a, b, and c where Russell used p, q, and r and $=$ where Russell used $\equiv$. Here Hp means the hypothesis of the proposition to be proved, $a = b . b \supset c$. From this one may get $a \supset b . b \supset a . b \supset c$, from which $a \supset c$ follows by eliminating conjuncts. The proposition follows from the deduction theorem. It was later shown to be a valid rule of inference in *Principia Mathematica* by Jacques Herbrand.

This rule is not used by Russell in either "The Theory of Implication" or *Principia Mathematica*. The abbreviation "Hp" does appear in $*5{\cdot}71$ and $*5{\cdot}75$ of *Principia Mathematica*, but it has a different use.

8. *The theory of implication*

"The Theory of Implication" [20], marks an important point in Russell's development of propositional logic. It contains much of the text which shows up in the expository part at the beginning of Part I, Section A of *Principia Mathematica*. Appendix A contains a detailed comparison of the two versions. The logic system, however, is not the same.

The text at the beginning of "The Theory of Implication" may be matched with some of the text in Part I, Section A. The first line of each stands out when the comparison is made. The first sentence of "The Theory of Implication" is, "The purpose of the present article is to set forth the first chapter of the deduction of pure mathematics from its logical foundations." In *Principia Mathematica* the sentence appears on page 90 but the word chapter has been changed to stage. This article is apparently part of a larger work. In his letter to Morley of 23 July 1905 Russell says, "If you do not feel that it is too much out of your line, I should like to follow it by a second article proceeding to the theory of classes. For, as it stands, it is a mere preface." The second article never appeared.

Page 90 of *Principia Mathematica* contains two long paragraphs. The corresponding part of "The Theory of Implication" contains essentially these two paragraphs with a third one which acknowledges ideas adopted from Peano and Frege. Russell says, "But although the symbolism is in the main Peano's, the ideas are more those of Frege.... The plan of taking implication and negation as our primitive ideas is also his."

This is the primary difference between "The Theory of Implication" and Part I, Section A of *Principia Mathematica* where disjunction and negation are primitive ideas.

The two ideas of negation, however, do not coincide exactly. In *Principia Mathematica* we find ([33], I: 93], "If p is any proposition, the proposition 'not-p' or 'p is false' will be represented by '$\sim p$'." In "The Theory of Implication" Russell says ([20], p. 164), "The proposition 'p is not true' is expressed by $\sim p$. Thus '$\sim p$' will be true if p is not a proposition, and if p is a false proposition; it will only be false if p is a true proposition."

The most interesting part (to me) of "The Theory of Implication" is Russell's statement of the rule of substitution mentioned above. This rule is not included in *Principia Mathematica* but is given explicitly here, with a number of examples of its use. Russell calls this $*2{\cdot}3$ and marks it as a primitive proposition. He gives an example of its use, stating, "... it is needed for the inference that, since '$p \supset p$' is true for any value of p, it is true if one substitute $p \supset q$ for p, so that '$p \supset q . \supset . p \supset q$' is true for any values of p and q. (The principle is to be extended to two or more variables.)" Russell had a formulation of the rule of substitution, and chose not to state it in *Principia Mathematica*.

The formal system of "The Theory of Implication" is based on seven axioms which employ only implication and negation. One of the axioms, $*2{\cdot}5$, is redundant. Jules Vuillemin ([30], p. 38) gives a proof from some of the other axioms.

The inference rules used are similar to those of *Principia Mathematica*. However, the rule which allows asserting the conjunction of two asserted propositions does not appear. The other inference rule which is proved in Part I, Section A, concerning equivalence with a commutative function, appears in essentially the same form. It is relegated to a note here as in *Principia Mathematica*.

The deduction theorem does not appear here either. Frege did not use it in his work and since the ideas here follow Frege (and *Principia Mathematica* follows the ideas here), Russell probably did not feel a need for it.

