Are substitutional quantifiers a solution to the problem of the elimination of classes in *Principia Mathematica*?

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INTRODUCTION

As we mean it, the eponymous question of this paper subdivides into three: (1) what, if any, are the obstacles to the elimination of classes in *Principia Mathematica* (*PM*)? (2) can quantifiers in *PM* be substitutionally interpreted? (3) if so, what are the effects, in terms of elimination, of a substitutional interpretation of quantifiers in *PM*? We take it that these three sub-questions, to be adequately answered, have to deal with the general economy of a theory of meaning, that is, with the complex matter of the interaction between logic, ontology and semantics. The first sub-question raises the matter in a rather complicated way: the doctrine of the elimination of pseudo-names presented in "On Denoting" (*OD*) applies directly to definite descriptions in the ramified theory of types.[1] But the question whether it also applies to classes in *PM* is the question whether the various logical "improvements", which in *PM* gradually lead from an intensional to an extensional logic, still preserve the theory of reference which initially allows for elimination. Yet the problem would be relatively simple, if it were only that of comparing and evaluating the semantical properties of two logical languages; but there is no such thing as two languages in *PM*. The task is rather like comparing and evaluating the semantical properties of one language when successively implemented by two different logics, one of them, moreover, having never been explicitly formulated as a theory. The second sub-question, on the other hand, raises the matter in the most straightforward way: a substitutional interpretation of quantifiers, if it amounts to a *re*interpretation of russellian quantifiers,[2] should still be compatible with the formal properties and the basic semantical features of the language of *PM*. This requirement is minimal in

[1] This will be established in the course of sec. 1 of this paper, but this claim obviously follows from the stronger claim, made in Couture [1983–a], that the language of the ramified theory of types is itself based on a generalization of the doctrine of definite descriptions.

[2] As far as references to *PM* are concerned, "russellian" and "Russell" are of course taken as abbreviations for "russell–whiteheadian" and "Russell and Whitehead", respectively.

two respects: first, if we want to claim that quantifiers *in PM* can be substitutionally interpreted, and second, if we want to make the further claim that such an interpretation is a solution to the problem of classes *in PM*. As for the third sub-question, because of the "if" it contains, it might never be met; but if it is, it will really just amount to putting together the facts one has to consider in order to answer the former two, that is, the theory of meaning which allows for elimination and the one which is allowed by the use of substitutional quantifiers.

There will be a one–one correspondence between these three sub-questions and the divisions of this paper; and the answer to the eponymous question will be: no.

To the first sub-question we have already given a detailed answer elsewhere.[3] But, since then, we have found a more powerful and elegant way to formulate it: we first reconstrue the characteristic formal features of an eliminative definition along the lines of the doctrine of definite descriptions; second, we compare in this respect the definition of classes and the definition of definite descriptions, as they are given in *PM*. We then show, without any detour, where, exactly, according to Russell's own standards, the definition of classes fails to be an eliminative definition.

In the second part of this paper, we will briefly recall Kripke's characterization of substitutional quantifiers and, after a discussion of Russell's theory of quantification, we will show to which extent, and under which further conditions not already contained in *PM*, a substitutional interpretation of quantifiers is possible in *PM*.

In the third part, finally, we will evaluate the ontological outcome of a substitutional interpretation of quantifiers in *PM* and its particular effects on eliminative definitions. We will conclude that substitutional quantifiers in *PM* reproduce the very problems initially raised by the definition of classes in *PM*, in a more explicit fashion, and not only for the definition of classes but also for the true eliminative definitions.

1. IS THERE A PROBLEM WITH THE DEFINITION OF CLASSES IN *PM*?

1.1 Eliminative definitions: the theory

According to *OD*, three observations have led to the discovery of *pseudo-names*: some expressions that we take as grammatical subjects, such as "some animal", "all men", "the actual king of France", have no denotation and thus, according to Russell, no meaning; the sentences where these expressions appear are meaningful (they express authentic propositions); to be authentic, these propositions are nonetheless incomplete, that is, their logical subject is a referentially undetermined entity.

These observations are respectively carried out by three formal features of the so-called *definitions* of "pseudo-names": (I) the *definiens* does not provide a substitute (or synonym) for the pseudo-name itself; (II) the *definiens* is a well-formed expression of a suitable logical language, that is, a language whose syntactical categories can fully express the logical theory which allows for the analysis of statements containing pseudo-names; (III) the logical subject of the *definiens* is not a logical name (a constant) but a bound variable.

In such definitions, *elimination* of pseudo-names thus needs to be achieved on three different grounds (or: elimination needs to have three aspects): (*a*) syntactical,

[3] See Couture [1983–b].

according to which the grammatical name not only disappears from the *definiens*, but does not even leave any traces, that is to say, expressions which could be seen as a substitute for it: the *definiens* then expresses the fact that the grammatical name has *no* meaning; (*b*) logical, according to which *there is* a logical language (a logical theory) which can express the meaning (the logical form) of a given grammatical statement without using one of its constituents: the *definiens* then shows the "pseudo-status" of such a grammatical constituent as a name or as a subject, but it does show that it is meaningfully used; (*c*) semantical, according to which the logical theory does not need any "nameable entity" whatsoever as the logical subject of the proposition expressed by the grammatical statement.

It is generally considered that a definition is eliminative if it has feature III, and reductive otherwise. It should be noticed, however, that a reductive definition may or may not have feature I, that this feature is not implied by feature III and that it is also required by an eliminative definition. On the other hand, feature II will obviously be required by any definition.

To show that the definition of classes in *PM* is an eliminative definition it is thus necessary and sufficient to show that it has the formal features I–III which express these three aspects of elimination. This, in turn, can be achieved without unnecessary detours, by simply comparing the definition of classes to the paradigmatic form of an eliminative definition in *PM*, that is, the definition of definite descriptions.

