Extension to geometry of *Principia Mathematica* and related systems II[1]

by Martha Harrell

Considering the lack of attention paid to this subject, it should be said that some reliable sources for its consideration have been found. It is surprising how many forward references to geometry are found in *Principia* itself. The paucity of manuscripts left by Whitehead can, fortunately, be remedied in part by resort to several published works of both Whitehead and Russell in most of which geometry is central, e.g., [30], [31], [35], [13], [16].[2] As to the value of attempting to discover Whitehead's results in *Principia* geometry and to build upon them, Russell was strongly positive on a number of occasions.[3] Russell's opinion was that it is important to the study of the principles of mathematics to include geometry. I concur, though our understanding of what this requires of us has changed dramatically since Russell. The commonly accepted twentieth-century view seems to be that no special study of the principles of geometry is required from the point of view either of logic or mathematical philosophy. The following study provides definite evidence in favour of a specialized study of the principles of geometry, especially when the influences of the general theory of relativity and quantum theory upon physical and mathematical geometry are considered. Our view of physical and mathematical geometry as separated by an iron curtain, so to speak, has been shaken by these physical theories. The dust has not yet settled. Perhaps our understanding of contemporary, i.e., twentieth-century, geometry depends upon certain physical theories and concepts as well as upon the relevant mathematical ones. The author examines various contemporary mathematical and physical systems of geometry in detail in *Elements and Applications of Contemporary Geometry* (1989).

[1] Paper I of this title was given at the Logic Colloquium '82 (Florence, Italy). It provided a brief account of *Principia* geometry based on Vols. I–III and Whitehead's letters to Russell of 1905–13. The author is indebted to K. de Bouvère, I. Grattan-Guinness, N. Griffin, A.C. Lewis, W.V.O. Quine, A. Riska, R. Tully and D.T. Whiteside, for helpful discussions relating to the present paper.

[2] Further relevant works considered in this paper are found in the References. Prof. Lowe suggests further Whitehead sources, *q.v.* see p. 18.

[3] See, e.g., his obiturary tribute to Whitehead in *Mind*, 57 (1948): 137–8.

Due to the two editions of *Principia*, criticisms of them, changes in Whitehead and Russell's opinions in geometry, and relevant post-*Principia* developments, a word about perspective is in order. Another logician has asked me what is meant by "geometry" in the title of this paper and which edition of *Principia* is used. By the former I mean *Principia* geometry according to the cited sources. The perspective relies upon the second edition. Published, extra-*Principia* sources of its authors' views are used as the primary foundation of speculative construction and are carefully related to the primary sources. Influential criticisms of *Principia* and significant post-*Principia* developments must be reserved for other occasions. Finally, although this work was prompted by questions about the effective applicability of *Principia* logic to its geometry in the projected Volume IV, this paper postpones a direct logical view of geometry. This paper is concerned with the preliminary work of demarcating the field of geometry and with providing the discipline with some specification adequate to *Principia* purposes.

In Sec. 1, *Principia* references to geometry are studied. In Sec. 2, the major views of Whitehead and Russell regarding geometry are cited with reference to Sec. 1. Whitehead's letters to Russell concerning Volume IV, the only primary sources of this work known extant, are studied here. In Sec. 3, related systems are considered. The conclusion is given in Sec. 4. A sequel to this paper is envisaged using new secondary sources, e.g., the Turnbull lecture notes above-cited and the 1914 letter of Whitehead to Russell, and providing further technical development of the subject.

1. *Principia*

In the Preface to the first edition, kept in the second, the authors acknowledge several other mathematicians in various matters. In matters of notation they named Peano chiefly; in those of logical analysis, Frege alone; in those of arithmetic and the theory of series, Cantor. When it comes, finally, to geometry, no one mathematician is acknowledged as their leading light, and Hilbert is not named at all ([37], I: viii).[4] They cite, in this order, von Staudt (see [28] and [29]), Pasch (see [7]), Peano (see [8] and [9]), Pieri (see [10]), and O. Veblen (see [27]). Von Staudt's methodology in projective geometry, in light of which that subject was developed as a "geometry of position", strongly influenced both Russell and Whitehead's treatments of this branch of geometry (see, e.g., [21], Chap. XLV; [31]). The methodological principle upon which von Staudt relied was that projective geometry is essentially non-quantitative and should thus be developed on a non-metrical basis. Of the three geometers of the turn-of-the-century Italian school, Pasch and Pieri are cited by Whitehead and Russell in their respective works in axiom systems of projective and so-called descriptive geometry, i.e., one in which parallel lines may exist. Modern affine geometry has replaced descriptive geometry. Peano, whose paper at the International Congress of Mathematicians in 1900 so impressed Russell, also figures in both Whitehead and Russell's studies in geometry. His major recognized contributions are an axiom system for descriptive geometry and a definition of congruence, the latter of which Whitehead contrasts with that in S. Lie's analysis (in [30], p. 45f.). Veblen's axiom system for geometry is relied upon by Whitehead

[4] Whitehead cites Hilbert's very influential *Grundlagen der Geometrie* [3] early in his [30], but considers Veblen's AS in [27] to supersede Hilbert's.

in the development of an axiom system of descriptive geometry (especially in [30]).

Some building blocks of geometry as apparently envisaged for *Principia* are introduced in Volume II. The most important are in Part V, especially the Summary, ∗205, ∗206, ∗215, and ∗234. These concern, respectively, the Summary on Series (Part V's topic), Maximum and Minimum Points, Sequent Points, Stretches, and Continuity of Functions. If Whitehead's method of extensive abstraction be brought in, as Prof. Victor Lowe's work counsels, ∗210, on Series of Classes Generated by the Relation of Inclusion, may support the development of geometry. Considering the introduction of certain points of series—maximum, minimum, sequent—in the general theory of series and the treatment of the continuity property of series, we could say that the prolegomena to geometry begins in Volume II.

Volume III provides a remarkable storehouse of concepts, theorems, definitions and brief discussion useful for a *Principia* account of the mathematical principles of geometry. It serves as the arithmetical heart of the prolegomena to geometry, from ∗275, Part V, on Continuous Series to the final theorem, ∗375·32 concerning Principal Ratios. In ∗275, probably due to Russell since he did most of the *Principia* work on series, the definition of continuity is due to Cantor. In contrast, Whitehead, as shown in Sec. 2 below, uses Dedekind's definition ([30], §9). Russell prefers Cantor's on account of its provision, *inter alia*, that two series continuous in his sense are ordinally similar; in any case, series continuous in Cantor's sense are also continuous in Dedekind's, but not vice versa.

In the Preface to Vol. III, the authors, after advising of the subjects to be treated therein, note for the first time that geometry is to be reserved for a separate, final volume ([37], III: v). This indicates that as of 15 February 1913, the date of the Preface, the volume was still expected to appear. The last known letter of Whitehead to Russell on this work is dated 10 Jan. 1914 (see [33]). No record of any unilateral or mutual decision to abandon Vol. IV by Whitehead or both authors is known. Russell comments in 1959 ([19], p. 99) that Whitehead's interest flagged. Clearly Whitehead was never satisfied with the *Principia* geometry work and thus never completed it. His dissatisfaction seemed to have directly involved a refusal to accept the apparent need for a Kantian or quasi-Kantian move for its completion. On my reading of the letters analyzed in Sec. 2, this most likely concerned Whitehead's belief that classes in intension were fundamentally required by Russell's theories of logic (primarily the theory of types) and of arithmetic in *Principia*. Further study is required to determine why Whitehead's view may have been anti-Kantian and exactly how an intensional theory of classes may be judged Kantian or quasi-Kantian.

The centerpiece of Vol. III regarding geometry is the theory of Measurement in Part VI, Sec. C on Quantity. In the Summary, its purpose is stated: "to explain the kinds of applications of numbers which may be called measurement" ([37], III: 233).

