Part II

Early Work in Mathematics and Logic

Bertrand Russell's *Essay on the Foundations of Geometry* and the Cambridge mathematical tradition

by Joan L. Richards

BERTRAND RUSSELL'S FIRST full-length philosophical work was *An Essay on the Foundations of Geometry*, published by the Cambridge University Press in 1897. At the time, Russell, who had graduated from Cambridge in 1894, conceived of this *Essay* as the first step in the development of a comprehensive neo-Hegelian philosophical programme encompassing both the sciences and social issues—the Tiergarten Programme[1]—and was initially quite satisfied that "problems concerning geometry had been disposed of".[2] Soon after its publication, however, Russell abandoned the Hegelian point of view within which the work was constructed. Thereafter, he consistently repudiated the *Essay* as fundamentally flawed: "Apart from details, I do not think that there is anything valid in this early book."[3] Thus, from one perspective the *Essay* stands as an historical curiosity—an isolated piece of Russell's work which led nowhere.

There is, however, another perspective one can take on the *Essay*, which takes into fuller account Russell's early development as a philosopher-mathematician and the major intellectual currents in which this development took place. From this point of view, the *Essay* represents an original contribution to a longstanding nineteenth-century discussion of geometrical foundations. Russell was a young man of twenty-five when it was published, fresh from a Cambridge education in mathematics and philosophy. In addition, his experience had been broadened by a year of study in Germany. With this in mind his *Essay* can be read as a powerful synthesis of a number of often apparently contradictory mathematical and philosophical treatments of the nature of geometrical foundations. In many respects, Russell was justified in the youthful pride he initially felt in his first publication.

It is not a completely straightforward matter to locate Russell's early geometrical

[1] For a fuller treatment of this programme, and of the place of *An Essay on the Foundations of Geometry* within it, see Nicholas Griffin, "The Tiergarten Programme", in these proceedings.

[2] Bertrand Russell, *My Philosophical Development* (London: Allen Unwin, 1959), pp. 37–8. Hereafter cited as *MPD*.

[3] *Ibid.*, p. 40.

ideas in their historical context, because few manuscripts have survived. Consequently, the task often involves relying heavily on Russell's autobiographical reconstructions which tend to be heavily prejudiced and retroactively distorted by his later negative evaluation of the project's worth. However, analyzing those materials which do survive,[4] and following the hints his memoirs contain into the contemporary surroundings, one can begin to understand the broader intellectual context in which the *Essay* was written, and to reconstruct the early development of Russell's geometrical ideas.

This endeavour leads in a variety of directions which organize what follows. The first section is an attempt to describe the broad intellectual context of Russell's Cambridge education. Although in his autobiographical writings Russell consistently denigrated this education, particularly in mathematics, it clearly had strong thematic effects on his early work. The second section focuses specifically on the philosophical tradition of non-Euclidean geometry, which set the stage for Russell's early work. This tradition was closely tied to the broad cultural mores of the Cambridge education. In addition to this philosophical tradition, Russell was increasingly influenced by a mathematical tradition which had moved away from philosophy in the final decades of the nineteenth century; this influence will be considered in the third section. Much of the impact of Russell's *Essay*, which is the focus of the fourth section, derived from his success in reconciling and synthesizing these philosophical and mathematical traditions.

Placing Russell's *Essay* in its nineteenth-century context reveals the strengths of the various traditions from which he was drawing. In particular, it places the development of mathematics in late nineteenth-century England in a somewhat different light. Looking back, Russell, like many of his contemporaries at Cambridge, had little but scorn for his education. A closer look suggests, however, that Russell's education ought not to be dismissed so cavalierly. Behind its peculiarities was a strong, well-conceived approach to the world which deeply influenced his early work.

I. THE BRITISH MATHEMATICAL TRADITION

In his retrospective writings, Russell consistently criticized the mathematical education he had received at Cambridge in the 1890's. Preparation for the first part of the Mathematical Tripos, the focus of which was primarily geometrical and applied, led him to such a dislike of the subject that despite being qualified to proceed to Part II, he divested himself of his mathematical books and devoted his fourth year to philosophy. He later remarked:

> The mathematical teaching at Cambridge when I was an undergraduate was definitely bad. Its badness was partly due to the order of merit in the Tripos, which was abolished not long afterwards. The necessity for nice discrimination between the abilities of different examinees led to an emphasis on "problems" as opposed to "bookwork". The "proofs" that were offered of mathematical theorems were an insult to the logical intelligence.... The effect of all this upon me was to make me think mathematics disgusting.[5]

In his approach to mathematical problems, Russell unambiguously rejected the Cambridge tradition early in his career. However, the epistemological assumptions which formed the underpinning for this approach to mathematics were in many ways central to his *Essay*.

Russell's criticism of his mathematical education at Cambridge is quite specific: the Cambridge mathematical education was too narrowly focused on the Tripos; there was too much emphasis on memorizing set solutions; and often the proofs given to mathematical theorems were logically inadequate. The first of these problems can be understood as a part of institutional history and growth through the century, which resulted in an increasing emphasis on the Tripos.[6] The second is arguably an epiphenomenon of the first, resulting from the attempt to render all mathematical knowledge susceptible to right and wrong answers, and hence clear marks.[7] However, the third, the failure to be strictly logical in proofs, was independent of the peculiar examination structure of the period. It reflects a carefully thought-out Cambridge tradition which treated mathematics in general, and geometry in particular, as a conceptual rather than a formal or logical study.

Within this conceptual view, geometry was an exact science—the science of space. Its definitional and axiomatic structure did not form the foundations of the subject; rather geometry exactly described the spatial subject-matter which was its object. To quote from a typical statement of this point of view:

> Euclid's definitions ... are attempts to describe, in a few words, notions which we have obtained by inspection of and abstraction from solids. A few more notions have to be added to these, principally those of the simplest line—the straight line, and of the simplest surface—the flat surface or plane. These notions we possess, but to define them accurately is difficult.[8]

It was a peculiarity of this spatial science that once the definitional and axiomatic structure was carefully laid out, one could proceed deductively to generate more and more complex theorems. Further, although logically derived, the truth of these theorems was founded as much in their exactness as conceptual descriptions of real space, as in the structure of their proof. Geometry's deductive proof structure was as much a crutch on the way to real intuitive understanding as a criterion for that understanding.

The theoretical justification behind having Russell and his classmates all take the geometrically oriented first part of the Mathematical Tripos before passing on to more modern, analytical subjects was that geometry, classically conceived, provided

[4] Most of the relevant are to be found in *The Collected Papers of Bertrand Russell*, Vol. I: *Cambridge Essays, 1888–99*, ed. K. Blackwell *et al.* (London and Boston: Allen and Unwin, 1983). These may be supplemented by additional ms. material in the Bertrand Russell Archives, McMaster University, Hamilton, Ont.

[5] Russell, *MPD*, pp. 37–8.

[6] For the development of the Tripos, see Sheldon Rothblatt, *Tradition and Change in English Liberal Education* (London: Faber and Faber, 1976). Also Arthur Denys Winstanley, *Early Victorian Cambridge* (Cambridge U. P., 1955), and Winstanley, *Later Victorian Cambridge* (Cambridge U. P., 1947).