1.2 Comparing "eliminative" definitions in PM

The definitions given in *PM*[4] for definite descriptions and for classes respectively are:

$$\text{(i)}\quad f[(\ x)(\phi x)]\,.=.\,(\exists c):\phi x\,.\equiv_x.\,x=c:fc$$

$$\text{(ii)}\quad f[(\hat{z}(\psi z)\,.=:(\exists\phi):\phi!x\,.\equiv_x.\,\psi x:f[\phi!\hat{z}]$$

In a uniform and more familiar notation (i) and (ii) become:

$$\text{(i}'\text{)}\quad f[\ x(\phi x)]=_{df}\exists c[\forall x(\phi x\Leftrightarrow x=c)\wedge f(c)]$$

$$\text{(ii}'\text{)}\quad f[\hat{z}(\psi z)=_{df}\exists\phi[\forall x(\psi x\Leftrightarrow\phi x)\wedge f(\phi\hat{z})]$$

We omitted the exclamation mark in (ii′), which was intended to indicate that what Russell called "a propositional function" is predicative; this property of propositional functions is not relevant to the following discussion.

Otherwise, our reformulation keeps track of the Russellian notation: $\phi\hat{x}$ is a variable "propositional function taken for itself"[5] (from now on: a variable *name-predicate*); ϕx is a variable and ψx a constant for "the undetermined values of a propositional function",[6] that is, sentences resulting from replacing x by a constant (from now on, respectively: a variable and a constant *open formula*).

[4] Russell [1925–27], Vol. I: p. 68 for the def. (i); *ibid.*, p. 190 for the def. (ii).
[5] *Ibid.*, p. 40.
[6] *Idem.*

I. Both (i′) and (ii′) have the characteristic form of a contextual definition: the *definienda* are the *sentences* where $x(\phi x)$ and $\hat{z}(\psi z)$ respectively appear, and the meaning of these sentences is made explicit in the *definiens* by means of the conditions of application of f, conditions which no longer involve the pseudo-names.

The syntactical aspect of elimination seems to be realized equally in (i′) and in (ii′). For the pseudo-name not only disappears in the *definiens* but also seems to "dissolve": the conditions of application of f, in each case, are complex and involve a *complex specification* of its argument so that neither c (in (i′)) nor $\phi\hat{z}$ (in (ii′)), nor any compound of expressions, also contained in the *definiens*, is likely to be a suitable synonym for the pseudo-name itself. Both definitions then seem to account for the fact that a pseudo-name has no separate meaning. The issue, of course, still depends on *which* relations are actually stated in the *definiens*; we will examine this matter in the next section in connection with features II and III.

II. According to the above specifications, a definition realizes the logical aspect of elimination if its *definiens* (1) is a well-formed sentence of a logical language, and (2) can fully express the propositional content of a statement containing a pseudo-name.

In the *definiens* of (ii′), each of these conditions raises a particular problem, to be solved in a uniform way.

Relatively to the first condition, the *definiens* of (ii′) presents a formal anomaly which consists in the existential quantifier ranging over both ϕx and $\phi\hat{z}$.

According to the formal account of the language of *PM*, ϕx and $\phi\hat{z}$ should have disjoint ranges of value, namely, sentences for the former and properties for the latter. In a well-formed sentence, they cannot, accordingly, be bound by the same quantifier.

The second difficulty, which concerns the fulfilment of condition (2), is the following: in many passages of *PM*, we find the statement that (ii) is a definition of "*the* class such that …", but at first sight nothing in (ii) secures the uniqueness of $\hat{z}(\psi z)$.

A comparison with (i′) provides a first indication of this fact. In (i′), uniqueness holds since the *definiens* states that *any* x for which ϕx is true *is* a certain c for which fc is true. The substatement of identity is crucial here to show that f could be true of just one individual. In order to secure the uniqueness of $\hat{z}(\psi z)$, we would have expected the *definiens* of (ii′) to establish, similarly, that *any* propositional function equivalent to ψx, when taken "for itself", is *identical* with $\phi\hat{z}$ for which $f(\phi\hat{z})$ is true.

The formal analogy with (i′) happens to be sound as a requirement of uniqueness in *PM*, for according to the ramified theory of types, propositional functions, equivalent as regards their values, still could be *different* when "taken for themselves". (ii′) as it is formulated, just states that f could apply to at least one (but possibly many) propositional function(s). To be comparable to (i) in this respect, (ii) should have been:

$$\text{(ii}''\text{)}\ f[\hat{z}(\psi z)]=_{df}\exists\phi[\forall\Theta(\forall x(\psi x\Leftrightarrow\Theta x)\Leftrightarrow\Theta\hat{x}=\phi\hat{z})\wedge f(\phi\hat{z})].$$

While (ii′′) conveys the intended uniqueness of $\hat{z}(\psi z)$, it is a very strange expression, given the *intensional* logic of *PM*; for it is possible from (ii′′) to prove:

$$\forall\phi\forall\Theta(\forall x(\phi x\Leftrightarrow\Theta x)\Rightarrow\phi\hat{z} = \Theta\hat{x}),$$

which is the axiom of extensionality.

The conclusion to be drawn from this result *is not* that, rather than deriving the axiom of extensionality *from a definition*, Russell finally gave up the clause of uniqueness in the definition of classes. The right conclusion is that Russell, having already *assumed* that propositional functions are different *iff* they are not equivalent (that is, having assumed a principle of extensionality), would have seen (ii′′) as a redundant formulation of (ii), as far as uniqueness is concerned; for by elementary logic, one can prove (ii′′) from (ii) *and* the axiom of extensionality.