In examining "The Theory of Implication" I had hoped to shed some light on some of the demonstrations of *Principia Mathematica*. Many of the demonstrations are identical, although the propositions have different numbers. In general "The Theory of Implication" gives less information about demonstrations, and in some cases gives no demonstration at all.

The role of "The Theory of Implication" in the development of *Principia Mathematica* is interesting, and worthy of further exploration. Russell describes the work, in a section headed "$*3$ Elementary Properties of Implication and Negation", as follows:

> The propositions that follow are all such as are actually needed in deducing mathematics from our primitive propositions. I shall omit proofs of some of the less important, and sometimes only briefly indicate the proofs where they are obvious. But in most cases I shall give the proofs in full, because the importance of the present subject lies, not in the propositions themselves, but (1) in the fact that they follow from the primitive propositions, (2) in the fact that it is the easiest, simplest, and most elementary example of the symbolic method of dealing with the principles of mathematics generally. Later portions—the theory of classes, relations, cardinal numbers, series, ordinal numbers, geometry, etc.—all employ the same method, but with an increasing complexity in the entities and functions, considered.

The first part of the paragraph sheds light on why some of the demonstrations are hard to follow.

9. After "The Theory of Implication"

In the time between "The Theory of Implication" and *Principia Mathematica* Russell underwent a shift in perspective concerning the primitive elements of propositional logic. As quoted above he viewed the notation of "The Theory of Implication" as coming from Peano but the ideas come more from Frege.

In a letter to Frege, dated 12 December 1904 [4], Russell presents some of the axioms of "The Theory of Implication". He gives a formal definition of negation (involving the universal quantifier) and a primitive proposition for negation (the principle of reduction of *The Principles of Mathematics*). At the end of "The Theory of Implication" ([20], p. 200) Russell explains his idea for a definition of negation and why he rejected it.

The next (now) published work by Russell [21], a paper read before the Cambridge Mathematical Club on 9 March 1907, contains the system of *Principia Mathematica*. The axioms are offered as alternative set to Frege's axioms, with no mention of the intervening set in "The Theory of Implication".

The system in *Principia Mathematica* probably came a little sooner. A form of one of the axioms, $p . \supset p \vee q$, is mentioned as a primitive proposition in a letter from Russell to Jourdain on 10 September 1906 [6].

The last major work before *Principia Mathematica* which contains a list of primitive propositions is "Mathematical Logic as Based on the Theory of Types" [22]. Here the *Principia Mathematica* system appears, complete except for working out the propositions. Russell comments, "In a previous article in [*The American Journal of Mathematics*], I took implication as undefinable, instead of disjunction, because it enables one to diminish the number of primitive propositions."

Thus the change in primitive propositions is a matter of taste, but taste dictated by one goal. One system is preferred over another if the former has a fewer number of primitive ideas. This single goal motivates the effort in propositional logic, to make the edifice of mathematics stand on as small a base as possible.

10. Conclusion

The work done in the propositional logic section of *Principia Mathematica* is a model of succinct expression. There are more than 200 numbered propositions in Part I, Section A. Many of the demonstrations contain parts which could stand alone. This raises the number of propositions to about 350—a monumental number by any standards.

It is the culmination of a long line of work which can be traced through "The Theory of Implication" to the founding work of Peano and Frege. From Peano Russell borrowed the notation and from Frege the concepts. When Russell was done we had a new work which went far beyond that of his two predecessors.

In the development of propositional logic Russell holds a premier position. All questions must be asked and answered in terms of *Principia Mathematica*. There are still many problems for the historian and mathematician to work on. I enjoy the opportunity to present some of my work and ideas.

Dover, New Hampshire

APPENDIX A. COMPARISON OF TEXT

I have prepared a comparison of some of the text in "The Theory of Implication", pp. 159–64, and the corresponding text in Part I, Section A of the second edition of *Principia Mathematica*, pp. 90–4. In this comparison "The Theory of Implication" is taken as the starting place. Deletions are indicated by italicized text *within curly brackets* (to distinguish them from normal italicized text), and additions are indicated by bold face. Thus to read the "Theory of Implication" version one should read the normal text and the italic text only. To read the *Principia Mathematica* version one should read the normal text and the text in bold face only. The footnotes have been numbered consecutively in "The Theory of Implication". Footnotes which appear only in *Principia Mathematica* have a number and a letter.