Part VI is described as new though based upon the method of Euclid in Book V of the *Elements* (p. v). Here the theory of proportion is found, developed from the account by Eudoxus. The authors consider the method as continued by Burali-Forti. If you have ever wondered why complex numbers are not mentioned in the published *Principia* under Generalizations of Number, Sec. A of Part VI, or if you considered it insignificant, you may be surprised by the Summary. The authors maintain that this subject belongs to geometry. Why? Because complex numbers do not form a one-dimensional series (p. 233). One is reminded of Russell's definition of geometry in the *Principles* as the study of two or more dimensions (Chap. XLIV, §352). Considering the acknowledgements to geometers in the Preface to *Principia*, the *Principles* appears of more than the historical interest to which Russell confined it in the Preface to the second edition in 1937. Whitehead defined the subject in quite different terms about six years after Russell wrote for the work's first edition. Geometry, Whitehead thought, is the science of cross-classification, a department in what may be called the general science of classification ([31], p. 4f.).

Let us review the four sections of Part VI, the final part of *Principia*.

Section A, regarding Generalizations of Number, we pass over.

Section B, on Vector-Families, is introduced by subordination to Section C, on Measurement. The purpose of Sec. B is to secure Sec. C by the introduction of hypotheses and proofs of propositions providing therefor. In the Summary of Sec. B ([37], III: 339), a magnitude is conceived as a vector, i.e., an operation, or a descriptive function in the sense of ∗30. An example is the relation "+1 gramme", i.e., the difference between 2 grammes and 1 gramme ([37], III: 339). By definition a gramme may not be a magnitude, since it is not a vector. A centimetre and a second are, however, considered to be vectors, since distances in space and time are. Units of mass or weight not being vectors, must be treated differently from units of space and time. Longstanding problems with lines of demarcation between mathematics and physics are suggested here. In contemporary relativistic physics, including quantum physics, we may call the last examples space-like and time-like intervals. These intervals seem to be vector quantities in the *Principia* sense.

Secs. A and B of Part VI constitute theories transitional between those of arithmetic together with branches of mathematics which may be built upon it, e.g., algebra and (algebraic) analysis, and geometry together with derivable branches, e.g., special geometries such as abstract Riemannian geometry and differential geometry, topology, and trigonometry. Of course, it is not now clear that any other branch of mathematics would have been derivable from geometry in *Principia*. Arithmetic and related branches of mathematics certainly appear to be granted more basic significance than geometry, not least because the work on geometry never saw the light of print. This tacitly accepted view popular from the turn of the century views geometry as in some way derivative from arithmetic. It is not proven, however, nor otherwise clear that with the completion of the *Principia* programme by a theory of geometry, and perhaps related disciplines such as topology, that Whitehead and Russell would not have agreed with Elie Cartan that one may start with geometry and derive certain concepts and techniques of analysis (e.g., the identification of vectors with derivatives) as well as the other way around.[5] A very important matter in such an attempt as the latter would be whether the theory or system of logic would differ from that found in the reverse procedure exemplified by *Principia*. A related question of special interest to mathematical logicians and philosophers would be the question whether any serious differences either in the logic or the mathematics would result from the reverse procedures. If the procedures turned out to be inverses of each other, with identical results but for their order of achievement, both theoretical and practical implications would be expected, e.g., greater simplicity of mathematical theories and improved ability to move from applied or practical to pure or theoretical mathematics. It appears accurate to view the *Prin-*

[5] See Cartan's work in the calculus of differential forms, *Oeuvres completes* (Paris: Gauthier-Villars, 1955).

cipia development from logic to arithmetic to geometry as one from pure to applied, from most to least abstract mathematics, in some results from mathematics to at least theoretical physics. The revolution in physics was in its infancy in the first decade of the twentieth century when Whitehead and Russell were at work on their landmark achievement. Both were deeply affected by it. It seems clearly to have hampered Whitehead's efforts at completing a theory of geometry adequately accounting for changed views of geometry due to the General Theory of Relativity, not to mention the Quantum Theory. Physics upset the logical analysis of space.

In Sec. C, of Volume III, the "pure" theory of ratios and real numbers of Sec. A is applied to vector-families. The derivation of the theory of measurement here is effected by means of attending to some one vector-family, $X\Updownarrow\kappa$, were κ is the vector-family in question, or $X\Updownarrow\kappa_\iota$, or $X\Updownarrow(\kappa\cup\mathrm{Cnv}``\kappa)$. Of particular importance to geometry, the authors note concerning this section, is the subject of "rational nets", developed at *354. Their importance is allied to the introduction of coordinates. Such nets are obtained from a given family, roughly, by selecting those *vectors* which are rational multiples of a given vector, then limiting their fields to the *points* which can be reached by means of them from a given point. The hypothesis that κ is a group is often used in proofs in this number; this is effected by taking κ to be a connected family and is termed κ_g. Based upon this number, *inter alia*, *356 proceeds to the theory of Measurement by Real Numbers. Finally in Sec. C, Existence-Theorems for Vector-Families are presented (*359, p. 452ff.). The axiom of infinity is needed for this development.

Sec. D, concerning Cyclic Families, is the final one of Part VI (Quantity). This, the authors have pointed out in the Summary of the Part, concerns such families of vectors as angles and elliptic straight lines ([37], III: 457). Owing to the fact that any number of complete revolutions may be added to a vector without thereby changing it, the theory of measurement for such families exhibits peculiar properties. There is no single ratio of two vectors, but many, from which one is selected as the principal ratio.

An angle is considered here as a vector whose field is all the rays in a given plane through a given point ([37, III: 457). Here we clearly detect the influence of the projected *Principia* account of geometry and uncover some of its basic terms and concerns within a definite theoretical framework. There is a null-vector on the basis of whose analysis the selection of the principal ratio is determined. The only geometrical figure in the work is found in this section, demonstrating vector-families in circular form (p. 459).

We are left, at *375·32, to complete the prolegomena to geometry, if more is needed, as well as the theory of geometry proper.

Let us turn to Whitehead's reports of progress in the work on geometry in his letters to Russell from 1905 to 1914.

2. WHITEHEAD AND RUSSELL

2.1 The fourth volume was to have been initially written by Whitehead alone, contrary to the practice with the first three volumes. This was due to the fact that he was working in the field at the time. Between 27 April 1905 and 1 Oct. 1913, Whitehead reported on his work to Russell in a number of letters, five of which are known to have survived ([33]). A sixth letter, dated 10 January 1914 and containing

important information, was recently discovered by Nicholas Griffin of McMaster University and an editor of *The Collected Papers of Bertrand Russell.*[6] These six letters comprise the entire extant corpus of original work on Vol. IV now known. Though woefully incomplete, they manage to reveal much of value in connection with extending *Principia* and related systems to geometry. In the former case, this requires connecting the parts of Vols. II and III discussed in Sec. I above with what Whitehead reports to Russell in the letters at hand.

Questions of mathematical philosophy, *viz.*, certain problems for the logicism which *Principia* was intended to realize, surface despite all efforts to avoid philosophy in a mathematical treatment of the principles of geometry. In the work of the first three volumes, Russell asserted that Whitehead left the philosophical issues to him ([19], p. 74). Both took up philosophical issues connected to geometry, with different results.

Let me give an overview of the work Whitehead reports in geometry in the six letters.[7] Disagreement between the two surfaces in 1905 over how to proceed with geometry, certainly a factor delaying Whitehead's progress. In letter (1), of 1905, Whitehead reports a crisis in the work (p. 1). He is excited over having determined that Oswald Veblen, presumably in [27], had recently taken up a perspective regarding descriptive geometry which suitably generalized and interpreted, should be taken up in the *Principia* treatment. Veblen's view is to take geometry to be the study of a single many-termed relation. The immediate need, then, the crisis, is the necessity to produce a notation suitable for triadic, tetradic, even n-adic relations for the theory of the principles of geometry. This Whitehead begins to set out in much detail in the third letter, written only three days after the first.