The assumption of extensionality also provides a coherent reading of the ∃-quantifier in (ii′). For, according to the principle of extensionality, propositional functions "taken for themselves" (name-predicates) are no longer logically relevant; only the values of propositional functions (sentences) are, and, as far as their semantical characterization is concerned, only the extensions of *predicates* are relevant.[7] Instead of referring to two different kinds of entities, name-predicates and open formulas can accordingly be seen as resuming for the predicate they contain the use/mention distinction, and the interpretation of (ii) in these conditions no longer raises the problem of the coherence of the ∃-quantifier.

There is a side effect, however. Since only the extensional properties of predicates are relevant to the truth-value of sentences, $\phi\hat{z}$ will have the same denotation as ϕx in a name position, and, consequently, any predicate applying to one will also apply to the other. To this extent, one could as well think of $f(\phi\hat{z})$ in (ii′) as a syntactical dummy; for actually it means nothing more than $f(\phi x)$. The significance of this fact for the question of elimination will fully appear in the next section, in connection with feature III.

The above remarks indicate that the definition of classes is given *together* by (ii′), which is formulated in the language of the ramified theory of types, *and* a so-called principle of extensionality, which was not initially part of that theory. This observation gives a new flavour to the logical aspect of elimination in (ii′); for it shows that the logical theory already at hand *does not* allow for the meaningful use of class-expressions and, à fortiori, for their elimination. It thus should be clear by now that, if elimination of classes is achieved at all, it will be thanks to an extensional logic which prescribes a reinterpretation of the language of *PM*.

But to this very extent, this language is no longer a language *suitable* in the sense previously defined (sec. 1.1); in particular, (ii′) contains $f(\phi\hat{z})$ which, according to the extensional logic which allows for a coherent reading of the ∃-quantifier, becomes a syntactical dummy. Consequently, (ii′), if it is found to express the meaning of a statement containing a class expression, could hardly be seen as making its logical form *explicit*.

[7] In the preface to the second edition of *PM* (I: xiv) Russell, summarizing the modifications to the intensional theory of *PM*, remarks that from now on, "a function can only occur in a proposition through its values". This means, among other things, that open sentences like Px do not stand any more for the "undetermined sentences" which result from replacing x by a constant, but for the well-defined collection of such sentences. This implies a departure from the intensional logic which will be made clearer in section 2.2 of this paper.

III. The semantical aspect of elimination corresponds to the fact that a proposition expressed by a statement containing a pseudo-name is an incomplete proposition, that is, a proposition whose logical subject is a referentially undetermined (non-nameable) entity.

This aspect of elimination is shown in (i′), by the following features: (1) c, the logical argument of f, is a bound variable, hence not a logical name; (2) the substatement $\exists c[\forall x(\psi x\Leftrightarrow x = c)]$, which indirectly links the logical and the grammatical argument of f, does not involve any individual constant.

The second feature is at least as important as the first one for the semantical aspect of elimination; for it secures the fact that c itself, in the present context, has no nameable substitute.

By contrast, the *definiens* of (ii′) contains the substatement $\exists\phi[\forall x(\psi x\Leftrightarrow\phi x)]$, which is the functional analog of $\exists c[\forall x(\phi x\Leftrightarrow x = c)]$ in (i′) and where ψx is a constant. Under the extensional reading which, we recall, is required by (ii′), $\forall x(\psi x\Leftrightarrow \phi x)$ implies $\psi\hat{z} = \phi\hat{z}$ and, therefore, $\exists\phi[(\forall x(\psi x\Leftrightarrow\phi x)\wedge f(\phi\hat{z})]$ implies $f(\psi\hat{z})$. That is to say, even if the logical argument of f in (ii′) is a bound variable, the *definiens* of the class actually *allows* for the substitution of a constant in the very context where $\hat{z}(\psi z)$ appears in the *definiendum*. To this extent, the semantical aspect of elimination (feature III) is not realized in the definition of classes; (ii′) at most shows that classes are *reducible* to some nameable entity.

This observation, together with our conclusions of the previous section, retrospectively throws some doubts on whether the syntactical aspect of elimination (feature I) is realized in (ii′). For if $f(\phi\hat{z})$ in (ii′) is a syntactical dummy, so is $f(\psi\hat{z})$, which is implied by (ii′). But, if $f(\psi\hat{z})$ means the same as $f(\psi x)$, then the definition of classes really allows for the direct replacement of $\hat{z}(\psi z)$ by ψx in the very same context.[8] To achieve this, a contextual definition was not required, and indeed, *in spite of its syntactical form*, the *definiens* of (ii′), under an extensional reading, does not satisfy the requirement that a pseudo-name has to "dissolve".

Our conclusion concerning the definition of classes in *PM* is, then, the following: (ii′) is not an eliminative definition since the features I and III, each of them being a necessary condition for an eliminative definition, are missing. This conclusion depends heavily on the fact that (ii′) is read extensionally and then shows at least a degenerate form of feature II; the alternative being that the "eliminative definition of classes" is not even a definition.

