Russell's zigzag path to the ramified theory of types

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THE RAMIFIED THEORY of types was five years in the making. A curious aspect of the historical evolution of the theory is that Russell began and ended with a theory of types. In this paper I give a survey of the foundational schemes which Russell considered and rejected in the five years during which his ideas on logic were in a continual state of flux. One of the themes that I wish to stress is the reluctance with which Russell adopted type distinctions. I shall go into this question in some detail after discussing the first version of type theory, which is sketched in Appendix B of *The Principles of Mathematics*.

1. The type theory of The Principles of Mathematics

The concept of logical type is introduced in Chapter x of the *Principles* (§104), and the idea is explored in some detail in Appendix B. The remarks made in Chapter x are not always consistent with the material of the appendix.

Russell begins by introducing the concepts of *range* and *type*. Every propositional function $\phi(x)$ is said to have a *range of significance*, that is, "a range within which x must lie if $\phi(x)$ is to be a proposition at all, whether true or false" (*Principles*, p. 523). Furthermore, objects are divided into *types*, for which it is postulated that if x belongs to the range of significance of $\phi(x)$, then the type of x is contained in the range of significance of $\phi(x)$. Russell then introduces the familiar hierarchy of types: individuals, classes of individuals, classes of classes of individuals, relations between individuals ... and so on.

So far, the theory looks very much like the simple theory of types as formulated by Ramsey, Chwistek, Tarski and Gödel. However, there are crucial differences. First, all ranges form a type, and in addition all objects form a type (p. 525); neither of these is true in the simple theory of types. Secondly, and more important, numbers form a type distinct from the hierarchy described above, and so do propositions. Russell considers imposing a type hierarchy on propositions, but rejects this suggestion as"harsh and highly artificial" (p. 528). As a direct consequence of the

type-free theory of propositions, the system is contradictory, as we can apply the Cantor diagonal argument to generate a paradox (p. 527).

The main inconsistency between Chapter x and Appendix B lies in the fact that in Chapter x Russell states that propositions of the form $x \varepsilon x$ are meaningless (pp. 105, 107), whereas in Appendix B, "x is an x" is said to be sometimes significant (p. 525). The reason for the discrepancy lies in the departures from the simple theory of types mentioned above. If r is the range of all ranges then $r \varepsilon r$ is true. This does not contradict the doctrine of types if it is postulated simply that for $x \varepsilon u$ to be significant, the type of u must be one greater than the type of x; the range of all ranges is simply a term of infinite type.

Russell's argument for making all ranges into a type is interesting. He argues simply that every range has a number, so *all ranges* form a type. This illustrates one of his main reasons for breaking type barriers; the Frege-Russell theory of number. I shall discuss this more fully in the next section.

2. Why Russell disliked type restrictions

Russell shared with Frege a conception of the nature of logic which can be summed up in the phrase: "logic is universal". That is to say, logic is a universally applicable theory which covers all entities, concrete or abstract.[1] More particularly, the quantifier "For all x ..." should be interpreted as ranging literally over all entities in the universe; the other logical operations such as negation are to have similarly universal scope. This conception of logic is to be contrasted with the tradition of the algebraic logicians who succeeded Boole. In the work of these logicians it is presupposed that the "universe" is simply a conveniently chosen "universe of discourse" (the term is De Morgan's [5]), appropriate to the reasoning formalized. This second conception is in fact the modern conception, and I would like to consider briefly the Frege/Russell view, which is presently quite heterodox.

There is certainly a good deal of persuasiveness about the idea that logic is a universal science. If a formula of logic is universally valid, it ought to apply to all conceivable entities, so that the quantifiers in the formula can be considered as having totally unlimited scope. However, this is not the point of view taken in modern approaches to the foundations of quantification theory. There meanings are assigned to formulas relative to a universe of discourse, which is assumed to form a *set* which thus appears somewhere in the cumulative type-hierarchy of Zermelo. This point of view certainly avoids awkward questions which arise when we consider the construction of interpretations for formal theories. However, the modern view is not without its own difficulties, as has been forcefully pointed out by Kreisel and Parsons ([10], [16]). The explanation of the meaning of the quantifiers sketched above cannot account for the use of quantifiers in the crucial case of set theory itself. In Zermelo's set theory of the universe of sets is not a set, so the above explanation does not apply. It is true, of course, that we can justify ourselves *ex post facto*, as it were, by appealing to the reflection principle within set theory itself (for an introduction to reflection principles, see Levy [12]); but note that we cannot claim to *understand* the content of the reflection principle without appealing to the concept of quantification over the universe of all sets.