In (2), of 1905, Whitehead presents to Russell, as he says "in substance" (p. 1), Pieri's axiom system for projective geometry (see [10]) considered first as the study of a three-termed relation of collinearity, then as the study of a four-termed relation of collinearity and separation.

In (3), also of 1905, the longest of the six letters at ten long handwritten pages, in addition to the proposed new notation for geometrical relations we find controversy with Russell over the basic entities needed in the geometry, an argument for an intensional view of the notation for relations, Veblen's definitions useful in the case of space, and a statement of Veblen's axioms of descriptive geometry in the new notation.

About five years and five months later letter (4) was written, in 1910. This one reports the progress of work to *505, concerning Axioms of Connection. The other specific topics and propositions mentioned are *500 on Associated Symmetrical and Permutative Triadic Functions, *502 on the Associated Relation of a Triadic Func-

[6] Prof. Griffin advised me of this in a letter of 4 December 1984; Russell's covering letter to Lady Ottoline Morrell is with the Whitehead letter, q.v. see note 8 below. The lecture notes of H.W. Turnbull from several courses of lectures of Whitehead in 1907 provide further material for this study. I am grateful to Dr. A.C. Lewis for this reference (see [25] and [32]). Of interest regarding *PM* geometry are notes on mathematics and mathematical reasoning, number theory, the idea of magnitude, theory of logic (based on *PM*), descriptive and projective geometry, distance, absolute and relational theories of space, geometry of existing space. In 1905 and 1906 at Cambridge, Whitehead offered several related courses of lectures; no record of these has been found.

[7] As noted in [33], Mrs. T. North Whitehead has kindly granted permission to quote from letters (1) through (5).

tion, and ∗504, Axioms of Permutation and Diversity. ∗505, Axioms of Connection, was not in final form, he said (p. 4). We are left with no record of the proposition-by-proposition development from ∗375·32, the last published, to ∗505. One would expect Volume IV to begin with ∗400, judging from the fact that geometry would begin in a new part and that V and VI began with ∗200 and ∗300, respectively. Whitehead cites ∗500 with reference to "the beginning of geometry" (p. 3). It could be that ∗400 began a Prolegomena to Geometry as in the case of arithmetic (Part II, Vol. I), but we have now no record whatever of this part.

The fifth letter (5) appears after a further three years, in 1913. Whitehead again reports progress, this time primarily in understanding what geometry is. He also reports having done "a lot of writing for Vol. IV" (p. 1). He continues:

> In fact I have found out which [*sic*] the science is about. The whole [subject] depends on the discussion of the connective properties of multiple relations. This is a grand subject. It merges into the discussion of Clν, where ν is a cardinal number, preferably inductive. I call such things 'multifolds'. (P. 1f.)

The two-page letter of January 1914, (6), shows that Whitehead planned, with Russell's consent, to include his paper on the relational theory of space virtually as it was in Volume IV. He adds that a second part is necessary to the paper. The reference is doubtless to either a different version or to his paper of that title for the Congress of Mathematical Philosophy at Paris, 1914, published in French in 1916 ([35]). While this throws new light on *PM* geometry, its revelation is not immediately obvious mathematically, since the French article, the only paper by this title known extant, is mostly discursive. Russell, in a letter to Lady Ottoline Morrell, expresses approval of this Whitehead work.[8] The copy of the paper Whitehead sent Russell is presumed lost, as is any reply from Russell if any were made. To include the French article in Volume IV would require it to appear in an introduction covering philosophical matters. Apparently the necessary second part of the paper would be technical, set forth in a notation for geometry following that set forth in letter (3) of April 1905.

2.2 In [30] and [31], published in the two years immediately following the year of the first three letters, *viz.* 1906 and 1907, Whitehead composed axiom systems for projective, descriptive, and metrical geometry. These divisions of the subject were also followed by Russell in his *Principles*, though they are not currently. Projective and descriptive geometry are interrelated in a number of ways. Following von Staudt, projective geometry has been developed by Whitehead non-metrically. Within descriptive geometry there is an Associated Projective Space: the axioms of projective geometry hold for certain ideal "projective elements" (§25), and to provide the superior generality of the projective axioms for use in descriptive geometry, given a descriptive space, a projective space is constructed of which the descriptive space forms a part (§15). No more than three dimensions are discussed in the case of either kind of space. Metrical geometry, including non-Euclidean geometries of Bolyai, Lobachevsky, and Riemann, constitutes the final development of descriptive

[8] Russell rated this Whitehead work highly. Russell letter #963, dated 12 January 1914, Lady Ottoline Morrell Papers, Harry Ransom Humanities Research Center, University of Texas at Austin.

geometry in [30]. The form of the axioms of metrical geometry is modified by Whitehead, but they are substantially those of Pasch and Peano for the straight line, primarily due to Pasch (see §§3–8). The first set of axioms for descriptive geometry, eleven axioms, provide for the ordinary properties of a straight line regarding points thereon and the division of a line by two points and by a single point. Compactness is a property of the straight line, but the Dedekind property (Dedekind continuity) is not. The Dedekind property is later secured as applying directly to the case of the descriptive line from Dedekind's original statement of the property. Further axioms for the systems are taken from Veblen [27], who took as indefinables "points" and a relation among three points he called "order". The view of geometry taken by Veblen and which so impressed Whitehead in 1905 (see letter (1)) here again influences Whitehead's conception of the subject fundamentally.

Three major axiom systems ("AS") are set forth by Whitehead in the letters to Russell from 1905–13, based upon viewing geometry as the study of properties of a many-term relation: An AS of fourteen axioms and nine definitions for projective geometry (G_{p_1}) giving Pieri's axioms in *Principia* notation based on a three-termed relation of collinearity (letter (2)); a second, tentative, AS of twelve axioms and eight definitions for projective geometry (G_{p_2}) based on Pieri and a four-term relation of collinearity and separation (letter (2)); and Veblen's AS of twelve axioms for descriptive geometry (G_d), all of which are definitions as well (letter (3)). All axioms are designated as hypotheses.[9] They are as follows. Note that the raised semicolon indicates that a relation R holds between its terms per letter (3), attachment p. 2.

I. G_{p_1}

1. $\mathrm{Hp}\,.\,R\,.=.\,(\ldots)R \geqslant 2$ Df.
2. $\mathrm{Hp}\,.\,R\,.=:R^{;}(abc)\,.\supset.\,R^{;}(cba)$ Df.
3. $\mathrm{Hp}\,.\,R\,.=:R^{;}(abc)\,.\supset.\,R^{;}(bac)$ Df.
4. $\mathrm{Hp}\,.\,R\,.=:R^{;}(abc)\,.\supset.\,a \neq b$ Df.
5. $\mathrm{Hp}\,.\,R\,.=:.\,R^{;}(abc)\,.\supset:R^{;}(abx)\,.\,x \neq c\,.\supset_x.\,R^{;}(acx)$ Df.
6. $\mathrm{Hp}\,.\,R\,.=:.\,a,b\,\varepsilon\,(\ldots)`R\,.\,a \neq b\,.\supset.\,\exists`R^{;}(ab;)$ Df.