2. A SUBSTITUTIONAL INTERPRETATION OF QUANTIFIERS IN *PM*

In this section we want to examine the minimal claim made in the substitutionalist proposals for the elimination of classes in *PM*, that is, that russellian quantifiers *can be* substitutionally interpreted. The claim certainly presupposes that a substitutional interpretation is not the intended interpretation of quantifiers in *PM* and that it will consequently require a modification in the semantics of russellian quantifiers. As we take it, the claim also entails that such a modification is a local one

[8] This was indirectly acknowledged in Russell [1925–27], I: xxxix: "... there is no longer any reason to distinguish between functions and classes, for we have, in virtue of the above, $\phi x \equiv_x \psi x \,.\supset.\, \phi\hat{x} = \psi\hat{x}$. We shall continue to use the notation $\hat{x}(\phi x)$ which is often more convenient than $\phi\hat{x}$; but there will no longer be any difference between the meaning of the two symbols."

which conflicts with neither the formal properties nor the basic semantical features of the language of *PM*. The examination of the minimal substitutionalist claim will then lead us to discuss the russellian theory of quantification in the perspective of the general economy of the language of *PM*. But before that, we will briefly recall the features of a substitutional interpretation of quantifiers as they emerge from Kripke's inductive characterization of truth for a language with substitutional quantifiers.[9]

2.1 Kripke's quantifiers

In Kripke's system, the inductive basis of a language with substitutional quantifiers is a language L_0, for which a definition of truth is assumed. L_0 contains no connectives (the sentences of L_0 are syntactically specified to be atomic sentences) and has a non-empty class C of expressions called the substitution class. The elements of C are called terms; they could be any class of expressions of L_0.

The notion of form of L_0 is then specified. Let x_1, x_2 ... be an infinite list of variables (not contained in L_0) and let A be a sentence of L_0. An expression A' obtained by replacing zero or more terms in A by variables is called a preform. If the result of replacing variables by arbitrary terms in a preform is always itself a sentence of L_0, the preform is a form.

L_0 is then extended to a language L. Sentences and forms of L_0 are atomic sentences and atomic formulas of L, respectively. L also contains the usual logical connectives and quantifiers.

Given the notion of an atomic formula (form of L_0), the notion of an arbitrary formula is defined inductively: an atomic formula is a formula; if ϕ and ψ are formulas, so are $\phi \wedge \psi$, $\sim\phi$ and $(\Sigma x_1)\phi$.

Formulas without free variables are called sentences of L; they include the atomic sentences (the sentences of L_0); $\phi \wedge \psi$ is a sentence iff ϕ and ψ are, $\sim\phi$ is a sentence iff ϕ is, $(\Sigma x_1)\phi$ is a sentence iff ϕ is a formula containing at most x_1 free. L also contains n-ary functors: these are expressions with n variables such that, when arbitrary terms replace the variables, a term results.

Granted that truth has been characterized of L_0, the truth conditions for sentences of L are extended in the following way:

(a) $\sim\phi$ is true iff ϕ is not;
(b) $\phi \wedge \psi$ is true iff ϕ is and ψ is;
(c) $(\Sigma x_1)\phi$ is true iff there is term t such that ϕ' is true, where ϕ' comes from ϕ by replacing all free occurrences of x by t.

So stated, the truth conditions of quantified sentences indeed refer to linguistic expressions; this is probably the best known feature of the theory of substitutional quantifiers. However it should be noticed that the linguistic expressions which are *semantically* relevant (which serve to determine the truth-value of the quantified statements) are sentences whose truth-values have already been determined (the so-called instances of quantified statements). The terms which are referred to, on the other hand, are syntactical substituends used to form these sentences, and, to this extent, they have no semantical role to play in the truth characterization of quantified statements. In particular, the substitution class is not to be conceived, by analogy with a certain process of characterizing truth for objectual quantifiers, as the domain (of linguistic expressions) where the variables take their value. Such an analogy would miss the most central feature of substitutional quantification, which is that the truth conditions of a statement with substitutional quantifiers do not directly involve any device related to a *definition* of truth. This is a condition to be fulfilled, even when such a definition is given for the language which provides the instances of quantified statements. And it is worth noticing, in this connection, that, in Kripke's system, the fulfilment of this condition is not a mere consequence of the fact that truth is assigned (rather than defined) for sentences of L_0.

[9] Kripke [1976], pp. 328–50.

Another important feature of Kripke's semantics for substitutional quantifiers is that it requires a hierarchy of expressions such that the substitution classes for quantifiers in L do not contain these very quantifiers. This requirement is met in a stronger form in Kripke's system by the fact that the substitution class for L is given together with L_0, which contains no quantifiers at all. The hierarchy is a condition for an adequate characterization of truth for L, to the extent that it guarantees the existence of a unique set S' of sentences true in L corresponding to the set S of true sentences in L_0. The connection between this condition and the avoidance of paradoxes is familiar: if, for instance, t in the clause (c) already contains Σ the truth of ϕ' might depend on $(\Sigma x_1)\phi$, and thus there will be a sentence whose truth-value has not been and cannot be determined.

This constraint on the substitution class for quantifiers will induce a multi-layered language if, for instance, we want substitutional quantifiers whose instances include quantified sentences of L. It will then become necessary, for the reason already mentioned, to extend L to a larger language L_1 with a new style of substitutional quantifiers whose instances are the sentences of L. This process can be repeated for L_2, L_3, ...

2.2 Russell's quantifiers

A syntactical description of a language with quantifiers is given in *PM* in terms of propositional functions and "matrix-functions" (or matrices).[10]

Propositional functions here correspond to what we called open formulas. For all practical purposes, a matrix and an open formula are the same; both contain free variables and are the "source", as Russell said,[11] of sentences and new open formulas. What distinguishes open formulas and matrices, according to Russell, is the process of generating expressions in which they are respectively involved and, above all, the kind of expressions which result from this process.

From an *open formula* of order n, one generates the collection of *elementary* sentences and open formulas belonging to the same order by substituting, respectively, for all and some variables, the appropriate *constants*. The appropriateness of con-

[10] The method of matrices by which the *intensional* hierarchy of propositional functions and propositions is established is exposed in Russell [1925–27], I: 50–5 and 162–3, and it is contrasted by Russell with the step-by-step method by which the *extensional* hierarchy of propositional functions and propositions is stated (*ibid.*, I: xxii). The latter Russell himself criticizes: "... it interferes with the method of matrices, which brings order into the successive generation of types of propositions and functions demanded by the theory of types, and ... it requires us, from the start, to deal with such propositions as $(y).\phi(a,y)$, which are not elementary."