[1] For a very interesting extended discussion of this aspect of Frege and Russell's thinking, the reader is referred to van Heijenoort [26].

A particularly important case where Russell wishes quantification to be completely unrestricted is in the definition of number. Recall that the famous Russell/Frege definition of number is as follows: the cardinal number of a class u is the class of all classes similar to u. Note that in this definition the word "all" is intended in its unrestricted sense. It is for this reason that in his early theory of types Russell postulates a separate type of numbers, because the range of the propositional function corresponding to the concept of cardinal number breaks through type barriers. It is for the same reason that *all ranges* form a type.

An important corollary of the view of logic as universal is the idea that the paradoxes should have a unified, common solution. Today this is also a heterodox view. Commentators on the work of Whitehead and Russell often quote in this connection Ramsey's remarks [19]. After enumerating a number of semantical paradoxes (the Liar, Burali-Forti, Richard and Berry paradoxes), Ramsey says that these paradoxes:

> ... are not purely logical, and cannot be stated in logical terms alone; for they all contain some reference to thought, language, or symbolism, which are not formal but empirical terms. So they may be due not to faulty logic or mathematics, but to faulty ideas concerning thought and language. If so, they would not be relevant to mathematics or to logic, if by 'logic' we mean a symbolic system ... ([19], pp. 20–1)

In fact, this analysis of Ramsey is quite questionable. As was shown in the work of Tarski, Gödel, Carnap and others, the concepts of syntax and semantics can be made as mathematically precise as any other notion in foundations of logic. Abstract syntax is no more an empirical subject than pure number theory. Thus there is no absurdity in looking for a common solution to all the paradoxes. It is a suggestive fact (and one that Russell stressed) that the paradoxes have a clear common structure; to be precise, all of them involve diagonalization. In fact, the final solution which Russell espoused, the ramified theory of types, gives a solution to the Liar paradox which closely resembles the widely accepted solution of Tarski, as has been emphasized by Church [2].

Russell had a further motive for not adopting type restrictions; he wanted to *prove* the axiom of infinity. If you examine the notorious proof of the axiom of infinity in *The Principles of Mathematics* (§339), you can see that it does not go through in the presence of type restrictions. Russell adopted the axiom of infinity late and with considerable reluctance; in fact, Whitehead and Russell do not commit themselves to the axiom in *Principia Mathematica*, instead using it as an explicit hypothesis where necessary ([27], II: 203). The reader is referred to Moss [13] and Grattan-Guinness [9] for interesting accounts of the historical background to this question.

3. *Naturalness of the type-free approach*

Before going on to discuss the details of Russell's attempts at foundational schemes prior to *Principia Mathematica*, I would like to emphasize something which is not usually stressed in accounts of the foundations of mathematics, namely the fact that there is a good deal to be said in favour of a type-free formulation of mathematics. Once again, we are dealing with an early position of Russell which is far removed from conventional wisdom. There are several areas of mathematics where a type-free approach seems the most natural. For example, in recursion theory, it is quite natural to think of an algorithm which operates on algorithms, in particular which can operate on itself. This idea is in fact central to the general theory of algorithms; Kleene's recursion theorem is based on this idea. Similar situations arise in group theory (where a group can act upon itself) and in category theory. Such mathematics is perhaps most naturally carried out in a type-free formalism like the lambda calculus.