The Text by Russell

The purpose of the present {*article*} **section** is to set forth the first {*chapter*} **stage** of the deduction of pure mathematics from its logical foundations. This first {*chapter*} **stage** is necessarily concerned with deduction itself, *i.e.* with the principles by which conclusions are inferred from premisses. If it is our purpose to make all our assumptions explicit, and to effect the deduction of all our other propositions from these assumptions, it is obvious that the first assumptions we need are those that are required to make deduction possible. Symbolic logic is often regarded as consisting of two coordinate parts, the theory of classes and the theory of propositions. But from our point of view these two parts are not coordinate; for in the theory of classes we deduce one proposition from another by means of principles belonging to the theory of propositions, whereas in the theory of propositions we nowhere require the theory of classes. Hence in a deductive system, the theory of propositions necessarily precedes the theory of classes.

But the subject to be treated in what follows is not quite properly described as the theory of *propositions*. It is in fact the theory of how one proposition can be inferred from another. Now in order that one proposition may be inferred from another, it is necessary that the two

should have that relation which makes the one a consequence of the other. When a propositions *q* is a consequence of a proposition *p*, we say that *p implies q*. Thus deduction depends upon the relation of *implication*, and every deductive system must contain among its premisses as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction. In the present {*article*} **section**, certain propositions {*concerning implication*} will be stated as premisses, and it will be shown that they are sufficient for all common forms of inference. It will not be shown that they are all *necessary*, and it is {*probable*} **possible** that the number of them might be diminished. All that is affirmed concerning the premisses is (1) that they are true, (2) that they are sufficient for the theory of deduction, (3) that I do not know how to diminish their number. But with regard to (2), there must always be some element of doubt, since it is hard to be sure that one never uses some principle unconsciously. The habit of being rigidly guided by formal symbolic rules is a safeguard against unconscious assumptions; but even this safeguard is not always adequate.

{*The symbolism adopted in what follows is that of Peano, with certain additions and changes. I have adopted his symbol for implication* (⊃), *and his use of dots instead of brackets, also his plan of numbering propositions with an integral and a decimal part.*[1] *But although the symbolism is in the main Peano's, the ideas are more those of Frege.*[2] *Frege's work, probably owing to the inconvenience of his symbols, has received far less recognition than it deserves. I shall not refer to him in detail in what follows, but whoever will consult his work will see how much I owe him. Especially I have adopted form him the assertion sign (cf. *1·1* infra), *the interpretation of* "p *implies* q," *(cf. *1·2* infra), *and the distinction between asserting a proposition for all values of the variable or variables, and asserting it for any values (cf. *7* infra). *The plan of taking implication and negation as our primitive ideas is also his.*

**1· Primitive Ideas.*}

***1 PRIMITIVE IDEAS AND PROPOSITIONS**

Since all definitions of terms are effected by means of other terms every system of definitions which is not circular must start from a certain apparatus of undefined terms. It is to some extent optional what ideas we take as undefined in mathematics; the motives guiding our choice will be (1) to make the number of undefined ideas as small as possible, (2) as between two systems in which the number is equal, to choose the one which seems simpler and easier. {*I*} **We** know no way of proving that such and-such a system of undefined ideas contains as few as will give such-and-such results.[3] Hence we can only say that such and such ideas are undefined in such and such a system, not that they are indefinable. Following Peano, {*I*} **we** shall call the undefined ideas and the undemonstrated propositions *primitive* ideas and propositions respectively. These ideas are *explained* by means of descriptions intended to point out to the reader what is meant but the explanations do not constitute definitions, because they really involve the ideas they explain.