Df. $R^{;}\overline{ab} = R^{;}(ab;) \cup \iota`a \cup \iota`b$
Df. $R^{;}\overline{abc} = x'[\mathsf{v}`R^{;;}\{(\iota`a)(\iota`c)(R^{;}\overline{bc})\}]$
Df. $R^{;}\overline{abcd} = x'[\mathsf{v}`R^{;;}\{(\iota`a)(\iota`x)(R^{;}\overline{bcd}\}]$
Df. $\Delta_R abc\,.=.\,a,b,c\,\varepsilon\,(\ldots)`R\,.\sim`R^{;}(abc)\,.\,a \neq b\,.\,b \neq c\,.\,c \neq a$
Df. $\mathrm{lin}`R\,.=.\,u'\{(\exists a,b)\,.\,a,b\,\varepsilon\,(\ldots)`R\,.\,a \neq b\,.\,u = R^{;}\overline{ab}\}$
Df. $\mathrm{ple}`R\,.=.\,p'\{(\exists a,b,c)\,.\,\Delta_R abc\,.\,p = R^{;}\overline{abc}\}$
Df. $\mathrm{tet}_R abcd\,.=.\,a,b,c,d\,\varepsilon\,(\ldots)`R\,.\sim`(\exists \nu)\,.\,\nu\,\varepsilon\,\mathrm{lin}`R\,.\,a,b,c,d\,\varepsilon\,\nu$

7. $\mathrm{Hp}\,.\,R\,.=:.\,a,b\,\varepsilon\,(\ldots)`R\,.\,a \neq b\,.\supset.\,\exists`\{(\ldots)`R - R^{;}\overline{ab}\}$
8. $\mathrm{Hp}\,.\,R =:\Delta_R abc\,.\,R^{;}(a'bc)\,.\,R^{;}(ab'c)\,.\supset.\,\exists`\{R^{;}(aa';) \cap R^{;}(bb')\}$ Df.
9. $\mathrm{Hp}\,.\,R\,.=:\Delta_R abc\,.\supset.\,\exists`\{(\ldots)`R - R^{;}\overline{abc}\}$ Df.

Df. $H_R{}^{;}(acbd)\,.=:R^{;}(adb)\,.\,R^{;}(adc):(\exists u,v)\,.\,R^{;}(ucv)\,.\sim`R^{;}(aub):$

[9] A sketch of the notation is given in (3), attachment pp. 2–6.

$$R^{;}(axu) . R^{;}(bxv) . R^{;}(byu) . R^{;}(ayv) . \supset_{x,y} . R^{;}(xdy)$$

10. $\mathrm{Hp} . R . = :. R^{;}(abc) . \supset : H_R^{;}(acbd) . \supset . c \neq d$ Df.

Df. $\mathrm{seg}_R\overline{abc} = x'\{x = b . \vee . (\exists y) . R^{;}(acy) . H_R^{;}[yb\{\text{¡`}H_R^{;}(ayc;)\}x\}$

11. $\mathrm{Hp} . R . = : R^{;}(abc) . \supset . R^{;}\overline{ac} = \mathrm{seg}_R\overline{abc} \cup \mathrm{seg}_R\overline{bca}$ Df.

12. $\mathrm{Hp} . R . = : R^{;}(abc) . \supset . \mathrm{seg}_R\overline{abc} \cap \mathrm{seg}_R\overline{bca} \cup \mathrm{seg}_R\overline{cab} = \Lambda$ Df.

13. $\mathrm{Hp} . R . = :. R^{;}(abc) . \supset : d \,\varepsilon\, \mathrm{seg}_R\overline{abc} . \supset . \mathrm{seg}_R\overline{adc} \subset \mathrm{seg}_R\overline{abc}$ Df.

14. An axiom on continuity

In [31] Whitehead adds a dimensionality axiom limiting the geometry to three dimensions and specifies a Dedekind continuity axiom. The dimensionality axiom is wanted for the introduction of coordinates. Axioms exactly corresponding to 11, 12 and 13 above appear as axioms of order, and others corresponding to 1 through 10 above, as axioms of classification. The dimensionality and continuity axioms are as follows:

D(XV) There exists a plane α and a point A, not incident in α, such that any point lies in some line possessing A and some point of α. ([31], p. 15)

C(XIX) If u is any segment of a line, there are two points A and B, such that, if P be any member of u distinct from A and B, segm (APB) is all of u with the possible exception of either or both of A and B which may also belong to u.[10]

II. G_{p_2}

1. $\mathrm{Hp} . R . = : R^{;}(abcd) . \supset . a \neq c . b \neq d$ Df.
2. $\mathrm{Hp} . R . = : R^{;}(abcd) . \supset . R^{;}(bcda)$ Df.
3. $\mathrm{Hp} . R . = : R^{;}(abcd) . \supset . R^{;}(dcba)$ Df.
4. $\mathrm{Hp} . R . = : R^{;}(abcd) . \supset . \sim\text{`}R^{;}(acbd)$ Df.
5. $\mathrm{Hp} . R . = : R^{;}(abcd) . R^{;}(abce) . \supset . \sim\text{`}R^{;}(adce)$ Df.
6. $\mathrm{Hp} . R . = ; a,c \,\varepsilon\, (....)\text{`}R . a \neq c . \supset . \exists\text{`}R^{;}(a;c;)$ Df.

Df. $R^{;}\overline{ac} . = (..)\text{`}R^{;}(a;c;) \cup (\iota\text{`}a \cup \iota\text{`}c) \cap (....)\text{`}R$

Df. $\Delta_R abc . = : a,b,c \,\varepsilon\, (....)\text{`}R . a \neq b . b \neq c . c \neq a . \sim\text{`}\exists\text{`}R^{;}(abc;)$

Df. $\mathrm{lin}\text{`}R . = . v'\{(\exists x,y) . x,y \,\varepsilon\, (....)\text{`}R . x \neq y . v = R^{;}\overline{xy}$

Df. $R^{;}\overline{abc} . = . \cup\text{`}(..)\text{``}R^{;;}\{(\iota\text{`}a);(R^{;}\overline{bc});\} \cup \iota\text{`}a \cup R^{;}\overline{bc}$

Df. $R^{;}\overline{abcd} . = . \cup\text{`}(..)\text{``}R^{;;}\{(\iota\text{`}a);(R^{;}\overline{bcd});\} \cup \iota\text{`}a \cup R^{;}\overline{bcd}$

Df. $\mathrm{ple}\text{`}R . = . p'\{(\exists a,b,c) . \Delta_R abc . p = R^{;}\overline{abc}\}$

Df. $\mathrm{tet}_R\overline{abcd} . = . a,b,c,d \,\varepsilon\, (....)\text{`}R . \sim\text{`}(\exists v) . v \,\varepsilon\, \mathrm{ple}\text{`}R . a,b,c,d \,\varepsilon\, v .$
$\sim\text{`}(\exists v) . v \,\varepsilon\, \mathrm{lin}\text{`}R . a,b,c,d, \varepsilon\, v$

7. $\mathrm{Hp} . R . = : c,d \,\varepsilon\, R^{;}\overline{ab} . c \neq d . \supset . a \,\varepsilon\, R^{;}\overline{cd}$ Df.
8. $\mathrm{Hp} . R . = : a' \varepsilon\, (..)\text{`}R^{;}(b;c;) . b' \varepsilon\, (..)\text{`}R^{;}(c;a;) . \supset . \exists\text{`}\{R^{;}\overline{aa'} \cap R^{;}\overline{bb'}$ Df.
9. $\mathrm{Hp} . R . = : (\exists a,b,c) . \Delta_R\overline{abc}$ Df.
10. $\mathrm{Hp} . R . = : (\exists a,b,c,d) . \mathrm{tet}_R\overline{abcd}$ Df.