Because of the difficulties which arise from a separate range of untyped propositions, the theory of Appendix B fails to give a satisfactory solution to the paradoxes. In the succeeding years, Russell was to attempt two main alternatives to the foundations of logic. The first was the direct construction of a type-free system of logic. The second was a radical attempt to eliminate classes altogether. The second ultimately led him back to a stricter and much more complicated system of type theory, the ramified theory of types. In the next two sections, I shall discuss these attempts.

4. *The zigzag theory*

Russell gave the intriguing name "zigzag theory" to his attempts at a foundational scheme in the year 1904 (letter to Jourdain 15 March 1906, [9], p. 79). The only published source for the theory is a survey paper on solutions to the paradoxes written in 1905. Russell's description of the ideas are very sketchy, but it is possible to make out a bare outline. First, the theory was to be completely type-free. Russell in fact claims that a type-free solution along the lines of the zigzag theory is forced upon us if we adhere to the Frege/Russell definition of number ([22], p. 147). Second, the paradoxes are to be avoided by direct restrictions on the comprehension axiom:

> In the zigzag theory we start from the suggestion that propositional functions determine classes when they are fairly simple, and only fail to do so when they are complicated and recondite. ([22], pp. 145–6)

Russell does not give much detail on the criteria adopted for simplicity. He does, however, explain the name for his theory. If $\phi!x$ is a propositional function which does not determine a class, then the extension of $\phi!x$ must differ from any given class u; thus we have:

$$(\exists y)(\phi!y \,.\, y \notin u)$$
$$\mathbf{v}$$
$$(\exists y)(y \,\varepsilon\, u \,.\, \sim \phi!y).$$

The idea here is that the propositional functions which do not determine classes are all derived by a "diagonal construction", so that they have a "zigzag" form as indicated above.

The zigzag theory has a precursor in some of the remarks made in *The Principles of Mathematics* (Sections 103, 104). In that book, though, Russell does not distinguish the theory clearly from the ideas of his early theory of types. In order to form an idea of Russell's system of 1904, we must turn to the unpublished manuscripts.

A fairly detailed sketch of zigzag theory was sent to Whitehead on 27 October 1904, together with a covering letter in which he described the enclosure as "a somewhat rambling MS, containing a mixture of rhetoric and aspiration. In my own feelings, it embodies a distinct advance: I have begun to feel the contradiction to be obvious or just what one might have expected...."[2] The manuscript is entitled "On Functions" and consists in twenty-eight sheets of a very rough draft for the foundational portions of the second volume of *The Principles of Mathematics*. The primitive concepts employed are those of *complex* and *constituent*; propositions are said to be complexes made by combining constituents. A complex may or may not be a function of its constituents. Thus $x \varepsilon u$ is not a function of x and u. On the other hand, $(x)\phi(x)$ is a function of the propositional function $\phi`\hat{x}$; Russell writes this as "$\vdash . \mathrm{Fo}_{\phi}\{(x)\phi`x\}$". The general idea is that if we can assert "$\mathrm{Fo}_x\{\ldots x \ldots\}$" then we can assert the existence of a class of all x such that ...x.... Russell's reasons for asserting the primitive proposition above are intriguing. He argues that any complex which we can apprehend is of finite complexity, i.e. has a finite number of constituents which can be described in a finite number of steps. It follows that the various values of x, which are infinite in number, are not constituents of $(x).\ \phi`x$; the only variable constituent in fact is $\phi`\hat{x}$.

The argument shows an interesting mixture of epistemology and logic. It also shows what Russell meant by the theory of "complexity"; he is in fact trying to form a theory of *humanly apprehensible* concepts. This is in marked contrast with Ramsey, for whom the fact that we cannot write down propositions of infinite length is "logically a mere accident" ([19], p. 41).

In this version of the zigzag theory, there is a class of all entities; furthermore each class is assumed to have a cardinal number. There are thus classes for which Cantor's theorem fails, as the class of all subclasses of a given class is assumed to be a class. These are "non-Cantorian" classes, to use the terminology of Rosser [20]. The cardinality of the universe and the cardinality of the class of all classes are said to be the same, as the Schröder–Bernstein theorem is considered to be universally valid (though Russell briefly considered rejecting it—see the sheets numbered 603 to 607 of the worksheets labelled "FN").