[11] Russell [1925–27], I: 50.

stants first refers to their order which should match the order of the variable(s) contained in an open formula: individual constants are of order 0 and should be substituted for variables of order 0 only; open formulas containing only individual variables are of order 1 and should be substituted for variables of order 1 only, and so on. But it also refers to the type of constants and variables. Briefly, types distinguish open formulas belonging to the same order according to the order of their relevant arguments. First-order open formulas could only take individuals as arguments; they will thus always be of type 0. But second-order open formulas could take as an argument either a first-order open formula or an individual; they will accordingly be of type 1 or type 0. According to this so-called ramification, there will be 2^{n-1} different types for open formulas of order n. In the notation of *PM*, orders and types were intended to be indicated by a system of subscripts.

Now the way to generate expressions from *matrices* does not involve the substitution of constants for variables, but "the replacement of free variables by '*apparent variables*'". Matrices are the "source" of quantified or *derived* expressions; from them one can generate a new collection of open formulas and sentences by quantifying respectively over some or all the free variables they contain.

Given the constraints resulting from the order and type of variables, quantifiers are restricted, and a quantified expression containing a bound variable of order n will be at least of order $n+1$.[12]

The striking fact about this syntactical description of a language with quantifiers is that it says nothing about the basis of the generative process in which open formulas and matrices are involved, and indeed, as it was often noticed, *PM* does not provide individuals and first-order constants, so that this process could lead to an actual language. However, *PM* contains a semantical (but no formal) description of what such a basis should be and of how, moreover, it would determine the interpretation of a language so generated.

According to Russell,[13] the basis of the language syntactically described in *PM* consists of "genuine judgments", that is, true statements of any degree of complexity but without quantifiers. One can think of the set of such judgments as a body of knowledge expressed by means of an observational language. The set of these statements is not part of the logical language, but it is understood to provide all the primitive constants available in a language, that is, constants not introduced by "meaning postulates" or explicit definitions. Primitive constants such as Px of $P\hat{z}$ are formed from a predicate P occurring in a judgment: no more than judgments are predicates part of the logical language; nonetheless, they are intended to determine the order and type of primitive constants such as Px and $P\hat{z}$ (their so-called "range of significance"); for, according to Russell, the order and type of these expressions follow from the meaning (or intension) of the predicates from which they are formed, which meaning is expected to be clearly understood since these predicates occur in sentences which express *judgments*.

From the primitive constants (including individual constants) one can generate (as described before: by replacing variables in open formulas or in matrices by constants and bound variables, respectively) open sentences and sentences which are different, hence *new*, relative to the initial set of judgments and open formulas. But these new expressions all keep track of the semantical information provided by "genuine judgments" to the exact extent that they must obey constraints on type and order (formation rules) which are themselves determined by the semantical features of judgments, namely, by the intension of the predicates they contain.

But "genuine judgements" are intended to provide more than the conditions of meaningfulness for a language. For, according to Russell, from knowing the intension of a predicate P, as it occurs in a judgment, one can know the truth-value of any sentence which is the value of Px. This view is related to the doctrine of the *priority of intension over extension*, which applies differently and more or less radically whether Russell is considering elementary or derived propositions.

Elementary propositions can be generated only when constants are available, that is, when the ramified theory of types is partly (as in *PM*) or fully interpreted. Let us suppose that a first-order open formula Px together with a list of individual constants is so available. Given the constraints of type and order, the set of elementary sentences p generated from Px is exactly the set of meaningful first-order elementary sentences involving P, that is, the set of all possible sentences p. For these sentences, the doctrine of the priority of intension over extension means that knowledge of the intension of P is required and suffices for the identification of its extension. Knowledge of the intension of P is, of course, presupposed by the very fact that the sentences p have been generated; for Px itself comes from P occurring in a "genuine judgment". Having such knowledge, one can identify the facts which are relevant to the truth-value of the sentences p, and one is also certainly able to determine, in presence of these facts, and for all particular cases represented by all sentences p, whether or not P applies. Russell does not deny that truth for elementary sentences can accordingly be characterized in terms of the extension of P. He will rather claim that such a characterization *follows* from the intensional grasp of constants together with the formal properties of his language. But from this point of view, there is no need at all, for Russell, to actually achieve such a characterization of truth or, for that matter, to provide the semantics of elementary sentences with a formal apparatus more powerful than the conditions of meaningfulness expressed by the ramified theory of types. Moreover, given his conception of elementary sentences, Russell certainly believed, as did others before Tarski, that an extensional characterization of truth must depend on an extra-logical inquiry which has nothing to do with the formal properties of a logical language.

These views, which we exemplified here for first-order elementary sentences, apply in *PM* to elementary sentences of any order, actually or theoretically available, that is, to sentences containing constants (primitive or not) in name positions.

The doctrine of the priority of intension over extension applies differently to quantified statements; for they have, said Russell, "a different kind of truth".[14]

In *PM*, $\vdash[\phi x] \supset (x)\phi x$.[15] The left-hand side of the horseshoe corresponds to what Russell calls "the assertion of a propositional function" (the assertion of an open formula), and this is the key notion for understanding Russell's views on quantification. Russell explains the conditions for the assertion of open formulas in the following way:

[12] The systematics of types in *PM* is explained with more details in Couture [1983–b].