In the present number, we shall first enumerate the primitive ideas required in this section; then we shall define *implication*; and then we shall enunciate the primitive propositions required in this section. Every definition or proposition in the work has a number, for purposes of reference. Following Peano, we use numbers having a decimal as well as an integral part, in order to insert new propositions between any two. A change in the integral part of a number will be used to correspond to a new chapter. Definitions will generally have numbers whose decimal part is less than .1, and will be usually put at the

[1] {*For an explanation of Peano's symbolism, cf. Whitehead, "On Cardinal Numbers,"* American Journal of Mathematics, *Vol.* XXIV, *No. 4.*}

[2] {*Cf. his "Grundgesetze der Arithmetik," Vol.* I, *Jena, 1893 Vol.* II, *1903.*}

[3] The recognized method of proving independence are not applicable, without reserve, to fundamentals. Cf. my "Principles of Mathematics", §17. What is there said concerning primitive propositions applies with even greater force to primitive ideas.

beginnings of chapters. In references, the integral parts of the numbers of propositions will be distinguished by being preceded by a star; thus "*1·01" will mean the definition or proposition so numbered, and "*1" will mean the chapter in which propositions have numbers whose integral part is 1, *i.e.* the present chapter. Chapters will generally be called "numbers."

PRIMITIVE IDEAS.

(1) *Elementary propositions.* By an "elementary" proposition we mean one which does not involve any variables, or, in other language, one which does not involve such words as "all," "some," "the" or equivalents for such words. A proposition such as "this is red," where "this" is something given in sensation, will be elementary. Any combination of given elementary propositions by means of negation, disjunction or conjunction (see below) will be elementary. In the primitive propositions of the present number, and therefore in the deductions from these primitive propositions in *2–*5, the letters *p*, *q*, *r*, *s*, will be used to denote elementary propositions.

(2) *Elementary propositional functions.* By an "elementary propositional function" we shall mean an expression containing an undetermined constituent, *i.e.* a variable, or several such constituents, and such that, when the undetermined constituent or constituents are determined, i.e. when values are assigned to the variable or variables, the resulting value of the expression in question is an elementary proposition. Thus if *p* is an undetermined elementary proposition, "not-*p*" is an elementary propositional function.

We shall show in *9 how to extend the results of this and the following numbers (*1–*5) to propositions which are not elementary.

{**1·1*} (3) *Assertion.*—Any proposition may be either asserted or merely considered. If I say "Caesar died", I assert the proposition "Caesar died"; if I say "'Caesar died' is a proposition", I make a different assertion, and "Caesar died" is no longer asserted, but merely considered. Similarly in a hypothetical proposition, *e.g.* "if $a = b$, then $b = a$", we have two unasserted propositions, namely "$a = b$" and "$b = a$" while what *is* asserted is that the first of these implies the second. In language, we indicate when a proposition is merely considered by "*if* so-and-so" or "*that* so-and-so" or merely inverted commas. In symbols, if *p* is a proposition, *p* by itself will stand for the unasserted proposition, while the asserted proposition will be designated by

$$\text{“}\vdash . p\text{”}$$

The sign "⊢" is called the assertion-sign[3a]; it may be read "it is true that" (although philosophically this is not what it means). The dots after the assertion-sign indicate its range; that is to say, everything following is asserted until we reach either an equal number of dots or the end of the sentence. Thus "$\vdash : p \supset . q$" means "it is true that *p* implies *q*", whereas "$\vdash . p . \supset \vdash . q$" means "*p* is true; therefore *q* is true".[3b] The first of these does not necessarily involve the truth either of *p* or of *q*, while the second involves the truth of both.

(4) *Assertion of a propositional function.* Besides the assertion of definite propositions, we need what we shall call "assertion of a propositional function." The general notion of asserting *any* propositional function is not used until *9 but we use at once the notion of asserting various special *elementary* propositional functions. Let *x* be a propositional function whose argument is *x*; then we may assert *x* without assigning a value to *x*. This is done, for example, when the law of identity is asserted in the form "*A* is *A*." Here *A* is left undetermined, because, however *A* may be determined, the result will be true. Thus

[3a] We have adopted both the idea and the symbol of assertion from Frege.