More details on zigzag theories are to be found in a fascinating manuscript in the McMaster archives. This is the collections of worksheets labelled "FN": the title page is missing, and we can only guess that the title of this set of working notes was something like "Fundamental Notions".[3] Some 303 sheets from a total of at least 888 survive from this manuscript, which constituted what Grattan-Guinness has aptly called a "logical diary" for the period of October to November 1904.[4] In these notes we can get a direct picture of Russell's thought in the process of formation. The notes show Russell's ideas as being in a state of continual turmoil (they were written during the period of intellectual deadlock described in the *Autobiography*, I: 151). Russell continually tries out varied schemes for primitive propositions; in effect he is looking for a set of instances of the comprehension axiom

[2] The ms. and accompanying letter are on file at the Russell Archives at McMaster University in file 230.030650.

[3] The remaining fragments of this manuscript are filed under the numbers 230.030700 and 230.030840. The separation of the manuscript is presumably due to Russell himself, who at some time abstracted parts he thought of value from his own rough notes.

[4] For the dating of this MS, see sheets numbered 591 and 841.

sufficient to derive the basic foundations for logic, but which avoids the paradoxes. At one point, it looks as if he has an adequate set of primitives, but by the end of the manuscript, most of his work has fallen apart under the corrosive attack of the paradoxes. For example, on fol. 752, Russell gives a proof that $u \subset v$ is a function of u and v. By fol. 761, this functionality assertion has been seen to lead to contradiction, and can be refuted; thus the class of all subclasses of a given class is no longer a class. Russell comments "Everything now is chaos" (fol. 760). It even appears that the centre piece of the whole enterprise, the definition of number, has to be sacrificed. Thus on fol. 762, Russell notes this fact and remarks: "If this fails, arithmetic totters".

Russell never succeeded in working out the zigzag theory in a consistent way. In fact, he was groping in the dark. He was attempting to set down an appropriate set of instances of the comprehension principle; but it was impossible to be sure that contradictions would not turn up unexpectedly to overturn the apple cart. Nevertheless in his 1905 survey he expressed guarded optimism in regard to such theories. Can these ideas in fact be worked out consistently? Some recent work of Aczel and Feferman provides interesting examples of type-free theories (see [1], [6]). However, in these theories, the Frege/Russell definition of cardinal number does not work (on this point, see Aczel [1], p. 42). Perhaps a nearer approach is provided by Quine's theory, NF or "New Foundations" (Quine [17], [18]). In this theory, there is a universal class, and the Frege/Russell definition of number is workable (for details of the development see Rosser [20]). This analogy was already noted by Gödel [7]. However, the restrictions on the comprehension axiom in NF are very unlike the restrictions considered by Russell, but instead stem from the device of typical ambiguity developed by Russell in the context of the theory of types.

5. *The substitutional theory*

The key advance, from Russell's point of view, came in 1905, with the famous theory of descriptions. This theory allowed Russell to dispense with a separate category of denoting terms which had been included in his theories. This success emboldened him to try the construction of a theory in which classes, like denoting terms, would be eliminated by contextual definition. This is the substitutional theory of classes and relations, which is sketched in the 1905 survey article. A footnote appended to the 1905 article dated 5 February 1905 reads: "From further investigation I now feel hardly any doubt that the no-classes theory affords the complete solution of all the difficulties stated in the first section of this paper." The theory is described more fully in a paper [23] read to the London Mathematical Society on 10 May 1906, but not published till 1973 (for some details of the paper's history, the reader is referred to Lackey [11] and Grattan-Guinness [8]).