[13] *Cf.* the theory of judgment sketched in Russell [1925–27], I: 41–7.

[14] Russell [1925–27], I: 45.

[15] *Ibid.*, I: 132.

> A function can be apprehended without its being necessary to apprehend its values severally and individually.... What is necessary is not that the values should be given individually and extensionally but that the totality of the values should be given intensionally....[16]

Such an intensional grasp of an open formula is, of course, the grasp of the meaning of the predicate from which it comes; it is a sufficient condition for its being asserted and, according to the theory of inferences in *PM*, for a quantified statement to be asserted. When we assert *Px* always, or *Px* sometimes (the russellian translation for "$\forall x(Px)$" and "$\exists x(Px)$", respectively[17]) we are making, according to Russell, an assertion about the generality of a predicate, and this should not be confused with an assertion about the particular cases where a predicate applies. The latter, such as *Pa*, is the assertion of a determinate value of *Px* and is true iff *a* is in the extension of *P*; but the former, such as $\forall x(Px)$ and $\exists x(Px)$, only presuppose the existence of these values and "refer ambiguously" to them.[18] Hence, the "different kind of truth" of quantified statements as opposed to elementary statements. For quantified statements, the priority of intension just means that no actual knowledge of the extension of a predicate is required in order to establish their truth-value.

The russellian account of quantified statements in *PM* is obviously a transposition, from *OD*, of the doctrine of referentially incomplete propositions, according to which sentences without logical names (constants in name positions) are nonetheless real propositions.[19] In the context of the intensional logic of *PM*, it appears that such a referential incompleteness (or indeterminacy of reference) is characteristic of propositions whose truth-value depends on the intension, rather than the extension, of predicates occurring in propositional functions.

The logical distinction made in *PM* between elementary and derived sentences thus summarizes a distinction which is both epistemologically and semantically grounded.

2.3 *How to substitutionalize russellian quantifiers*

The sharp distinction between elementary and derived sentences appears to be a *prima facie* refutation of the claim that russellian quantifiers, in the intensional logic of *PM*, *are* substitutional quantifiers: according to this distinction, there is, literally, no class of substitution for a russellian quantifier. This is not so, as it is sometimes said, because there are no primitive constants in *PM*; this situation is related to the logicist concern of *PM* and could easily be overcome, as we pointed out, by an otherwise russellian language. The reason why there are no classes of substitution is more deeply rooted in the theory of meaning which prescribes both the semantics and the logic of quantifiers; the truth of quantified statements, being "of a different kind", does not depend on the truth of sentences containing only constants. To this extent, these sentences are not instances of quantified sentences, and, accordingly, constants, when available, do not constitute a class of substitution for it.

Indeed, according to Russell's account, the existence of a least one elementary sentence is presupposed by the very fact that a quantified sentence has been generated in the language; since the "totality of values" of an open formula can be "intensionally apprehended", a quantified statement could even refer to a *collection* of elementary sentences; and, since such an intensional grasp as well reveals the degree of generality of the predicate contained in an open formula, a quantified statement could assert, of such a collection, that it contains no true sentence, only true sentences, or at least one true sentence. In Russell's view, the truth-value of a quantified statement is certainly not independent of the truth-value of elementary sentences, but this does not suffice to make substitutional quantifiers of russellian quantifiers; for the truth conditions of substitutional quantifiers, as we have seen, require more than a collective reference to sentences, they also require more than the presupposition that a certain collection of sentences could be formed; they require that one actually exhibits the sentences which are relevant to its truth-value. And this is precisely what is prohibited by Russell's account of quantified statements.

However, the russellian conception of quantification shows at least two important similarities to Kripke's formal conditions for a substitutional interpretation of quantifiers, and it is not unreasonable, in view of these similarities, to think that russellian quantifiers *can* be substitutionalized, simply by interpreting them directly in terms of the (theoretically) available constants and elementary sentences.

The first of these similarities is suggested by the fact that the ramified theory of types induces categories of expressions where both constants and variables could be considered exclusively as type-and-order bearers. In the process of forming elementary sentences from open formulas, for instance, constants are not to be taken as values, but as substituends for free variables. The same conception would naturally apply to bound variables if we were to provide quantifiers with classes of substitution.

The result, similar to Kripke's, will be that in the truth characterization of quantified statements, the semantically relevant role will be played by sentences whose truth is (assumed to be) already characterized. Thus, Russell's, as well as Kripke's, conditions of truth for quantified statements would completely avoid the semantical devices related to a *definition* of truth such as the domain of value for variables, or the notion of a predicate to be satisfied, which are responsible for an objectual interpretation of quantifiers.

The second similarity between russellian and substitutional quantifiers is suggested by the syntactical properties that russellian quantified sentences inherit from the matrices, namely that they will always be at least of order $n+1$ when the order of the bound variables they contain is at most n. In Russell's account, this constraint is related to the so-called "vicious-circle principle",[20] and its result, if russellian quantifiers are taken as having a class of substitution, is that, like Kripke's quantifiers, they will never be part of their own class of substitution. They could, of course, as for Kripke's quantifiers, be part of the class of substitution of *another* quantifier, that is, a higher-order quantifier. This particular similarity between substitutional and russellian quantifiers has been pointed out by Kripke himself.[21]

Another constraint on russellian quantifiers, not to be found in Kripke's system,

[16] *Ibid.*, I: 39–40.
[17] *Ibid.*, I: 41–2.
[18] *Ibid.*, I: 17–18 and 40.
[19] For a more detailed account of the connection between the doctrine of definite descriptions and the ramified theory of types, we would suggest Couture [1983–a].

[20] *Ibid.*, I: 39–40.
[21] Kripke [1976], p. 368.

but which would put an additional restriction on the "class of substitution", is, of course, the constraint on types for any bound variable whose order is greater than 1.