Df. $H_R^{;}(acbd) . = :. b,c \,\varepsilon\, R^{;}\overline{ad} \cap (....)\text{`}R :$

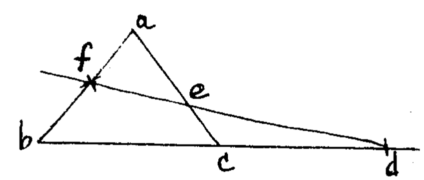
$(\exists u,v) : \exists\text{`}R^{;}(ucv;) . \sim\text{`}\exists\text{`}R^{;}(aub;) :$

$p \,\varepsilon\, R^{;}\overline{av} \cap R^{;}\overline{bu} . q \,\varepsilon\, R^{;}\overline{au} \cap R^{;}\overline{bv} . \supset_{p,q} . \exists\text{`}R^{;}(pqd;)$

11. $\mathrm{Hp} . R . = : a,b,c \,\varepsilon\, (....)\text{`}R . a \neq b . b \neq c . c \neq a . \supset : H_R^{;}(acbd) . \supset . R^{;}(acbd)$
12. $\mathrm{Hp} . R .$ (Axiom on continuity)

III. G_d

1. $\mathrm{Hp} . R . = . \mathrm{Nc}\text{`}(...)\text{`}R \geqslant 2$ Df.
2. $\mathrm{Hp} . R . = : R^{;}(abc) . \supset . R^{;}(cba)$ Df.
3. $\mathrm{Hp} . R . = : R^{;}(abc) . \supset . \sim\text{`}R^{;}(bca)$ Df.
4. $\mathrm{Hp} . R . = : R^{;}(abc) . \supset . a \neq c$ Df.
5. $\mathrm{Hp} . R . = : a \neq b . a,b \,\varepsilon\, (...)\text{`}R . \supset . \exists\text{`}R^{;}(ab;)$ Df.
6. $\mathrm{Hp} . R . = : c,d \,\varepsilon\, R^{;}\overline{ab} . c \neq d . \supset . a \,\varepsilon\, R^{;}\overline{cd}$ Df.
7. $\mathrm{Hp} . R . = : \mathrm{Nc}\text{`}(...)\text{`}R \geqslant 3 . \supset . (\exists a,b,c) . \Delta_R(abc)$ Df.
8. $\mathrm{Hp} . R . = : \Delta_R(abc) . R^{;}(bcd) . R^{;}(cea) . \supset . \exists\text{`}\{R^{;}\overline{de} \cap R^{;}(a,b)\}$ Df.



Whitehead notes that an order definition is not assumed.

9. $\mathrm{Hp} . R . = : (\exists x,y,z) . \Delta_R(x,y,z) . \supset . (\exists p,d) . p \,\varepsilon\, \mathrm{ple}\text{`}R . d \,\varepsilon\, (...)\text{`}R - p$ Df.
10. $\mathrm{Hp} . R . = : (\exists x,y,z,u) . \mathrm{tet}_R(xyzu) . \supset . (\exists a,b,c,d) . (...)\text{`}R \subset \Pi_R(abcd)$ Df.

Note: "$\Pi_R(abcd)$" indicates three-dimensional space.

11. $\mathrm{Hp} . R . = :. (...)\text{`}R \,\varepsilon\, \mathrm{cls\,infin} . \supset : (\exists a,c) :$
$a,c \,\varepsilon\, (...)\text{`}R . a \neq c : \sigma \,\varepsilon\, \mathrm{cls\,infin} . \cap \mathrm{cls}$
$\text{`}R^{;;}\{(R^{;}\overline{ac});(R^{;}\overline{ac})\} . \iota\text{`}a \cup \iota\text{`}c \cup R^{;(a;c)}$
$\subset \cup\text{`}\sigma . \supset . (\exists p) . p \,\varepsilon\, \mathrm{cls\,fin} \cap \mathrm{cls}\text{`}\sigma . \iota\text{`}a \cup \iota\text{`}c \cup R^{;}(a;c) \subset \cup\text{`}p$ Df.

Whitehead notes that this is the continuity axiom in the Heine–Borel form.

12. $\mathrm{Hp} . R . = :. \alpha \,\varepsilon\, \mathrm{ple}\text{`}R . a \,\varepsilon\, \mathrm{lin}\text{`}R \cap \mathrm{cls}\text{`}\alpha . \supset : (\exists c) .$
$c \,\varepsilon\, \alpha : 1,1' \,\varepsilon\, \mathrm{lin}\text{`}R \cap \mathrm{cls}\text{`}\alpha . c \,\varepsilon\, 1 \cap 1' . 1 \cap a = \Lambda . 1' \cap a = \Lambda . \supset 1,1' . 1 = 1'$ Df.

Whitehead notes that this is the axiom for Euclidean space.

Euclidean geometry is descriptive according to Whitehead ([31], p. 8).

In *The Axioms of Descriptive Geometry* ([30], Hafner ed., p. 1) Whitehead noted three methods by which order may be managed in that subject. Order is in G_d the focus of attention, as classification is in G_p. The three are (1) by taking the class of points lying *between* any two points as a fundamental idea, as by Peano; (2) by taking a straight line to be basically a serial relation involving two terms and whose field forms the class of points on the straight line, i.e., by taking a class of relations to be the fundamental starting point of G_d, as by Vailati and Russell; and (3) by taking a single three-termed relation of order as fundamental and making G_d the study of

[10] See also the AS of 19 axioms of G_p in [31, Chaps. II and IV. Closed and open series, respectively, characterize the types of order of G_p and G_d according to [31], p. 6 (*cf.* [37], Part V, Vols. II and III). Theorems corresponding to the axioms of G_p hold true of projective entities treated in G_d (see [30], p. 27).

its properties. Whitehead chooses Veblen's method rather than Vailati and Russell's. In [30] Whitehead states the Veblen axioms 1 through 10 shown in III. G_d above, and eight definitions, in English with symbols for points, segments, and spaces (pp. 7–9). The counterpart to axiom 11 in III above is stated in terms of closed series in a form adapted from Dedekind's original formulation, rather than in the Heine–Borel form given in III above (p. 9f.). The counterpart to axiom 12 in III above is stated similarly to 12 but in English (p. 11). The letter stating the G_d axioms is dated 30 April 1905 (Mill House, Grantchester), and the preface to the *Axioms of Descriptive Geometry* is dated March 1907 (Cambridge). Before metrical geometry is taken up, at the end of this work, projective space and geometry are associated with descriptive space and geometry, and a general theory of correspondence, axioms of congruence, and infinitesimal rotations are introduced (Chaps. II through VII). Congruence is presented first by means of Pasch's ten axioms (p. 44f.), then by means of Lie's congruence groups of motions (p. 45).

What we obviously lack in the letters is an AS for metrical geometry. No AS is provided for metrical geometry in any of the cited letters to Russell or in any other known technical work of Whitehead. From the *Universal Algebra* (1898) ([36]) to the *Axioms of Descriptive Geometry* Whitehead paid more attention to the development of non-metrical than to that of metrical geometry. In the latter work he continues to introduce the theory of distance in geometry last among those figuring in geometry. This procedure emphasizes mathematical, or abstract, geometry rather than physical geometry. In Chap. VIII of [30], he derives distance from the theory of congruence by considering one fact about the anharmonic ratio of a range of collinear points and another about a definite congruence group (p. 69). The so-called "characteristic addition property" of distance for collinear points and the characteristic invariability of distances in a congruence transformation result. Coordinates (three) of any point on a given line P_1P_2 are stated. Distance on a line is introduced for the elliptic case providing for a system of G_m including the whole of projective space. Reference is made in this connection to Riemann's suggestion for the treatment of distance with closed lines of finite length (p. 70). For the hyperbolic case, distance is then defined, with reference to Lobachevsky and J. Bolyai; this distance equation defines metrical geometry of the hyperbolic type (p. 71). Ordinary Euclidean geometry figures in this division of G_m. Equations for the measurement of the angle between planes follow the same procedure as for that of distance. According to Whitehead G_m is "in fact the investigation of the properties of a particular congruence group" (p. 73). In fact, according to this view, no additional geometrical axiom is needed to obtain metrical properties from G_p. This is not, however, the case for G_d, due to certain properties of a congruence group in G_d.