[3b] Cf. *Principles of Mathematics*, §38.

when we assert *x*, leaving *x* undetermined, we are asserting an ambiguous value of our function. This is only legitimate if, however the ambiguity may be determined, the result will be true. Thus take, as an illustration, the primitive proposition *1·2 below, namely

$$\text{“}\vdash : p \vee p . \supset . p,\text{”}$$

i.e. "'*p* or *p*' implies *p*." Here *p* may be *any* elementary proposition: by leaving *p* undetermined, we obtain an assertion which can be applied to any particular elementary proposition. Such assertions are like the particular enunciations in Euclid: when it is said "let ABC be an isosceles triangle; then the angles at the base will be equal," what is said applies to *any* isosceles triangle; it is stated concerning *one* triangle, but not concerning a definite one. All assertions in the present work, with very few exceptions, assert propositional function, not definite propositions.

As a matter of fact, no constant elementary proposition will occur in the present work, or can occur in any work which employs only logical ideas. The ideas and propositions of logic are all *general*: an assertion (for example) which is true of Socrates but not of Plato, will belong to logic,[3c] and if an assertion which is true of both is to occur in logic, it must not be made concerning either, but concerning a variable *x*. In order to obtain, in logic, a definite proposition instead of a propositional function, it is necessary to take some propositional function and assert that it is true always or sometimes, *i.e.* with all possible values of the variable or with some possible value. Thus, giving the name "individual" to whatever there is that is neither a proposition nor a function, the proposition "every individual is identical with itself" of the proposition "there are individuals" will be a proposition belonging to logic. But these propositions are not elementary.

(5) *Negation.* If *p* is any proposition, the proposition "not-*p*," or "*p* is false," will be represented by "$\sim p$." For the present, *p* must be an *elementary* proposition.

(6) *Disjunction.* If *p* and *q* are any propositions, the proposition "*p* or *q*," *i.e.* "either *p* is true or *q* is true," where the alternatives are to be not mutually exclusive, will be represented by

$$\text{“}p \vee q.\text{”}$$

This is called the *disjunction* or the *logical sum* of *p* and *q*. Thus "$\sim p \vee q$" will mean "*p* is false or q is true"; "$\sim(p \vee q)$" will mean "it is false that either *p* or *q* is true," which is equivalent to "*p* and *q* are both false"; "$\sim(\sim p \vee \sim q)$" will mean "it is false that either *p* is false or *q* is false," which is equivalent to "*p* and *q* are both true"; and so on. For the present, *p* and *q* must be elementary propositions.

The above are all the primitive ideas required in the theory of deduction. Other primitive ideas will be introduced in Section B.

{**1·2 Implication.*} ***Definition of Implication.*** When a proposition q follows from a proposition *p*, so that if *p* is true, *q* must also be true, we say that p *implies q*. The idea of implication, in the form in which we require it, can be defined. The meaning to be given to implication in what follows may at first sight be somewhat artificial; but although there are other legitimate meanings, the one here adopted is, {*if I am not mistaken,*} very much more convenient than any of its rivals. The essential property that we require of implication is this: "What is implied by a true proposition is true". It is in virtue of this property that implication yields proofs. But this property by no means determines whether anything, and if so what, is implied by a false proposition, {*or by something which is not a proposition at all.*} What it does determine is that, if *p* implies *q*, then it cannot be the case that *p* is true and *q* is {*not true.*} false, *i.e.* it must be the case that either p is false or *q* is true. The most convenient inter-

[3c] When we say that a proposition "belongs to logic," we mean that it can be expressed in terms of the primitive ideas of logic. We do not mean that logic *applies* to it, for that would be true of any proposition.