[7] The clearest arguments along these lines can be found in Isaac Todhunter, "Elementary Geometry", in *The Conflict of Studies and Other Essays* (London: Macmillan, 1873), pp. 136–92. See also Charles L. Dodgson, *Euclid and His Modern Rivals* (London: Macmillan, 1885), pp. 1–12.

[8] [Olaus Henrici], *Encyclopaedia Britannica*, 9th ed. (1879), s.v. "Geometry".

an excellent training for the developing intellect. This was a two-pronged claim. To quote Charles Dodgson's defence of that education, classical geometry was a subject which would both "exercise the learner in habits of clear definite conception, and enable him to test the logical value of scientific argument."[9] Both conceptualization and logical argument were skills which the young scholar would then be able to apply in the whole variety of other fields to which he might later turn his attention.

In the emphasis on clear conception as the goal of geometrical study, the typical nineteenth-century view of geometry as the science of space was institutionally recognized and reinforced. Although in Britain, as elsewhere in Europe, new mathematical and intellectual currents had begun to suggest the cogency of less conceptually oriented views of geometry, these currents were self-consciously and rigidly excluded from the basic educational programme. To quote from Todhunter's defence of the conceptual, Euclidean emphasis of Cambridge mathematics, despite the recognition that other forms of mathematics were being developed:

> a youth may be advantageously trained in the rigorous methods of Euclid; and yet when in mature life he is speculating on ideal secants and circles at infinity he may be quite emancipated from his early restrictions.[10]

It was against the bounds of these "early restrictions", which emphasized clarity of conception over logical development, that Russell reacted so vehemently after the Tripos, selling all of his mathematical books and vowing never to study mathematics again.[11]

Russell's disgust at the logical inadequacy of the Cambridge conceptual view of geometry must be balanced against another aspect of that approach to which he was highly receptive. This was bound up with the peculiar nature of conceptual geometrical truth. In the middle of the century, when the basic outlines of the Tripos as Russell took it were laid down, a large part of the justification for the emphasis on geometry lay in the perception that whereas in other sciences truth was merely contingent, in geometry it was *necessary*. To quote from William Whewell, the major mid-century reformer and defender of the Tripos, "one of the most important lessons which we learn from our mathematical studies is a knowledge that there are such truths and a familiarity with their form and character."[12] Students were educated in geometry not merely to strengthen their reasoning powers but also to develop the ability to recognize necessary truths wherever they might be found.

The introduction of non-Euclidean geometry into England in the decade following the publication of Darwin's *Origin of Species* seriously threatened this position, and the British assurance that necessary truth was humanly attainable. At the same time, geometers began considering alternative mathematical systems wherein Euclid's parallel postulate did not hold, raising doubts about the necessary truth of Euclid's science. If such geometries were taken as descriptions of alternative, conceivable spaces, Euclidean statements about parallels were no longer *necessarily* true.

The implications of this kind of thinking in the mid-Victorian context were clearly

[9] Dodgson, *Euclid*, p. 7.
[10] Todhunter, "Elementary Geometry", p. 145.
[11] *MPD*, p. 38.
[12] William Whewell, *Of a Liberal Education in General* (London: Parker, 1845), p. 163.

spelled out in the addresses of William Clifford, England's most promising young mathematician of the 1870's, who was instrumental in introducing non-Euclidean geometry to the British community. Clifford maintained that with the development of non-Euclidean geometry the concept of space had become ambiguous, and geometrical truth merely contingent. Since Euclidean geometry was the only clear example of a study where real, necessary truth had been obtained, doubts about its validity meant to Clifford that there was no remaining counter-example with which to answer the limited view of knowledge and human aspirations he propounded. As a consequence of the development of non-Euclidean geometry, Clifford told his audience at the Royal Institution in 1873,

> the knowledge of Immensity and Eternity is replaced by knowledge of Here and Now. And ... the idea of the Universe, the macrocosm, the All, as subject of human knowledge, and therefore of human interest, has fallen to pieces.[13]

Clifford's conclusions were widely discussed and soon became part of the common coin of the late Victorian debates about the relation of science and religion.[13]

It is essentially in this context that Russell developed his earliest interest in geometry. Describing the adolescent thoughts he entertained more than two decades after Clifford's speech, Russell wrote:

> I discovered that, in addition to Euclidean geometry, there were various non-Euclidean varieties, and that no one knew which was right. If mathematics was doubtful, how much more doubtful ethics must be! If nothing was known, it could not be known how a virtuous life should be lived. Such thoughts troubled my adolescence, and drove me more and more towards philosophy.[15]

As Clifford presented it, the non-Euclidean challenge was clear and conclusive. In fact, however, the situation was much more complicated and the kinds of issues Clifford raised and Russell brooded over were repeatedly considered from different perspectives by philosophers in the 1870's and after.

2. THE PHILOSOPHICAL TRADITION IN NON-EUCLIDEAN GEOMETRY

Although the initial development of geometries in which Euclid's parallel postulate did not hold is traditionally traced to the work of Nicholai Lobachevskii and János Bólyai late in the 1820's, it was the subsequent work of two Germans, Bernhard Riemann and Hermann von Helmholtz, which triggered widespread discussion of the nature of geometrical truth. Late in the 1860's, these men both published papers in which they tried to analyze the basic spatial concept in order to determine which parts were logically necessary, and which parts were experientially determined. Both asserted that the properties distinguishing Euclidean from non-Euclidean spaces

[13] William Clifford, "The Postulates of the Science of Space", *Contemporary Review*, 25 (1875): 363.
[14] The broader discussion is explored in Imre Toth, "Gott und Geometrie: Eine victorianische Kontroverse", in *Evolutionstheorie und ihre Evolution*, ed. Heinrich Dieter (Schriftenreihe der Universitat Regensburg, Vol. 7).
[15] Bertrand Russell, "A Turning-Point in My Life", *The Saturday Book*, 8 (1948): 143.

were not part of the logically necessary structure of space, but were experientially determined. Thus their analyses raised the question of whether Euclidean geometry was necessarily or only contingently true.

This non-Euclidean tradition originated in Riemann's "Habilitationsvortrag" which he read to the faculty at Göttingen in 1854. Although he reportedly impressed Gauss at that time, his ideas had little impact until they were published posthumously in 1867. In this work, Riemann's stated goal was the analysis of the spatial concept. This he attempted by reducing spatial properties to a basic numerical structure, the three-dimensional manifold. He introduced the case as follows:

> I have ... set myself the task of constructing the notion of a multiply extended magnitude out of general notions of magnitude. It will follow from this that ... space is only a particular case of a triply extended magnitude.[16]

He continued to say that it must be empirical facts which lead us to choose Euclidean space from among the many alternatives which fit the most general notion of a three-dimensional magnitude.

> But hence flows as necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience. Thus arises the problem, to discover the simplest matters of fact from which the measure-relations of space may be determined.... These matters of fact are—like all matters of fact—not necessary, but only of empirical certainty; they are hypotheses.[17]

Riemann's primary concern in his analysis of manifolds was to find what elements of our geometrical construction are hypotheses and what parts are components of any manifold and hence necessary. His paper was highly abstract and difficult to interpret in the context of common notions of space, which might explain the initial lack of reaction to it in 1854.