It is this branch of the three main branches identified by Whitehead and Russell *c.*1900–14 (projective, descriptive, and metrical) which most seriously concerns differences between mathematical and physical geometry. These concern in particular what Russell calls interpretation, e.g. in [13] and [16], now studied in formal semantics and in model theory. This branch, then, due to crucial new developments in physics, seems to have borne the brunt of these developments' effects upon Whitehead's theory of *PM* geometry. According to Professor Victor Lowe:

> He said more than once (though never, I think, in print) that these works [those of Whitehead's second period, *c.*1914–23] were all preliminaries to Volume IV of *Principia Mathematica*, which for a long time (even after he went to America) he hoped to complete in such a way that Minkowski, Einstein, and the growth of logic after 1910 would be taken care of. ([4], p. 177)[11]

Lowe had earlier enumerated Whitehead's works indirectly concerned with the *PM* geometry programme:

> Concerning his work on a fourth volume of *Principia Mathematica*, which was to have been written by him alone, Professor Whitehead seems to have felt that the results of most value were either sufficiently contained in or superseded by the published writings which are listed below as 1916–2 ["La Théorie Relationniste de l'Espace"], 1919–1 [*An Enquiry Concerning the Principles of Natural Knowledge*], 1920–1 [*The Concept of Nature*], 1922–1 [*The Principle of Relativity*], and 1929–1 (Part IV) [*Process and Reality*]. ([5])

In the letter to Russell of 1914, cited above, Whitehead has referred to work at least part of which we presume was published in 1916 under the title "La Théorie relationniste de l'espace" [35]. While the published French paper does not contain any axiom systems for mathematical, or abstract, geometry, it does contain several quite definite statements apparently applicable to the extension to geometry of *PM*. It is most helpful to have learned of Whitehead's 1914 letter to Russell verifying the correctness of this application. At the beginning of the paper ([35], Fitzgerald, p. 167), Whitehead distinguishes four meanings of the word "space" and states that they extend to the word "geometry" in so far as the latter is the science of the properties of space. Here we clearly have to consider not only mathematical space but physical and perceptual space as well: the types of spaces are complete apparent, immediate apparent, physical, and abstract (mathematical). In *PM* the authors' aim was to provide a mathematical treatment of the principles of mathematics, including geometry. Thus, it would be only abstract geometry with which *PM* would be concerned, at least directly. As shown in sec. I of this paper, there are sections in Volumes II and III of *PM* evidencing the authors' concern for certain physical applications of their expected theory of geometry, e.g., in Part VI, Sec. B, on Vector-Families, where the magnitude "+1 gramme" is introduced (sec. I above). In works cited below in this section, Russell clearly distinguishes à priori from empirical or physical geometry numerous times. Difficulties for a *PM* treatment of geometry seem to arise with this Whitehead paper, however, with the definition of a point in terms of extensive abstraction from regions or volumes and his apparent view that geometry, at least to some extent, concerns the physical world. He has been led closer to foundations of physics and to applied mathematics and farther from pure mathematics and logic (*cf.* [6], p. 179).

Unfortunately, I must omit detailed discussion of axioms and definitions given in this article. A few points should be noted, however. The *PM* notation for relations was used, and certain relations and classes of relations were defined (see secs. III and IV) and used in the development of the relational theory of space. The following statement indicates Whitehead's general viewpoint at the time:

[11] Prof. Lowe notes that one discussion of these matters he had with Whitehead took place on May 14, 1941.

> Geometry as a mathematical theory has usually taken as a point of departure all or part of the fundamental spatial entities, points, curved or straight lines, surfaces, and volumes. It takes them as simple primitive ideas, i.e., in abstract language as "variables" which are not logical functions of more simple variables. But if the relational theory of space is adopted either for the apparent world or for the physical world, this cannot be the first stage of geometric research. For the relative theory of space, it is essential that points, for example, be complex entities, logical functions of those relations between objects constituting space. For, if a point is a simple thing, incapable of being logically defined by means of relations among objects, then the points are indeed absolute positions. Then the relation of "being at a point" must be a primitive relation incapable of definition, and thus, one must take as the only ultimate fact of geometry the primitive relations of objects to their absolute positions. But this is nothing else than the absolute theory of space which, nominally at least, has almost universally been abandoned. Then, the first occupation of geometricians searching for the foundations of their science is to define points as functions of relations between objects. ([35], sec. III: Fitzgerald, p. 174)

Perhaps the effect of this work upon the *PM* geometry programme would be accurately described as revealing to Whitehead perceived inadequacies of the mathematical theory of geometry he had developed by then to ∗505—Axioms of Connection—of *PM* for physical applications. We have here, it seems to me, the question, in both technical and philosophical forms, whether every entity mentioned and every proposition stated in physical geometry should have provided for it in advance a more general definition or statement in mathematical geometry, so that the former is a mere derivation from or application of the latter, or whether such a scheme exceeded then-current and even, worse yet, attainable knowledge. It was certainly true that early twentieth-century developments in theoretical physics had already convinced Whitehead of some gaps in the accepted edifice of mathematical geometry. Mathematics and logic needed to catch up with physics indicating some inadequacy in the mathematics or the logic, or both.

Geometry was of serious concern to both Whitehead and Russell from the 1890's onwards. Nearly every major work of both concerns the subject in some way. Below let us consider Russell's views relevant to the present task in our effort to determine, as precisely as possible, how the *Principia* extension to geometry might be realized.

2.3 Russell claimed in 1901 ([17]):

> All knowledge must be recognition, on pain of being mere delusion; arithmetic must be discovered in just the same sense in which Columbus discovered the West Indies, and we no more create numbers than he created the Indians.

Created or discovered, arithmetic was completed for *PM*. Geometry was not.

In the *Essay on the Foundations of Geometry*, Russell's treatment of geometry is more apt for the case of physical geometry, and for physical space, than his treatment in the *Principles*. His concern with the homogeneity of space required for both projective and metrical geometries, is a case in point. The modern relativistic physicist, whether cosmological or reistic or quantum in outlook, is very much concerned with this property as well as with isotropy and continuity. Prof. Morris Kline, in the introduction to the 1956 edition, cites Russell's rejection of non-homogeneous space on logical grounds, and that of four-dimensional and non-Euclidean properties of space on empirical grounds (p. [ix]). Riemann, as Klein points out, foreshadowed relativity theory with the suggestion that when matter is taken into account, homogeneity disappears. So it is in the general theory of relativity; as Klein puts it, "the matter in space becomes absorbed by the geometry of space-time so that the nature of space-time varies from one region to another in accordance with the matter in it" (*ibid.*). Due to developments after 1896, Klein suggests reconsideration of what he calls, in keeping with the *Essay*, the à priori in geometry; Whitehead would be more likely to term this area mathematical geometry. Whatever it may be called this is the area to be treated by *PM* geometry. The supposed mere application of mathematical geometry to the case of physical space turned out to be a great challenge, as yet unmet on the broad scale envisaged by *PM*.

In the *Principles* Russell attempts for the first time ever to so broad an extent to set forth all basic geometrical axioms and postulates as well as all geometrical definitions in terms of "general logical concepts" ([21], §378) and uses mainly the theories of series and relations. Of course, Frege had made the original similar attempt for arithmetic before *PM*, but not for geometry. Regarding projective and descriptive geometry, Whitehead's AS's with definitions have advanced from Russell's treatment in the *Principles* from the point of view of a logistic philosophy of mathematics. Russell's consideration of metrical geometry provides far more detail than Whitehead's in [30] and, contrary to Whitehead's consideration there, evident concern for "actual space" (see §393, e.g.). Whereas Whitehead saw no necessity of new axioms for the introduction of distance relations and measures in metrical geometry, Russell's view is that three new axioms and one new indefinable are needed (§399). The stretch is introduced as a quantity, and the axioms are such as make stretches of points measurable. The logic is used to define all the classes of entities known as spaces to mathematicians and to deduce from the given definitions all propositions of the associated geometries. The continuity and infinity of a space can be arithmetically defined, which suits the apparent *PM* view of geometry as derivative from arithmetic, given a logic as well. Absolute space is affirmed, in contrast to Whitehead in [35] (see Russell's summary in §436). Late in 1898, Russell noted his view that arithmetic involves all the ideas of geometry except dimensions.[12]

Several post-*PM* Russell works indicate bases for its development in keeping with that of *PM* arithmetic, carrying the development to connections with topology. In *The Analysis of Matter* (1927) ([13]), *My Own Philosophy* (1946) ([18]), and *Human Knowledge* (1948) ([16]), he suggests the development of geometry based on interpretation and Peano arithmetic of the real numbers. In *My Philosophical Development* (1959) ([19], p. 99ff.), he discusses his notion of structure in connection with Whitehead's unfinished work for *PM* geometry. Let us review each of these two courses of development in turn.