pretation of implication is to say, conversely, that {*unless* p *is true and* q *is not true*} **if either *p* is false or *q* is true, then** "*p* implies *q*" is to be true. Hence, "*p* implies *q*" {*will be a relation which holds between any two entities* p *and* q *unless* p *is true and* q *is not true, i.e. whenever either* p *is not true or* q *is true.*[4] *The proposition "p implies q" is equivalent to "if p is true, then q is true", i.e. "'p is true' implies 'q is true'"; it is also equivalent to "if q is false, p is false". When p is in fact true, "implies" may be replaced by "therefore", i.e. in place of "p implies q", we may say "p is true, therefore q is true". For "implies" we use the symbol "$\supset$", thus*

$$\text{"}p \supset q\text{" means "}p \text{ implies } q\text{"}$$

"$p . \supset . q \supset r$" *means "p implies that q implies r", etc. The chief advantage of the above interpretation is that it avoids hypotheses which are otherwise necessary. We wish, for example, to assert "$p \supset p$". If implication can only hold between propositions, it is necessary to preface "$p \supset p$" by the hypothesis "p is a proposition"; and whenever we want to use "$p \supset p$" in a particular case, we shall have to prove first that we are applying it to a proposition.*[5] *But this is highly inconvenient. Again, paradoxes result from restricting the meaning of implication. For example, it will be admitted that "if p and q are true, then r is true" is equivalent to "if p is true, then if q is true, r is true", i.e. "if p is true, then q implies r". Also it will be admitted that if p and q are true, then p is true. Hence, by the above admission, if p is true, then q implies p; and here q is not subject to any limitation, for even if q is not a proposition, it must be admitted that if p and q* were *both true,* p would be *true. Hence, unless a true proposition* p *is to be implied by* every *entity* q, *one at least of the above obvious propositions will have to be denied.*} **is to be defined to mean: "Either *p* is false or *q* is true." Hence we put:**

$$*1{\cdot}01.\ \ p \supset q . = . \sim p \vee q \quad \text{Df.}$$

Here the letters "Df" stand for "definition." They and the sign of equality together are to be regarded as forming one symbol, standing for "is defined to mean."[5a] Whatever comes to the left of the sign of equality is defined to mean the same as what comes to the right of it. Definition is not among the primitive ideas, because definitions are concerned solely with the symbolism, not with what is symbolized; they are introduced for practical convenience, and are theoretically unnecessary.

In virtue of the above definition, when "$p \supset q$" holds, then either *p* is false or *q* is true; hence if *p* is true, *q* must be true. Thus the above definition preserves the essential characteristic of implication; it gives, in fact, the most general meaning compatible with the preservation of this characteristic.

{**1·3 The Variable. A single letter, unless specially defined to have a certain constant meaning, will always stand for an independent variable. The possible values of an independent variable are always to include all entities absolutely. The reason for this is as follows. If we affirm some statement about* x, *where* x *is restricted by some condition, we must mention the condition to make our statement accurate, but then we are really affirming that the truth of the condition implies the truth of our original statement about* x; *and this in virtue of our interpretation of implication, will hold equally when the condition is not fulfilled. The "universe of discourse", as it has been called, must be replaced by a general hypothesis concerning the variable, and then our formulae are true whether the hypothesis is verified or not, because an implication holds whenever its hypothesis is not true. The old theory*

[4] {*Cf.* *5·53 *and* *5·54 *below.*}

[5] {*In my "Principles of Mathematics" I adopted the interpretation "p and q are propositions, and p is* false *or q is true", instead of, as here,* p *is* not true *or* q *is true". Thus "$p \supset q$" was false if p was not a proposition, and "$p \supset p$" was equivalent to "p is a proposition". The advantages of the present interpretation may be seen by comparing the primitive propositions of *2 below with those of §18 of the above mentioned work.*}

[5a] **The sign of equality not followed by the letters "Df" will have a different meaning, to be defined later.**

of the "universe" had the defect of introducing tacit hypotheses, thus making all enunciations incomplete, since a hypothesis does not cease to be an essential part of a proposition merely because we do not take the trouble to state it.