This was not the case with the writings of Riemann's compatriot, Herman von Helmholtz, whose first geometrical papers appeared in 1868. Like Riemann, to whose work he was explicitly indebted, Helmholtz reduced space to an analytically defined three-fold manifold and tried to distinguish between the logically necessary elements of the manifold and those which were added to it when considering space. But, unlike Riemann, Helmholtz was a physiologist and primarily interested in how humans developed a concept of space. Therefore, he went beyond mathematical analysis to a more concrete consideration of what experiences could generate human spatial conceptions.

The experiences Helmholtz specified as the basis for constructing a Euclidean spatial concept were the motions of rigid bodies. The experience of moving rigid objects through space without deformation was universal enough that all infants, even blind ones, had had it. In addition, Helmholtz demonstrated that these experiences were analytically sufficient to ground all of the hypotheses Riemann had specified as necessary to distinguish space from any other numerical manifold.

[16] Bernhard Riemann, "Über die Hypothesen welche der Geometrie zu gründe liegen", *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 13 (1867): 133–4; trans. W.K. Clifford, "On the Hypotheses Which Lie at the Bases of Geometry", *Nature*, 8 (1873): 14.

[17] *Ibid.*, p. 134; Clifford, pp. 14–15.

Helmholtz used these insights as a plausibility argument to strengthen his claim that the concept of space was developed by human infants from their earliest experiences. The culmination of this line of thought appeared in two papers, published in *Mind* in 1876 and 1878. These papers were the starting-point for most English and continental philosophical discussions of geometry in the late nineteenth century.

In his first *Mind* article, Helmholtz argued that space was not an à priori intuition but rather an empirical concept generated from experience. The argument was constructed on two somewhat different levels. At the more concrete level Helmholtz argued that the ability to generate Euclidean geometrical relations from the movement of rigid bodies suggested that these experiences were the basis of the spatial concept. However he also recognized that non-Euclidean geometries could be generated equally easily from rigid body motions. This meant, Helmholtz argued, one had to take yet another empirical step to generate Euclidean geometry. It was only with the help of mechanics that one was ultimately able to fix space as Euclidean. Helmholtz concluded his argument at the philosophical level with the assertion that Euclidean geometry could not claim the status of necessary truth; epistemologically it was no different than empirical mechanics.[18]

Helmholtz's 1876 article sparked an immediate response from the Dutch philosopher, J.P.N. Land. In 1877, Land published a rebuttal based on the argument that non-Euclidean geometries were not *conceptually* equivalent to Euclidean ones. Mathematically and even perceptually describable they might be, but imaginable they were not. He argued:

> In the present case, the first question is whether any sort of space besides the space of Euclid be capable of being imagined.... [W]e are told of spherical and pseudospherical space, and non-Euclideans exert all their powers to legitimate these as space by making them imaginable. We do not find that they succeed in this, unless the notion of imaginability be stretched far beyond what Kantians and others understand by the word.[19]

Following this line of reasoning, Euclidean geometry retained its special place as the single clearly imaginable space in which experience takes place.

In his 1878 article, Helmholtz tried to meet this objection and render non-Euclidean geometries imaginable by "fully representing the sense-impressions which the object would excite in us according the known laws of our sense-organs under all conceivable conditions of observation."[20] He introduced flat beings living on various two-dimensional surfaces as a way of illustrating how experiences might be different in spaces of non-zero curvature. In addition, he used worlds behind curved circus mirrors and in gazing globes as pictures of three dimensional non-Euclidean worlds. With these descriptions Helmholtz tried to establish that non-Euclidean spaces were imaginable and therefore conceptually equivalent to Euclidean ones.

Like most late nineteenth-century philosophical treatments, Russell's early writings about non-Euclidean geometry took off directly from the Helmholtz–Land dis-

[18] Hermann von Helmholtz, "The Origin and Meaning Geometrical Axioms", *Mind*, 3 (1876): 301–21.

[19] J.P.N. Land, "Kant's Space and Modern Mathematics", *Mind*, 2 (1877): 41.

[20] Hermann von Helmholtz, "The Origin and Meaning Geometrical Axioms (II)", *Mind*, 3 (1878): 215.

cussion. The first time he directly considered the epistemological status of non-Euclidean geometry was in a paper set for him in Ward's course on metaphysics. The assignment was:

> Discuss a) meaning b) possibility of mentally representing other space-relations than those of Euclid.—Explain (look at Helmholtz) and discuss Helmholtz's distinction between geometry based on transcendental intuition and geometry based on experience. [Specific references were then made to the Helmholtz–Land papers.][21]

Russell introduced the epistemological problem as follows:

> In Euclidean geometry certain axioms are assumed (such as especially the axiom of parallels) which depend upon the nature of space and which are held to derive their validity from the impossibility of picturing a case in which they fail. This impossibility is denied by Meta-Geometry.[22]

He then considered whether Helmholtz had succeeded in enabling one to picture non-Euclidean geometries. Although he admitted that "if Helmholtz should say that he can picture them, I see no way of disproving his assertion", Russell did not accept the arguments as persuasive. The analytical ones may demonstrate the consistency of such spaces but not their imaginability, and Russell brushed aside the more concrete examples saying: "The analogy of flat-fish living on the surface of a sphere seems irrelevant, if only because we are not flat-fish, and I do not see why the line of sight of such a flat-fish should not be just as much a Euclidean straight line as ours." He concluded with the comment:

> ... I do not see how we can avoid the conclusion, that geometry, to be geometry, ... must depend ultimately on space-intuitions ..., further that such space intuitions are *for us* necessarily Euclidean; and that therefore the speculations of meta-geometry have no epistemological importance.[23]

Russell persisted in this evaluation of the epistemological significance of non-Euclidean geometries in his next treatment of the nature of space, the manuscript entitled "Observations on Space and Geometry" written in Berlin between March and 6 June 1895. He began this manuscript with a detailed criticism of the treatment of geometry given by the German metaphysicians Heinrich Lotze and Franz Erhardt, who defended Kantian views of space against the empirical claims of Helmholtz and the non-Euclidean geometers. Although he was distressed by their mathematical clumsiness, Russell basically agreed with them about the à priori conceptual nature of spatial knowledge.

In addition to his analysis and critique of Lotze and Erhardt, Russell considered the works of Riemann and Helmholtz. He was critical of both. Riemann, he acknowledged, had done a masterful job of reducing the axioms of space to their barest analytical roots. However, the philosophical problem, as Russell saw it, was

[21] Russell, *Collected Papers*, I: 124.
[22] *Ibid.*, p. 126.
[23] *Ibid.*, p. 127.

how this reduction was related to the spatial *concept*. In this context, Russell pointed out, Riemann's work is of doubtful relevance because

> we have, throughout the argument, no point of contact with actual perception.... Mathematically, Riemann's form is probably as good as any that can be imagined; but philosophically it seems to me very ill fitted to settle what space-conception we require to fit our space-perceptions; and this is the question on which turns the truth to fact of any Geometry, as opposed to mere logical self-consistency.[24]

Russell had a different problem with Helmholtz, because the physiologist had directly linked the development of Euclidean concepts with common experience. Like Land, Russell focused his remarks on Helmholtz's arguments for the conceptual equality of Euclidean and non-Euclidean spaces. Helmholtz's elaborations of the experiences one would have in non-Euclidean space, Russell argued, served merely to describe these spaces, it did not render them imaginable.