In *The Analysis of Matter* Russell considers several matters of importance to determining his likely view of *PM* geometry: (1) the hypothetical nature of geometry considered as part of pure mathematics, (2) empirical and non-empirical interpretations of geometry, (3) a construction of points, (4) topology with an associated

[12] This statement is found in "Various Notes on Mathematical Philosophy", now in the Russell Archives. I am indebted to N. Griffin for this reference.

geometry. Regarding (1), he emphasizes the similarity between a logical analysis of physics and the methods of geometry considered as part of pure mathematics. What one begins with are not "axioms" supposed to be "true" but hypotheses containing variables (p. 2). One then attempts to prove that the entities with which one is concerned have the properties asserted of them in geometry or physics. It is extremely useful to discover a few primitive hypotheses, as we may call them though Russell does not, from which a science's entire deductive system follows. The entities initially taken as primitive may be replaced by complicated logical structures, as in pure mathematics in the definitions of cardinal numbers, ratios, real numbers, etc. A like result may occur in physics, e.g., in the definition of a "'point' of space-time" (p. 2). For example, Peano's five basic propositions for finite integers show that arithmetic, and by extension analysis, is deducible from those five propositions and three undefined ideas, zero, number (or finite integer), and successor. The Peano propositions may be seen as specifying properties of the undefined ideas, logical properties, though, not mathematical ones. These propositions are ultimately about the terms of any progression, not about "definite logical objects called numbers" (p. 4). We call the terms 0, 1, 2, 3, ..., terms of *any* progression, making those terms then "variables". To make the terms constants we must choose some interpretation, i.e., some definite progression. The general process of "interpretation" is that of specifying an important set of entities for the undefined entities of a deductive system. The philosophical import of a science must be in part determined by such a process. Whitehead has presented all basic geometrical propositions in the AS's cited above in this section as hypotheses, so he and Russell agreed in this matter. Whitehead, however, to my knowledge has not emphasized either Peano arithmetic, which Russell does repeatedly, or the process of interpretation. Regarding (2), the interpretation of geometry, what is striking regarding our present subject is the difference between an important and an unimportant interpretation of a science or system. Any geometry, Euclidean or non-Euclidean, Russell claims, in which every point has real coordinates, can be interpreted as applying to a system of sets of real numbers—i.e., a point may be taken to *be* the series of its coordinates. This is an acceptable and convenient interpretation of geometry as studied in pure mathematics. It is not, however, an *important* interpretation.

> Geometry is important, unlike arithmetic and analysis, because it can be interpreted so as to be part of applied mathematics—in fact, so as to be part of physics. It is this interpretation which is the really interesting one, and we cannot therefore rest content with the interpretation which makes geometry a part of the study of finite integers. (P. 5)

In this work geometry is treated as part of physics and regarded as dealing with objects not either mere variables or definable in purely logical terms. Its initial objects are satisfactorily interpreted only in terms of entities of the empirical world. The distinction between empirical and non-empirical, or à priori, geometry is constantly maintained by Russell in the works considered in this section, as is a similar one by Whitehead in different terms, usually mathematical or abstract vs. physical geometry. It is important in considering such distinctions of types to consider the question whether there is a further one between an interpretation and an application of a given geometry. Russell seems to have both distinctions, though not as clearly as would assist in the specification of *PM* geometry.

The following difference between geometry and arithmetic is remarked in Chap. II in emphasizing the hypothetical character of geometry studied in pure mathematics:

> No one before the non-Euclideans perceived that arithmetic and geometry stand on a quite different footing, the former being continuous with pure logic and independent of experience, the latter being continuous with physics and dependent upon physical data. (P. 21)

This does not seem to accord with Russell's own views, however. The statement should confine itself to empirical or physical geometry, for non-empirical geometry should be on a par with arithmetic as here described rather than geometry.

(3) While crediting Whitehead with the conception of a method specifying "points" in terms of sets of finitely extended events (p. 290), Russell (Chap. XXVIII) reworks Whitehead's method to construct "points" and "point-instants" by extending his own of "partial overlapping" in [20]. The problem of applying his method of "partial overlapping" to the "point-instant" of physics, he says, is a problem in topology. A topological treatment of space-time is offered as part of the logical analysis of then-current physics (the 1920's). An associated geometry is brought in as noted above, topic (4) of interest in this work. Russell wants to define points in terms of "events" where the latter have a one-to-one correspondence with certain neighborhoods. Only two primitive entities are needed, a point and neighborhoods of a given point (collections of points). Hausdorff's definitions of metrical space and topological space are used. The fundamental relation in the construction of points is a five-term relation of "co-punctuality" (overlapping) which holds between five events when there is a region common to all of them. A point is defined as "a co-punctual group which cannot be enlarged without ceasing to be co-punctual" (p. 299). The existence of such points may be demonstrated by assuming that all events (or at least all those co-punctual with a given co-punctual quintet) can be well-ordered. This follows if Zermelo's axiom is true. Russell credits F.P. Ramsey with helping to convince him of the truth of the well-orderedness of events (*ibid.*). Topology, including point-set topology are required, then, to demonstrate the logical structure of physics, rather than just that of geometry. The same may hold for empirical or physical geometry.

The geometry associated with the constructed topology is, he says, a pre-coordinate geometry, a non-metrical or non-metrizable geometry we may say. Here are found propositions about a configuration such as would remain true if it were subjected to any kind of continuous deformation. The kind of space–time order underlying the general theory of relativity is topological rather than geometrical, far less rigid than, say, in projective geometry. The geometry associated with this topological space-time includes points in a continuous space-time of points generated from an initial assumption of $\aleph_0$ events, by means of the relation of co-punctuality and logical inclusion. The extension of the geometry so as to include surfaces, volumes, and four-dimensional regions requires no further theoretical underpinning. Dimensions are briefly mentioned, in regard to which Poincaré's inductive definition is taken to be most suitable to Russell's purpose.

In *My Own Philosophy*, written in 1946 ([18]), Russell offers us a clear view of the outcome for the case of geometry of his study of the principles of mathematics.

It disproved Kant's view that we have à priori knowledge of space (actual space, presumably) and showed geometry, to the extent it is à priori, to be a development of arithmetic. One should begin, then, with the study of numbers, the central theory of which is Peano's system from which not only arithmetic and analysis but also all geometry, which belongs to pure mathematics, follows. Peano's three concepts and five propositions serve well as hostages for all pure mathematics, to the extent that pure mathematics depends upon the concept of number. As noted in passing in [13] (Chap. XXVIII) and emphasized in [16] (p. 238, New York ed.), certain ordered sets of real numbers can define a three-dimensional Euclidean space, e.g.; the real number system appears quite often in Russell's studies in geometry. Peano's three concepts or undefined ideas are definable by what Russell called complicated logical structures in [13], i.e., in terms definable by logic, rather than number theory. Frege is, of course, due the credit for demonstrating the logical analysis of the concept of number, at least for logicists.