In such a formula as (say) "$x . = . p \supset q$", the x *is still an* independent *variable, because we are not only concerned with the value of* x *that makes the formula* true, *but also with all other values that make it false. On the other hand, "$p \supset q$" is a* dependent *variable.*

**1·4 Propositional Functions. Any statement about a variable* x *will be expressed by*

$$(C)(x)$$

or by (A)(x), (B)(x), *etc. Similarly any statement about two variables* x *and* y *will be expressed by*

$$(C)(x, y)$$

or by (A)(x, y), (B)(x, y) *etc. and so on for any number of variables. Such expressions are functions whose values are propositions, hence we call them* propositional functions. *Thus "$p \supset q$" is a propositional function of* p *and* q, *"$p \supset q$" is a propositional function of* p. *There are other kinds of functions in mathematics, but they are not required for the theory of implication.*

When an expression of the form (C)(x) *is asserted, what is meant is that the expression in question is true for* any *value of the variable* x. *Thus, e.g.*

$$\text{"}\vdash . p \supset p\text{"}$$

means. "For any *value of* p, p *implies* p". *But such an expression as* (C)($p \supset q$) *does not assert that* (C)(x) *holds for* any *argument, but only for such as are of the form* $p \supset q$, *e.g. if* (C)(x) *is "x is a proposition", then*

$$\text{"}\vdash . (C)(x)\text{" is false, but}$$
$$\text{"}\vdash . (C)(p \supset q)\text{" is true.}$$

Generally, "$\vdash . (C)((A(y))$" means: "For any value of y, (C)((A)(y)) *is true", or "For any value of* y, (A)(y) *is a value of* x *for which* (C)(x) *is true". The* C *in* (C)(x) *has no meaning by itself; all that its use assumes is that* (C)(x) *and* (C)(y) *are to have similarity of* form, *although this form is not always capable of separate specification apart from any argument* x.

**1·5 Negation. The proposition "p is not true" is expressed by*

$$\sim p.$$

Thus "$\sim p$" will be true if p *is not a proposition, and if* p *is a false proposition; it will only be false if* p *is a true proposition.*

*One other primitive idea will be introduced in *7. Two others, or perhaps three, complete what is required for the whole of pure mathematics. Thus, six are necessary for the theory of implication, and eight or nine for all pure mathematics.*}

APPENDIX B. SOME MAJOR WRITINGS

Russell used three sets of axioms for propositional logic in his major writings. A set was introduced in each of *Principles of Mathematics* (PoM), "The Theory of Implication" (TI), and *Principia Mathematica* (PM). Listed below are some of Russell's published writings on logic in the period from *Principles of Mathematics* to *Principia Mathematica*, Volume I. This list is adapted from Kenneth Blackwell's compilation which appears in Appendix B of *Essays in Analysis [11]*. The numbering scheme there has been retained for ease of reference. The list below excludes reviews and unpublished work. The axiom system used in each case is noted.

PoM	1903	(a)	The *Principles of Mathematics*
	1904	(a)	Meinong's Theory of Complexes and Assumptions
		(b)	The Axiom of infinity
		(c)	Non-Euclidean Geometry
	1905	(a)	The Existential Import of Propositions
		(c)	On Denoting
		(e)	Sur la relation des mathématiques à la logistique
	1906	(a)	On Some Difficulties in the Theory of Transfinite Numbers and Order Types
		(b)	On the Substitutional Theory of Classes and Relations
TI		(c)	The Theory of Implication
		(g)	Les Paradoxes de la logique
		(i)	The Nature of Truth
	1907	(a)	On the Nature of Truth
PM		(b)	The Regressive Method of Discovering the Premisses of Mathematics
		(c)	The Study of Mathematics
	1908	(a)	Transatlantic "Truth"
		(b)	"If" and "Imply"
PM		(d)	Mathematical Logic as Based on the Theory of Types
	1909	(a)	Pragmatism
PM	1910	(b)	*Principia Mathematica*, Vol. I

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