> We really derive no more knowledge of it [non-Euclidean space] from this analogy, than a man born blind may have of light; he may be perfectly well-*informed* on the subject, and may even be able, theoretically, to work out mathematical problems in Optics; but he cannot, in any ordinary sense of the work, *imagine* light—he has the *conception*, but lacks the living *intuition*.[25]

Whereas Russell criticized Riemann's work as too abstract for epistemological relevance, Helmholtz's was too concrete.

Russell's early defences of the à priori nature of spatial knowledge were thus argued largely within the confines of the issues laid out twenty years before in the Helmholtz–Land articles. The key to his position lay in his finely tuned sense of the meaning of terms like "conceive", "imagine", or "intuit". He directed his arguments towards the defence of a Kantian philosophical position, but he can also be seen as examining and reaffirming the underlying point of view which had informed the undergraduate training he had received at Cambridge.

3. THE MATHEMATICAL TRADITION OF PROJECTIVE GEOMETRY

From a mathematical perspective, what is striking about Russell's early treatments of non-Euclidean geometry is how narrowly he confined his focus to the issues raised by Riemann and Helmholtz. This constancy of focus was maintained even though in the historical section of his Berlin "Observations" he emphasized that after the early 1870's the metrical ideas which underlay Riemann's and Helmholtz's analysis of space had been largely abandoned in favour of a projective approach. This change resulted largely from the successes of the German mathematician, Felix Klein, who

[24] Bertrand Russell, "Observations on Space and Geometry" (ms. notebook dated Berlin, June 1895), pp. 34–5 (65–8). The bulk of this notebook is printed under this title in *Cambridge Essays*, pp. 258ff. The two sections that are not printed are early drafts of chapters that appeared in Russell's *Essay*. All quotations are taken from the Russell Archives typescript of those sections. Page nos. refer to the typed pages; the ms. page nos. are given in parentheses.
[25] *Ibid.*, p. 43 (85).

developed projective geometry to the point where it had come to dominate geometrical thinking in the final quarter of the nineteenth century.

In 1895, Russell characterized this fundamental shift in mathematical emphasis as follows:

> The fundamental distinction, between this [projective period] and the preceding [metric] period, is one of method alone—whereas Riemann and Helmholtz dealt with metrical ideas, and took as their foundations the measure of curvature and the formula for the linear element (both purely metrical), the new method takes as its foundations the formulae for transformation of coordinates, which are required to express any given motion, and begins by reducing all so called metrical notions (distance, angle, etc.) to projective forms. The treatment derives, from this reduction, a methodological unity and simplicity before impossible.[26]

Here Russell presents projective geometry as an elegant and powerful analytical method to solve geometrical problems. As such it was mathematically very attractive.

Philosophically, however, it was virtually meaningless, a point Russell emphasized in the sentence preceding the above quotation. "Of this [projective] period," he wrote, "I can only treat with great brevity, as there are few fresh philosophical ideas involved; the advance is mainly technical, and impossible to render in non-mathematical terms."[27] By interpreting projective geometry as a method, Russell distanced it from the fundamental spatial concept which formed the object of his philosophical interest. He was not completely comfortable with this dismissal, however. "Nevertheless," he continued, "I will give a brief sketch of the two most important advances made by it, namely [Arthur] Cayley's projective Geometry and [Sophus] Lie's treatment of motions; ... at the same time I will endeavour to point out certain philosophical difficulties in the former."[28]

Thus Russell's presentation of projective geometry in the "Observations" begins ambivalently. Although he emphasized that it held little philosophical interest, he continued to consider the subject in the next fifteen pages of manuscript. Philosophically assessing projective geometry was clearly a perplexing task. A brief historical review of its nineteenth-century development will help to clarify the nature of Russell's difficulty.

From its inception in France at the turn of the century, projective geometry was closely tied to foundational questions. Blithe acceptance of the usefulness of such troublesome mathematical entities as $\sqrt{-1}$ had allowed eighteenth-century analysts to generate enormously powerful results. However, in the early nineteenth century it was clear that there was still no sound foundational approach which could justify their reality and hence their use. In the 1820's, the newly developed discipline of projective geometry was pursued largely because it was seen as a way to geometrically interpret the eighteenth-century legacy of puzzling analytical results. A significant group of French mathematicians, who subscribed to a conceptual view of mathematical truth similar to that of late nineteenth-century Cambridge, believed

[26] *Ibid.*, p. 50 (103).
[27] *Ibid.*
[28] *Ibid.*, p. 51 (104).

projective geometry provided answers to the foundational difficulties posed by analytical development.

This belief was based on the fact that projective geometry, which was the study of those aspects of figures which remained invariant through transformations of projection and section, both allowed rigorously conceptual geometry to achieve the same generality which had been so powerful in analysis, and provided geometrical interpretations for mysterious analytical entities like $\sqrt{-1}$.

These two features can best be seen through an example. The fundamental "principle of continuity", first explicitly stated by Jean Victor Poncelet in 1822, asserted that theorems proved for a geometric figure were true for correlative figures formed from the original by a continuous transformation. Thus, Poncelet's principle asserts that figures 1, 2, and 3 are essentially equivalent since they are formed from each other by the continuous motion of sliding circles C_1 and C_2 apart.

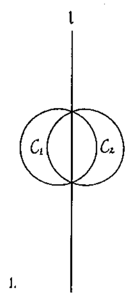

Figure 1.

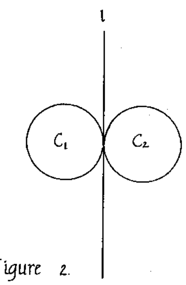

Figure 2.

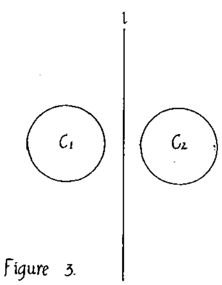

Figure 3.

While these figures would be treated as significantly different in Euclidean geometry,[29] the projective geometers emphasized their constancy under transformation. This allowed them to see general results which had been obscured by Euclidean specificity.

One such general result can be illustrated by looking at the radical axes of C_1 and C_2, line l in figures 1–3. This line has a number of interesting properties, some of which, like the Archimedian theorems on the shoemaker's knife (figure 4), had been recognized in Euclidean geometry through the inspection of particular cases.

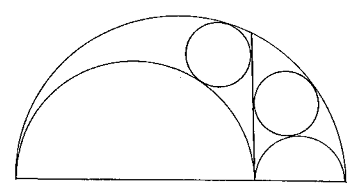

Figure 4.

However, in the general case, it has the more important property that the tangents

[29] So, for example, in Book III Euclid has two propositions (10 and 13) to treat the points of intersection of circles which cut each other and those which are tangent. From a projective standpoint, these two cases are the same; the apparent difference arises because when the circles are tangent, the two points of intersection are coincident and thus appear as one.

from *any* point on the radical axis to the two circles will be equal, illustrated in figure 5.

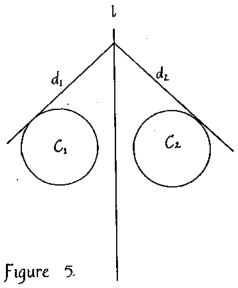

Figure 5.