Let us pursue the Peano theory of the real number system in the service of geometry. We lack the counterpart to the logical analysis of the concept of number, that is the logical analysis of the concept of point. Apparently, here a point should be interpreted as an ordered triad of reals, the totality of which form a three-dimensional Euclidean space. Such an interpretation of point suits standard Cartesian analytic geometry and at least part of contemporary algebraic geometry. However, if points have some structure other than real number-based structure, as in Russell's topology-*cum*-geometry relating to the space-time continuum of relativistic physics (see [13], Chap. XXIX), the analysis and interpretation of the concept of point will not be done with just by Peano's system. Does this mean that such additional structure must be empirical? This may be a defensible position, but there is another. Another would be to maintain that the à priori or mathematical concept of point should allow a structure of points such that any acceptable empirical point-concept could be modelled in the mathematical theory of points. If points, as numbers, are defined in logical, rather than mathematical, terms, the level of generality increases. Thus, given a basic system for geometry comparable to Peano's for arithmetic, we should be capable of determining what entities geometrical propositions are about other than points. If the Peano system for real numbers suffices as a basic system for geometry, as Russell suggests here, we only need to build from that. If so, geometry will be derivable from arithmetic. Cartan's suggestion of a reverse procedure is interesting to consider here with regard to reversible derivability of geometry and arithmetic. We would expect only partial derivability in the direction of arithmetic from geometry, but, apart from empirical geometry, perhaps complete derivability in the direction of geometry from arithmetic. Whitehead and Russell's view, that geometry consists of two entirely distinct studies, still maintained in [18], must be brought to bear on Cartan's conjecture.

In *Human Knowledge*, Russell considers empirical and non-empirical (logical) interpretations of geometry (see esp. pp. 237–9). The empirical interpretation he considers to be an unsolved problem about the exactness of mathematical assertions, one which, in 1948, had been forgotten. With the non-empirical interpretation, all Euclidean geometry is deducible from arithmetic, i.e., from the theory of real numbers. He asserts that both Euclidean geometry and every form of non-Euclidean geometry are provably applicable to every class having the same number of terms as the real numbers. Dimensionality and the question whether a geometry resulting from such an application is Euclidean or non-Euclidean are dependent upon the ordering relation selected. Only empirical convenience selects some particular ordering relation out of the infinite number available. Thus, in an empirical interpretation, the ordering relation, as well as the terms ordered, must be defined empirically. An example of such an empirical interpretation he himself provided is that based on co-punctuality in his topological theory of space–time points in [13].

A decade later, in *My Philosophical Development* ([19]), he considers his notion of structure in relation to Whitehead's work on *PM* geometry. In *PM* relation-arithmetic, when two relation-numbers are ordinally similar, they generate the same "structure". We can generalize the dyadic-relation-based notion of structure considering triadic to n-adic relations, e.g., the relation *between*. Relations P and Q, we shall say, have the same structure if their fields can be correlated so that whenever, say, x, y, z in that order have the relation P, their correlates, say, u, v, w have the relation Q, and vice versa. Russell considers structure important for empirical as well as logical reasons. Identity of logical structure implies identity of logical properties.

3. RELATED SYSTEMS

Let us consider first related systems of geometry, *viz.*, a Whitehead AS, Hilbert's AS, and an AS of Tarski for elementary geometry (non-set-theoretical Euclidean geometry) and secondly related systems in the sense of Gödel in [1] and [2].

For *PM* geometry let us take the sum of Whitehead's G_{p_2} (with D(XV) and C(XIX), its associated G_m, and G_d. Let us call this AS G_W. This system is intended to comprehend all geometry.

Probably the most influential related, though less comprehensive, system is Hilbert's, intended as a new formulation of Euclidean geometry (see [3]). Though written in ordinary German, it aimed to be rigorous and complete. The outlook is Kantian and formalist. It is the metamathematics rather than the geometry of the *Grundlagen* which has had the greatest long-term influence upon mathematics; metamathematics is less developed in both Whitehead's and Russell's treatments. Of special importance is the investigation into the consistency of geometry. Hilbert's AS, to be called G_H, consists of twenty axioms, implicit definitions of the concepts of spatial objects, basically "point", "line", and "plane", and relations, "belongs to" (i.e., "is incident with", "lies on" and even "is a point of"), "is between", "is congruent to", and "is parallel to". Theorems are deduced and derivative notions are introduced between axioms. There are twenty axioms in five groups: axioms of I. incidence (1–8), II. order (1–4), III. congruence (1–5), IV. parallels (1), and V. continuity (1–2). The AS, as in G_W, is a hypothetico-deductive system. The axioms of continuity are an axiom of measurement or Archimedean axiom using segments and an axiom of linear completeness regarding the system of points on a line.

Regarding the consistency of axiomatic Euclidean geometry, Hilbert constructed a model of the AS in the domain of arithmetic. If the consistency of arithmetic is granted, that of his AS of Euclidean geometry follows. We have a relative consistency proof, then. In 1900, he put forward an AS for the real numbers, claiming that the consistency of geometry will be decided on the basis of that of the real number system (see [26], Hilbert—1900, p. 641). Although G_H is far less comprehensive than G_W, it is intended to be extendable, apparently, to comprehend all geometry; ordinary Cartesian geometry and a model of non-Euclidean geometry figure in Hilbert's investigations into the independence of his axioms.

Alfred Tarski's AS for elementary geometry in [23] descends from G_H in its initial exclusive concern with Euclidean geometry (see [24] for the metamathematical result upon which the system in [23] is based), and in its like concern with the field of geometry. Tarski considers elementary geometry to be that part of Euclidean geometry which is statable and provable without the use of set-theoretical devices ([23], Hintikka, p. 164). The geometry is a theory with standard formalization, in the sense of Tarski, Mostowski, and Robinson (see [24]), in the first-order predicate calculus. Only points are taken as primitive entities, and two relations are basic: a ternary predicate β used to denote betweenness and a quaternary predicate δ to denote equidistance. The AS ξ_2 contains twelve individual axioms and the infinite collection of all elementary continuity axioms. Tarski investigates the representation problem, completeness, finite axiomatizability, and decidability of his system.

In *Mathematical Logic* ([11], p. 279), Quine suggested a method of reducing geometry to logic. The method would identify geometrical entities with the arithmetical counterparts with which analytic geometry identifies them. The more abstract parts of geometry would fall under the general category of arithmetical analysis. Quine regards this account as the most convenient in the matter of the application of geometry to nature (see also [12]).

Regarding related systems, we should recognize that the system of *PM* with or without geometry is now only indirectly influential. A type-theoretical treatment of geometry can be partially determined from the work of sec. 2. This is not developed here for lack of space. The characterization of the field of geometry has two primary problems: that of sub-field classification and treatment and that of kind of formalization. Whitehead and Russell, e.g., solved these differently from Hilbert. Gödel's modified *PM* system for arithmetic in [1] and [2] are instructive here.

The related systems discussed here, and other influential contemporary systems, have carried systematic work in geometry forward. A full characterization of the elements of contemporary systems of geometry suitable as a basis of any full formal account of this field of mathematics has not been forthcoming. Whether this is the result of such limitations on mathematical theorizing as uncovered by Gödel, of lack of expected utility, of lack of perceived need, or other causes is unclear.

4. CONCLUSION

The record reviewed above demonstrates the significance of geometry to the original *PM* programme. While one may have thought its significance historical only, such a judgment is too narrow. Geometry continued to thread its way through later works of both Whitehead and Russell. Clarifying their views during their *PM* collaboration sheds light on differences which evolved in their respective work. A further significance should be attached to *PM* geometry on account of the continuing lack of a widely accepted theory, or theories, of geometry as comprehensive as those of arithmetic. We may look forward with positive expectations to results providing comprehensive characterization of the whole field as well as to results providing metalogical and metamathematical specifications regarding such properties as completeness, decidability, provability, and realizability. An investigation of the problem of application or *Anwendungsproblem* for geometry should also be undertaken

in conjunction with contemporary formalizations and may be expected to contribute to a comprehensive theory of geometry.

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