To substitutionalize russellian quantifiers, given the formal properties they already have in the intensional logic of *PM*, may thus appear an easy task; all it needs is to take constants (of suitable type and order) as forming their class of substitution.

It is important to notice, however, that this task is still easier if we start from the second edition of *PM*. For there the logic is extensional, and Russell, accordingly, had already given up the idea which precludes a substitutional interpretation of quantifiers in intensional logic, namely, that the truth-value of a quantified statement depends on the intension rather than on the extension of predicates, or, in other words, that the truth-value of quantified statements does not depend on the truth-value of elementary sentences. In this context, and given the effect of the principle of extensionality on the interpretation of the language of *PM*,[22] one may even be right in claiming that russellian quantifiers *are* already substitutional quantifiers.

These could be comforting remarks to those who are interested in narrowing the formal connection between russellian and substitutional quantifiers, but they also point to the logical and semantical departure from the initial theory of *PM* which is involved in substitutionalization. On the other hand, if one is to invoke the extensional logic of *PM* in order to claim that a substitutional interpretation does not conflict with russellian (extensional) quantifiers, one should also be ready to discover that the former could not any more than the latter be a solution to the problem of the elimination of classes.

3. ELIMINATION: THE SUBSTITUTIONAL EFFECT

A language, according to Quine, is committed to the existence of the entities which are the values of the bound variables in a true quantified statement.[23] Such a criterion of ontological commitment is valid for a "canonical language" and relies characteristically on an objectual interpretation of quantifiers.

Substitutional quantifiers, as we have seen, are rather non-committal as to what concerns such things as the values for variables or the notion of a predicate being satisfied by objects of a certain domain. The truth conditions of a statement with substitutional quantifiers rather refer to syntactical substituends for variables and to sentences whose truth-value is already determined.

As Kripke pointed out, however, substitutional quantifiers are not "a guarantee of ontological freedom": a substitutional interpretation of quantifiers can avoid the question of how truth for quantified statements is to be defined only if this question has already been answered for a language without quantifiers which contains the instances of quantified statements. If any ontological entity needs to be referred to

[22] It is clear that if ϕx and $\phi\hat{z}$ have to be construed in an extensional (rather than intensional) perspective (*cf.* sec. 1.2), the truth conditions of $\forall x(\phi x)$ and $\exists x(\phi x)$ should refer (directly or indirectly but not ambiguously) to elementary sentences.

[23] Quine [1953], p. 103.

by the truth conditions of these instances, then a language with substitutional quantifiers is committed to the existence of exactly these entities.

As far as ontological commitment is concerned, the contrast between a language with objectual quantifiers and a language with substitutional quantifiers can then be summarized in the following way. In a language with objectual quantifiers (e.g.: Quine's canonical language), any expression occupying a name position in a sentence will refer to ontological entities. In a language with substitutional quantifiers, only name positions in a sentence without a quantifier are occupied by expressions which could (but need not) refer to ontological entities; name positions, in quantified statements, are filled by names of linguistic expressions.

From this it follows that if substitutional quantifiers are not a "guarantee of ontological freedom" they could be much more economical, ontologically speaking, than objectual quantifiers. For name positions could be less numerous in the instances of a quantified sentence than in that sentence itself: an objectual interpretation of $\exists\phi\forall x(\phi x \Leftrightarrow Px)$ needs to admit entities corresponding to the values of ϕ and x, while the instances of the same sentence just refer to the values of x for which $\phi' x \Leftrightarrow Px$ is true (where ϕ' is the substituend for ϕ).

The ontological outcome of substitutionalizing the russellian quantifiers should be calculated from the fact that, in the initial russellian semantics, not all name positions are occupied by names. Only "genuine entities" are nameable, only constants can be names, and, whenever they appear in a name position, they should be considered as real (logical) names. Bound variables, even in a name position, are not names.

Since a substitutional interpretation of quantifiers avoids taking expressions in name position *in a quantified sentence* as referring to ontological entities, and, since russellian bound variables already do no such thing, at this level, the only noticeable effect of substitutionalizing russellian quantifiers will be, indirectly, on the interpretation of constants in a name position, if any are contained in a quantified sentence. This could lead to a non-negligible ontological economy if these constants do not reappear in a name position in the instances of quantified sentences. In the previous example, according to *PM*, Px would be in a name position (since $\Leftrightarrow$ is a higher order predicate), and as a constant it then certainly refers to a given propositional function (more precisely, to a *collection* of sentences which are the values of the propositional function Px). Under a substitutional interpretation, only the sentences in which P occurs will be referred to (but not as a collection), and, in the instances of this quantified statement, P will occur in a purely predicative (non-referential) position.

But economy on this side could be counterbalanced by the ontological multiplication which results from taking bound variables as standing for expressions. In the same example as above, the substituends for x will be constants; in instances of the quantified sentence they will be in a name position, and, where the russellian language was not ontologically committed before, it will now be referring to individuals as "genuine entities".

This result, in spite of the ontological economy which could possibly be realized by substitutionalizing russellian quantifiers, is mostly undesirable; for it will obtain precisely in the so-called eliminative definitions. These definitions are *contextual* definitions; their *definiens* is intended to make explicit the condition of application of a given predicate and is therefore expected to contain an irreducible name position. In Russell's account such a name position is filled by a bound variable and thus does not refer to genuine entities. But, if bound variables have to stand for expressions, then such a name position will finally be filled by a constant. The undesirable effect could be directly observed on the definition of a definite description:

$$f[\ x(\phi x)] =_{df} \exists c[\forall x(\phi x \Leftrightarrow x = c) \wedge fc].$$

Under a substitutional interpretation the *definiens* will read: there is a term c' such that the result of replacing x by any term x' and c by c' in $(\phi x \Leftrightarrow x = c) \wedge f(c)$ is a true sentence of the language.