Thus the projective approach greatly facilitated immediate recognition of geometric properties by examining classes of Euclidean cases that were the product of a continuous transformation of these cases.

Moreover, these generalizations provided interpretations for previously meaningless mathematical symbols like $\sqrt{-1}$. An instance of this can again be illustrated by considering the radical axis. In figure 1, this line connects the two points of intersection of the circles. In figure 2, these two points have slid together and are coincident. Here, the radical axis is the common tangent of the two circles. In figure 3, one can still define a radical axis which has the same properties as those of figures 1 and 2. However, here the points of circular intersection are nowhere to be found in the real plane.

In this case, the early projective geometers said that the points of intersection became "imaginary". This term not only suggested their disappearance from the real plane but linked the geometrical situation with the analytical one. This connection can be seen by recalling that the analytical equations of circles C_1 and C_2 are:

$$C_1: \quad (x-a_1)^2+(y-b_1)^2-r_1^2=0$$
$$C_2: \quad (x-a_2)^2+(y-b_2)^2-r_2^2=0.$$

The equation for a line passing through the points of intersection of C_1 and C_2 is of the form $C_1-C_2=0$, or, expanding:

$$2(a_1-a_2)x+2(b_1-b_2)y = r_1^2-r_2^2 = a_1^2-a_2^2+b_1^2-b_2^2=0.$$

Finding the points of intersection requires solving a second-degree equation. Depending on whether the solution to this equation is real, zero, or imaginary, its solution can be represented by the geometric cases in which two circles intersect, are tangent, or do not intersect. In this way the projective method generated unambiguous meaning for the imaginary numbers, and a foundation that had had previously been lacking.

Despite this kind of success at increasing geometrical generality and generating geometrical analogues for previously uninterpretable algebraic symbols, the study of projective geometry did not thrive long in France. The interpretation of mathematics that supported it, in which rigour is based on conceptual interpretation was

superceded by the more formal approach of Augustin Cauchy by the end of the 1820's.[30]

In England, on the other hand, Cauchy's formal view of mathematical foundations was not widely accepted, particularly after the middle of the century when geometry was generally reaffirmed as the backbone of Cambridge mathematical education. This movement represented a re-emphasis on the conceptual approach to mathematical foundations. It was in this environment that projective geometry took root in England in the second half of the century.

The British projective tradition was greatly strengthened by the work of Arthur Cayley, whose aged form overshadowed Cambridge mathematics when Russell was a student there. In 1859, Cayley published his "Sixth Memoir on Quantics" in which he presented a projective geometrical interpretation of algebraic results he had published in five previous memoirs. He also presented his development of a projective definition of distance. This was a critically important contribution because by emphasizing the notion of equality under transformation, projective geometers had sacrificed the notion of distance. Since distance was an integral part of spatial reality in the Euclidean system, this was an intuitive weakness in the understanding one could attain through projective geometry.

Cayley's approach to the definition of distance within projective geometry centred on the "cross-ratio". Since a line segment of any length could easily be projected onto one of any other, earlier projective geometers held that distance had no projective reality. This conclusion was tempered somewhat by one apparently metrical relationship which was preserved under projection, the cross-ratio, which appears in figure 6.

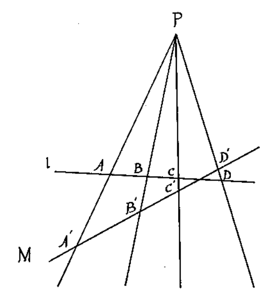

Figure 6.

It had early been recognized that this relationship among *four* collinear points was invariant through all the projective transformations.

Cayley capitalized on this invariance in his attempt to projectively define distance. He first designated a particular conic in the space as the "Absolute". Any line in the space would intersect this conic in two points, the points at infinity. The distance between any other points on the line was defined as a function of the cross-ratio of the two points and the points at infinity.[31] This value was projectively invariant and

[30] For a detailed treatment of the development of projective geometry in France, see Lorraine J. Daston, "The Physicalist Tradition in Early Nineteenth Century French Geometry", *Studies in the History and Philosophy of Science*, 17 (1986): 269–95.

[31] For Cayley it was an arc-cosine function. Klein later focused on a more convenient logarithmic function.

Cayley carefully established that it exhibited the same algebraic properties as the common Euclidean distance function. With this success, he claimed to have found the roots of classical geometry, including its metric, in projective geometry. He confidently concluded his "Sixth Memoir" with the statement: "Metrical geometry is thus a part of descriptive geometry, and descriptive geometry is *all* geometry."[32]

Cayley developed his ideas before he knew about non-Euclidean geometry, but they were easily extendable. In 1871, Felix Klein showed that non-Euclidean geometries could also be generated from projective space by changing the forms of Cayley's Absolute. With this insight, the German provided a powerful unifying framework for the study of all forms of geometry.

Klein's success at interpreting not only Euclidean but also non-Euclidean spaces through his definition of distance was compelling enough to rekindle Continental interest in projective geometry, as well as to contribute to the British tradition. These two groups developed highly divergent interpretations of its *meaning*, however. On the Continent, where the formal view of mathematics was strong, the foundations of projective geometry were increasingly seen to lie in the abstract theory of "throws" developed by Karl von Staudt in his *Geometrie der Lage* of 1847. In Britain, on the other hand, the emphasis remained on conceptual clarity.

Russell renounced his mathematical education at Cambridge before taking the second part of the Tripos which would have included projective geometry.[33] His introduction to the subject seems to have been provided by Klein's *Vorlesungen über Nicht Euclidische Geometrie*, which he praised as "one of the very best textbooks I have ever come across."[34] The view of projective geometry he gleaned from this source was the Continental one, axiomatically grounded in Karl von Staudt's theory of throws.[35] Although Russell was clearly fascinated by the subject, the formally based development of projective geometry he found on the Continent had little relevance to his philosophical interest in the à priori bases of the spatial concept.

It is telling that the one place Russell did find philosophical issues involved in projective geometry was in the work of Cayley. Although he treated it only briefly in his Berlin-based "Observations", there was a significant British tradition focusing on the conceptual bases of projective geometry which placed the subject clearly in the area of Russell's philosophical concerns. This tradition grew out of Cayley's and Klein's success at generating metrical geometries, both Euclidean and non-Euclidean, from projective space.

In Britain, as on the Continent, Klein's work strengthened the conviction that projective geometry was *all* geometry. Since, for the British, any legitimate geometrical statement was about space and grounded in a real interpretation, projective geometry began to be seen as a direct description of space itself. This tradition moved away from the view—reflected in Russell's opening paragraph (see quotation

[32] Arthur Cayley, "A Sixth Memoir on Quantics", *Philosophical Transactions of the Royal Society of London*, 149 (1859): 90. (Cayley used the term "descriptive" where we now use "projective".)

[33] For a detailed treatment of the structure of the Tripos when Russell took it, see W.H. Besant, "The Mathematical Tripos", *The Student's Guide to the University of Cambridge*, 5th ed. (Cambridge: Deighton, Bell, 1893).

[34] Russell, "Observations", p. 57 (121).

[35] Karl Georg Christian von Staudt, *Geometrie der Lage* (Nürnberg: Bauer und Raspe, 1847). This is the point of view taken by Felix Klein in *Vorlesungen über Nicht-Euklidische Geometrie* (Göttingen: 1892), Chaps. I and II. Russell explicitly refers to these chapters in his "Observations", p. 106.

of fn. 5)—that projective geometry was merely a different *method* by which one could approach classical Euclidean space, and treated "projective space" as a reality.

This attitude became increasingly articulate and British mathematicians like H.J.S. Smith began to characterize projective geometry as "the direct contemplation of space".[36] Not only did this buttress a considerable interest in projective geometry, it also provided a mathematical solution to Riemann's and Helmholtz's radical attack on the necessary truth of geometry which had begun to be felt by 1870. If one transferred the claims of necessary truth from Euclidean to projective space, the arguments of Helmholtz and Riemann became irrelevant. Whatever the basis for choosing Euclidean or non-Euclidean spaces, the ultimate conceptual validity of geometry transcended it by lying in projective space. This argument, based on a conceptual view of mathematical foundations, allowed Britain's mathematical community to transcend the whole philosophical debate stemming from the Helmholtz–Land articles.

It was not a completely satisfactory solution, however. There were at least two closely related tensions generated within the conceptual interpretation of projective space. One concerned the conceivability of the imaginary points and points at infinity. This was closely related to the conceivability of distance in the Cayley–Klein definition. The other concerned the possible circularity of the arguments by which distance was introduced into projective space to begin with. Both problems were considered repeatedly by British mathematicians in philosophical moments during the 1870's.[37]

A major problem was clearly articulated in Arthur Cayley's 1883 Presidential Address to the British Association for the Advancement of Science. There Cayley pointed out the serious philosophical problems with the imaginary points in projective space. When projective geometry was considered only as a *method* by which one approached Euclidean space, these points could be interpreted as indications of particular circumstances in that space—for example, that two circles did not intersect. But when projective geometry was considered as a description of *conceivable* projective *space*, the difficulty of clearly conceiving these points presented a pressing philosophical challenge.

The problem of conceiving imaginary points in projective space had obvious implications for conceiving distance in that space since the definition of distance included these points. In his "Sixth Memoir" Cayley had claimed to establish the notion of distance on "purely descriptive [projective] principles".[38] He took pains to demonstrate that the definition he had developed displayed the same *algebraic* properties as the ordinary definition of distance. It was not immediately obvious, however, how his definition related to the intuitive notion of distance.

Within the conceptual tradition, however, this was the requirement that had to be met. In concluding his "Sixth Memoir" with the claim that projective geometry "was *all* geometry", Cayley was asserting the real, conceptual primacy of his projective notion of distance. This claim included the somewhat peculiar implication

[36] H.J. Stephen Smith, "On the Present State and Prospects of Some Branches of Pure Mathematics", *Proceedings of the London Mathematical Society*, 8 (1876–77): 8.

[37] For a detailed treatment of the development of projective geometry in England, see Joan L. Richards, "Projective Geometry and Mathematical Progress in Mid-Victorian Britain", *Studies in the History and Philosophy of Science*, 17 (1968): 297–325.

[38] Arthur Cayley, "Sixth Memoir" (1859), p. 561.

that distance was a relationship among four points rather than two: "the theory in effect, is that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connexion with another figure, viz. the conic termed the Absolute."[39]

A tendency to view this approach with reservations can be illustrated by an article entitled "The Non-Euclidean Geometry", written by the Irishman, Robert Ball. In it he presented the Cayley–Klein interpretation of metrical geometry as follows: "To take the first step in the exposition of this theory, it is necessary to replace our ordinary conception of distance, or rather of *the mode in which distance is measured* by a more general conception." He then introduced the fundamental quadric—Cayley's Absolute—and the logarithmic definition of distance. "It cannot be denied", he continued, "that there *appears* to be something arbitrary in this definition when read for the first time. But as the reader proceeds he will find that it is at all events *plausible*, even though he may not go so far as to agree with those who consider that any other conception of distance is imperfect."[40]

As these passages suggest, Ball was not just engaged in assessing the mathematical usefulness of the Cayley–Klein definition of distance, but was also struggling with its conceptual plausibility. His efforts are indicative of the difficulties involved in reconciling intuitive spatial ideas with those which were being generated mathematically in projective geometry.

The Cayley–Klein definition of distance and its attendant Absolute, imaginary points and points at infinity, were not only difficult to grasp conceptually. When carefully examined, the very introduction of this metric into essentially non-metric projective space seemed suspect. The process seemed circular. While the distance between two points was defined as a function of their cross-ratio with points on the Absolute, this ratio itself seemed to demand a prior notion of distance. Projective space seemed to come from metric space only to move back into it with the more complicated definition of distance.

When the "Sixth Memoir" was republished in his *Mathematical Papers*, Cayley included a "Note" examining this issue.[41] He briefly pointed to a number of different perspectives one could take on it, but remained noncommittal. One of the approaches he mentioned, without comment, was that taken by Ball in an article entitled "On the Theory of the Content". In this 1887 paper, Ball abruptly abandoned the conceptual approach to projective geometry he had struggled with in 1879. In the later paper, he attempted to deal with the problem of circularity by developing a formal theory of the "Content", thus unabashedly splitting geometrical forms from their spatial meanings. He generated a series of definitions and axioms describing this content which included "objects", their "ranges", their "extents" and the "intervene" as abstract entities. Having introduced this system and its basic axioms, he proceeded to match the elements of his content with those of space—points with objects, lines with ranges, planes with extents, and distance with the intervene.

With this theory, Ball hoped to entirely remove the discussion of distance from the conceptual sphere. His intervene was defined wholly by axioms and other

[39] *Ibid.*, p. 90.
[40] Robert S. Ball, "The Non-Euclidean Geometry", *Hermathena*, 3 (1877–79): 502.
[41] Arthur Cayley, *The Collected Mathematical Papers* (Cambridge: at the U. P., 1889), II: 605.

abstractly defined elements of the system, and thus carried no intuitive baggage.

> The theory of the content is no doubt equivalent to ... the theory of geometry of elliptic space. I have indeed, myself, frequently used these latter expressions, but am glad to take this opportunity of renouncing them. I have found the associations they suggest very misleading; and, in fact, it was only by employing some such conceptions and terminology as that used in the present Paper that I succeeded in understanding the subject. *It is especially needful to avoid confusing the sign with the thing signified.* The space-points are only the signs of the objects—they are not the objects themselves—and, if we are not careful to preserve this distinction, obscurity will arise about the Intervene.[42]

Ball's formal solution to the problem of circularity placed him far outside of the conceptual tradition of British geometry. Perhaps because it was so radical, the paper does not seem to have had an immediate effect on the mainstream of British mathematical thought. Cayley mentioned it in his "Note", but only to say that "in it the same problem is addressed."[43] British mathematicians continued to be committed to conceivability in projective geometry, trusting that future insight would resolve the apparent difficulties.