The truth conditions of such an assertion minimally require that the suitable substituends for x and c be identified, so that the sentences which are referred to as being true are themselves exhibited. Such a sentence could be $(\phi a \Leftrightarrow a = b) \wedge fb$. Nothing yet has to satisfy ϕ and f, but a and b are constants in a name position, and the truth conditions of the above certainly refer to the "genuine entities" which have to be ϕ and f in order for this sentence to be true. Moreover, one of these constants just occurs as an argument for f which, on the left-hand side, applies to $x(\phi x)$. Now the point of contextual definitions in connection with elimination is precisely to show that the predicate applying to a grammatical name does not, in the logical language, apply to a constant (feature III in section 1.1). Where the language with russellian quantifiers thus revealed the presence of a pseudo-name, it will now, thanks to substitutional quantifiers, reveal that $x(\phi x)$ is a synonym for a real logical name. Under such an "improvement" of the russellian semantics, any application of the technique of elimination will turn out to be a reduction.

At this point it should be clear that, as far as the very idea of a substitutional *solution* to the problem of classes is concerned, there is a misunderstanding, either about the problem to be solved or about the powers of a substitutional interpretation of quantifiers.

By the same process as described above, a substitutional reading of: $f[\hat{z}(\psi z) =_{df} \exists\phi[\forall x(\phi x \Leftrightarrow \psi x) \wedge f(\phi\hat{z})]$ will contribute to dismiss the name position for both ϕx and ψx in the first subformula of the instances of the *definiens*. But in the second subformula of these very instances, a substituend for $\phi\hat{z}$ will be a constant in a name position, thus denoting a "genuine entity" to which the truth conditions of the entire instances have to refer.

It is of little importance here what exactly the nature of this entity is; for elimination will fail as soon as a logical name, whichever it is, occurs in the very same context where an alleged pseudo-name previously occurred. And by russellian standards, a substituend for $\phi\hat{z}$ in $f(\phi\hat{z})$ will be such a logical name. But, from a logical point of view, it could be of some interest to remark that the use of substitutional quantifiers is not completely neutral in that respect, and that it here forces an interpretation of $\phi\hat{z}$ which could be very close to its extensional interpretation.

A first indication in this sense is related to the very conditions of a substitutional interpretation of (ii). One cannot use a substitutional quantifier, any more than a russellian quantifier, in (ii), without some sort of assumption connecting ϕx and $\phi\hat{z}$; in particular, one should be sure that at least one term of the class of substitution for $\Sigma\phi$ could be a substituend *for both* ϕx and $\phi\hat{z}$. A simple way to secure the coherence of a substitutional quantifier in this respect is to take $\phi\hat{z}$ as "ϕx"; which,

we recall, was also the natural thing to do, but then, for logical reasons, in an extensional reading of (ii). Now, it seems that the so-called "quotational device" even if it does not require to be so motivated, when used in connection with substitutional quantifiers, at the semantical level cannot avoid one of the consequences of an extensional interpretation of (ii). For to take $\phi\hat{z}$ as "ϕx" is indeed to admit, as we did in section 1.2, the interchangeability of $\phi\hat{z}$ and ϕx in a name position (where the ramified theory of types was acknowledging two disjoint ranges of values.)

On the other hand, it is trivially the case, as we noticed at the end of section 2.3, that, in a substitutional as well as in an extensional interpretation, the only relevant aspect of propositional functions is what Russell called their values; instances of quantified statements *are* such values and nothing else than instances is involved in the semantics of (substitutionally) quantified statements. It is equally the case, in a substitutional as well as in an extensional interpretation, that propositional functions do not only always "occur in propositions through their values", but through their values being themselves truth-valued; in both cases then, "elementary sentences" which are the values of propositional functions should constitute a well-defined collection for which truth is entirely characterized. These constraints do not *determine* how truth has to be defined for elementary sentences; but they restrict considerably the possibilities. If we are to rely on Russell's semantical account of elementary sentences, Px in a name position and now, by transitivity, $P\hat{z}$ will denote the extension of a predicate P; this is also ascertained in the preface to the second edition of *PM*, where Russell remarks that "a function of function [like $f(\phi\hat{x})$] is always extensional",[24] and also that "there will no longer be any difference between the meaning of the two symbols $\hat{x}(\phi x)$ and $\phi\hat{x}$".[25]

These last remarks give additional support to our conclusion (although not an indispensable one).

We have shown above (section 1), that the definition of classes is not eliminative; for it allows, by implication, the direct replacement of a class expression $\hat{z}(\psi z)$ by the constant ψx, which, by all russellian standards, violates the elimination requirements. A substitutional interpretation of (ii′), by displacing the semantical question from the assertion made by a quantified statement to the truth conditions of its instances, not only allows by implication the above translation, but actually provides it. It thus makes the failure of elimination very explicit, and, to this extent, our conclusion is that the theory of substitutional quantifiers proves to be useful as an analytic reconstruction of the problem raised by classes in *PM* but not as a solution to it.

We have also shown, in the course of our argument, that the definition of classes in *PM* requires a reinterpretation of the initial language, so that what appears to be at least a reductive definition will actually be better described as the introduction of *ad hoc* entities (extensions) in the formal ontology of *PM*. Our last remarks were to suggest that the substitutional interpretation of (ii′) tends to reproduce also that aspect of the problem raised by the definition of classes in *PM*.

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[24] Russell [1925–27], I: xiv.
[25] *Ibid.*, I: xxxix.

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