Russell first addressed these issues in his "Observations", which reveal his Continental introduction to projective geometry. There, he cavalierly brushed aside the conceptual problem posed by imaginary points. Their central role in Cayley's projective theory of distance served merely to emphasize to him the irrelevance of that theory to the essential spatial concept. In his words: "... for philosophical purposes the reduction [of metrical to projective space through the Cayley–Klein theory of distance] is irrelevant, since it depends, usually, upon imaginary points and figures ... which have no reality apart from symbols."[44] He found formal interpretations of its meaning equally irrelevant. Russell was particularly impressed by Ball's non-intuitive development of the "Content" which Russell acknowledged allowed Ball to clear up the nagging problem associated with the apparent circularity of distance in non-Euclidean space. "But," he continued, "with the confusion, the supposed Euclidean interpretation also disappears; Sir R. Ball's Content, if it is to be in any sense a *space*, must be a space radically different from Euclid's."[45] Russell's position was that, while formal mathematical developments and models could be invaluable in clearing up logical difficulties in mathematical arguments, they had no bearing on philosophical issues concerned with the spatial concept.

Clearly, in 1895 Russell did not see any hope for conceivability in projective geometry. Instead, his philosophical treatment of space remained rooted in metrical conceptions. Thus, in the Berlin "Observations", Russell defended the position that space was known à priori. His defence is couched in terms almost entirely defined by the Helmholtz–Land articles.

The 1897 *Essay*, on the other hand, reveals a dramatic shift in this defence. Here Russell defended the same philosophical position from a radically different stand-

[42] Sir Robert Stawell Ball, "On the Theory of the Content", *The Transactions of the Royal Irish Academy*, 29 (1887–92): 151.
[43] Cayley, *Collected Papers*, II: 605.
[44] Russell, "Observations", pp. 51–2 (104).
[45] *Ibid.*, p. 53 (110–11).

point, by placing projective geometry in a central position. Projective space became an integral part of the à priori spatial concept. The *Essay* brings the essential thesis of the "Observations" more closely into line with developments in the conceptual British mathematical tradition that he had essentially ignored in his earlier work.

4. *An Essay on the Foundations of Geometry*

There are few surviving manuscripts relevant to the development of Russell's spatial ideas in the two years separating the "Observations" from the published *Essay*. In August of 1895, he submitted a fellowship dissertation on the subject which has now been lost. He later reported that Whitehead, the mathematical reader, "criticized it rather severely, though quite justly, and I came to the conclusion that it was worthless."[46] Nonetheless, he was awarded the fellowship, and did not abandon the subject. By all accounts, *An Essay on the Foundations of Geometry*, in which Russell attempted to defend a Kantian view of space by specifying its à priori structure, is a revised version of the fellowship dissertation. Without the dissertation itself, or a fuller description of the objections raised to it, Russell's development can only be traced through a comparison of the *Essay* with the "Observations".

The first chapter of Russell's *Essay* is historical and borrows heavily from the "Observations". There are, however, some significant departures from the earlier work, particularly with respect to projective geometry. In the 1897 work rather than asserting its philosophical irrelevance, Russell stated:

> The third [projective] period differs radically alike in its methods and aims, and in the underlying philosophical ideas, from the period which it replaced. Whereas everything, in the second [metric] period, turned on measurement ... these vanish completely in the third period, which, swinging to the opposite extreme, regards quantity as a perfectly irrelevant category in Geometry.... The ideas of this period, unfortunately, have found no exponent so philosophical as Riemann or Helmholtz, but have been set forth only by technical mathematicians.[47]

Russell's *Essay* can be seen as an attempt to provide the philosophical treatment of projective geometry which had been lacking so long.

In this, which Russell effected at a high technical level, he can be seen as bringing fruit out of a fertile combination of his mathematical and philosophical knowledge and acumen. The philosophical details of his construction are grounded in an impressive understanding of both projective and metric geometries as his contemporaries were pursuing them. He can equally well be seen as consolidating and reinterpreting the informal philosophical orientation from which British mathematicians had been pursuing projective ideas since the 1860's. The basic structure of his argument, in which projective space forms the à priori substructure for Euclidean space, reflects in a somewhat distorted form the British perspective which had emerged in the 1870's.

In the *Essay*, Russell's defence of the à priority of the spatial concept began with the recognition that projective geometry involved the development of fundamental

[46] Russell, *The Autobiography of Bertrand Russell*, Vol. I (Boston: Atlantic—Little, Brown, 1967): 186.
[47] Russell, *An Essay on the Foundations of Geometry* (Cambridge: at the U. P., 1897), pp. 27–8.

spatial ideas. In this sense he moved from his earlier view that the study merely represented a new method with which to approach classical Euclidean space to the more British view identifying the geometry with its own space. He adopted the same basic stance wherein projective geometry was "the direct contemplation of space".

In doing this, however, Russell was less naively concrete than many British mathematicians who had preceded him. He did not insist that each projective entity have a real spatial interpretation, but rather focused on the qualitative conceptions lying behind the form of projective space. Therefore, Russell's late century defence of the particularly British view that projective geometry described a conceptually real projective space entailed radically attacking some of the basic points on which it rested. In particular it entailed dismissing imaginary points as mere analytic artifacts. More fundamentally, it required not identifying but strictly *separating* the notions of distance as they appeared in projective and metric geometries in order that they not be considered as arbitrarily interchangeable.

Russell's consideration of projective *space* led him directly into the question of the reality of the portions of that space which had hitherto been so problematic, notably the imaginary points. Whereas in the "Observations" he had considered the reality of these points only to substantiate the philosophical irrelevance of projective geometry, in the *Essay*—where he was taking projective space seriously—he had to consider their ontological status directly. Using the following analogy, he concluded that they were meaningless analytical artifacts: "As well might a postman assume that, because every house in a street is uniquely determined by its number, therefore there must be a house for every number."[48]

Russell's approach to the problem of distance involved an even more radical break with British tradition. The issue which re-focused Russell's attention seems to have been conventionalism. This view was being championed by Henri Poincaré, a French polymath equally conversant with mathematical and philosophical ideas. Early in the final decade of the century, Poincaré began publishing a series of articles which eventually formed the core of his geometrical discussion in *Science and Hypothesis*. An early one of these, translated and published in *Nature* in 1892, seems to have alerted Russell to serious problems with the dismissive treatment of projective geometry he had given in the "Observations".

In the *Essay*, Russell presented the conventionalist position as follows: "'What ought one to think', he [Poincaré] says, 'of this question: Is the Euclidean Geometry true? The question is nonsense.' Geometrical axioms, according to him, are mere conventions: they are 'definitions in disguise'."[49]

Poincaré had argued this point of view in the context of metrical geometry. He had interpreted the various consistency models which allowed one to treat non-Euclidean geometries in Euclidean terms as "translations". When viewed in this manner, the differences among Euclidean and non-Euclidean geometries had no more intrinsic meaning than did differences among languages; as languages expressed a common meaning, the various geometries merely represented different conventional systems for dealing pragmatically with a common spatial reality. The implication of this position was that there were no essential differences among geo-

[48] *Ibid*., pp. 44–5.
[49] *Ibid*. Russell's quotations are from Poincaré, "Non-Euclidean Geometry", trans. W.J.L., *Nature*, 45 (1892): 407.

metrical systems, and consequently, no geometrical system could capture a better or more true conception of space than any other.

Russell's rejection of conventionalism had been foreshadowed in his "Observations". There, he briefly ascribed a conventionalist view to Klein who, in Russell's words, "tends to regard the whole controversy [about the truth of geometrical systems] as a mere question as to the definition of distance, not as to the nature of space—for by mere alterations in this definition we can, from a Euclidean plane, derive all these different systems."[50] Russell countered this claim by denying that the "space" in which these definitions were being introduced was truly a space at all—being analytically defined it had no legitimate spatial referent.[51]

In the *Essay*, where he was treating projective space as the fundamental underpinning of the spatial concept, however, Russell had to develop a different response. The projective treatment of non-Euclidean geometry Cayley had developed seemed highly vulnerable to conventionalist interpretation. If the various metrical geometries could be obtained merely by varying arbitrary definitions of distance in à priori projective space, the choice among them could easily be seen as meaningless and conventional. This possibility simply had not occurred to Cayley who staunchly maintained the à priori truth of Euclidean geometry throughout his mathematical career.

Russell met this challenge by assigning apriority to distance. He argued that distance was a relationship between two points, not among four, and that the whole study of metrical geometry was intricately intertwined with the development of this basic idea. Projective geometry, on the other hand, studied the qualitative features of space without reference to the quantitative notion of distance at all. It was possible to work with coordinates in this system, defining them wholly qualitatively or projectively, as von Staudt had. Since projective coordinates were strictly qualitative they bore no relation to real distance. Although functions could be defined on them which had the mathematical properties of quantitative distance, these could never be properly "distance" functions.

The feature supporting this radical separation lay in the fact that projective distance was a relationship among four points rather than between two. Cayley and those who followed him had interpreted this to indicate the inadequacy of the two-point relationship; Russell, on the other hand, firmly defended the two-point relationship and instead denied the legitimacy of identifying it with its four-point projective counterpart. He argued that by naming functions "distance" because they had algebraic properties *like* those of metrical distance, projective geometers had created great confusions for themselves. Their apparently metrical relations were not true distance, but only "conventional symbols for purely qualitative spatial relations". To quote from Russell's explanation of this view:

> Distance in the ordinary sense is ... that quantitative relation, between two points on a line, by which their difference from other points can be defined. The projective definition, however, being unable to distinguish a collection of less than four points from any other on the same straight line, makes distance depend on two other points besides those whose relation it defines. No name remains, therefore, for distance in the ordinary sense, and many projective Geometers, having abolished the name, believe the thing to be abolished also, and are inclined to deny that *two* points have a unique relation at all. This confusion, in projective Geometry, shows the importance of a name, and should make us chary of allowing new meanings to obscure one of the fundamental properties of space.[52]

[50] Russell, "Observations", p. 52 (107).

[51] *Ibid.*, p. 53 (110–11).

In this argument, the connection between qualitative projective space and quantitative metrical space suggested by the Cayley–Klein theory of distance was only apparent because the distance between two points was a fundamental metrical property only palely imitated by the analytic manipulation of cross-ratios. In the "Observations" this line of reasoning had led Russell to reject the *spatial* relevance of projective geometry; in the *Essay*, where he took the space that geometry defined as à priori, he used it to argue that "distance" could not be properly defined in that space. Thus, in the *Essay*, Russell renegotiated the mathematical and philosophical matrix of the 1890's in order to defend the basic view which had been a mainstay of British work for so long that geometry including projective geometry, was a conceptual study.

It was a somewhat strained defence, depending as it did on separating much of what passed as geometry from its spatial referent while leaving other parts strictly connected. The whole notion of qualitative projective space was kept as real and judged to represent a significant conceptual advance over what had gone before. On the other hand, the imaginary points and "distance" of projective geometry became mere analytical artifacts or accidents of language. Some of geometry remained the necessary study of space, but much of it was merely technical analysis, developed separately from space or any other independent reality.

Convinced that geometry was ultimately grounded in spatial concepts, Russell did not see that the kind of separation between mathematical and conceptual developments he was proposing might pull the two entirely apart. Yet this was to be the particular fate of Russell himself, and of British mathematics in general. The places where Russell broke from the British tradition that identified geometrical signs with spatial referents anticipate of the radical break into logicism he was to make a few years later.

5. THE FATE OF RUSSELL'S *Essay*

The development of the geometrical ideas which appeared in the *Essay on the Foundations of Geometry* took place in several intellectual traditions. The earliest and, throughout, the strongest influence was the British tradition of conceptual mathematics. This tradition both informed the intellectual culture in which Russell located his first interest in the epistemological status of non-Euclidean geometry, and shaped his formal mathematical education as a Cambridge undergraduate. Although the inadequacy of that education is a persistent theme in his retrospective evaluations of his intellectual development, his work continued to bear the mark of its philosophical underpinning.

In the year after he finished Cambridge, Russell enthusiastically worked in a second tradition, German geometry. Despite his consistent praise for German as opposed to British mathematics, in his philosophical works written during this period the themes of his British training emerge and temper his evaluations of the

[52] Russell, *Essay*, p. 36.

continental tradition. Upon his return to Britain to revise his fellowship dissertation into a book, these themes become even more pronounced. Much of the strength of the *Essay* derived from his understanding of his British mathematical heritage.

This point was clearly recognized by Russell's most enthusiastic French reviewer, Louis Couturat. In lauding Russell's work, Couturat did not allow his praise to fall wholly on Russell as an individual. He placed the credit largely on the British education Russell had received.

> Qu'un tel esprit ne se soit pas rencontré en France, il est permis de le regretter, mais non de s'en étonner: la faute en est, non aux hommes, mais aux institutions, à cet absurde système de la bifurcation qui continue à régner dans l'organisation de nos études, et à la déplorable scission qui en résulte entre la Philosophie et les connaissances scientifiques, qui en sont l'aliment nécessaire. C'est donc à un Anglais qu'était réservé l'honneur de résumer et de tirer au clair les décourvertes et les progrès de la Géométrie moderne [projective geometry], et d'en profiter la Théorie de la connaissance.[53]

Unfortunately, although in correspondence Couturat specifically asked Russell to comment on his British education, we do not have Russell's reply.[54]

In later life, however, Russell would certainly have objected, since that tradition was the one most supportive of the view that ultimately geometry was the conceptual study of space. This entire orientation was so far from his later view that in *My Philosophical Development* he wrote:

> There is one major division in my philosophical work: in the years 1899–1900 I adopted the philosophy of logical atomism and the technique of Peano in mathematical logic. This was so great a revolution as to make my previous work, except such as was purely mathematical, irrelevant to everything that I did later. The change in these years was a revolution; subsequent changes have been the nature of an evolution.[55]

Russell's intellectual revolution was not merely private. The radical change in his point of view roughly coincided with the demise of the Cambridge system which had nurtured it. Within a decade of the publication of the *Essay* the stranglehold of the Tripos was broken. With it went many other aspects of the education Russell had received, including the emphasis on conceptual mathematics as an exemplar of knowledge and the key to a strong liberal education.

These alterations in the intellectual landscape destroyed the setting for Russell's *Essay*, leaving it an anachronistic period piece. When viewed from the nineteenth century to which it belongs, however, Russell's first work on geometry stands as a strong and original contribution to a long mathematical and philosophical tradition. Many of its deepest insights represent the culmination of fifty years of British geometrical thinking.

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[53] Louis Couturat, "Essai sur les fondements de la géométrie par Bertrand Russell", *Revue de métaphysique et de morale*, 6 (1898): 354–5.
[54] Couturat to Russell, 3 Oct. 1897, in Russell Archives, McMaster University, Hamilton, Ont.
[55] Russell, *MPD*, p. 11.