In the version of the theory outlined in [23], Russell postulates a class of entities, and allows quantification over entities. The basic primitive notion in the theory if "$p/a;x!q$" to be read "q results from p by substituting x for a in all those places (if any) where a occurs in p". Russell draws a sharp distinction between *substituting* (replacement of one constant by another) and *determination* (assignment of a constant as the value of a variable). This distinction, incidentally, shows a connection with the zigzag theory, in that the contrast drawn here is a kind of "syntactical" version of the distinction between "being a constituent of" and "being a function of" which are central in the zigzag theory (letter to Whitehead, October 1904). Classes are to be eliminated contextually by the definition:

$$x \varepsilon \alpha . = . (\exists p, a)(\alpha = p/a \,.\, p/a\text{'}x) \quad \text{Df.}$$

Note that the symbol "p/a" is an incomplete symbol and has no meaning in isolation; p/a is a non-entity for Russell and cannot be quantified over in a meaningful way. The Russell paradox can now be dealt with as $x \varepsilon x$ cannot be eliminated contextually if x is an entity, because $x = p/a$ must be false for all entities p and a.

The idea of "type" re-emerges in the substitutional theory in an indirect way. Classes of classes are described as matrices which arise by performing a double substitution, for example $q/(p/a)$. On eliminating classes contextually, we find that "$p/a = q/(b/c)$" leads to nonsense because of an unfilled argument place on the right-hand side ([22], p. 177). Through this indirect "syntactical" argument Russell is led to introduce a hierarchy which formally at least resembles his original ontologically conceived hierarchy of types.

If it were worked out the resulting theory would presumably resemble the simple theory of types, though with added notational complications. The intricate syntactically oriented character of the system was in fact the subject of adverse criticism by Whitehead (see the letter quoted by Lackey [11]). However, it would seem that it was internal difficulties rather than external criticism which led to the theory being replaced by a still more elaborate scheme.

6. *The vicious-circle principle and the ramified theory of types*

The "no-classes" theory provided, in Russell's view, a satisfactory solution to the set-theoretic paradoxes. However, Russell needed a uniform solution to *all* the paradoxes, and the substitutional theory was superseded because of difficulties arising from the Epimenides paradox. In his 1905 substitutional theory Russell does not give a fully worked out theory of propositions. He simply asserts that propositions are entities and allows quantification over all entities ([22], pp. 168, 188).

The difficulties which arise from this naive view of propositions are made clear in Russell's polemical reply to Poincaré in 1906 [24], [22]. To deal with the Epimenides paradox, Russell espouses the "vicious-circle principle", in the form: "Whatever involves an apparent variable must not be among the possible values of the variable" ([22], p. 204). The theory of propositions which results is not clearly spelt out. Russell makes the vague suggestion that statements which result from quantification over propositions are to be treated as non-entities and eliminated by contextual definition, like classes. Although the paper does not contain a doctrine of types for propositions, it does contain what can be regarded as the first statement of an Axiom of Reducibility (p. 212).

The contextual elimination of classes, however, cannot be extended to a contextual elimination of all forms of quantification over higher-order expressions. It is necessary to quantify over propositional functions to eliminate classes, so that the suggestion made in the 1906 reply to Poincaré bore no fruit. Instead Russell was led to the final form of type theory, the ramified theory of types with the axiom of reducibility.

The ramified theory has been discussed in great detail in many papers and books,

so that I shall not discuss it in more than a sketchy way. It is sufficient to recall that each propositional function is assigned an *order*, functions of higher order being derived by quantifying over those of lower order. The range of significance of a propositional function contains only terms of a lower order than the function itself. With these restrictions it is impossible to derive ordinary number theory, because the induction axiom in its usual form violates the restriction on order; for a formal proof of impossibility see Myhill [14]. Consequently, Russell is forced to adopt the Axiom of Reducibility which says that a function of any order is equivalent to a predicative function of the variable x, that is, a function whose order is one greater than that of x. With this addition, Russell's foundational scheme was complete and all that remained was the complete development provided in *Principia Mathematica*.

It is a commonly held view that the introduction of the hierarchy of orders, followed by the formal elimination of hierarchy via the Axiom of Reducibility, is self-defeating. If the Axiom of Reducibility reinstates the paradoxes (it is argued) then it cannot be justified. On the other hand, if the paradoxes are avoided then since the resulting theory amounts in effect to the simple theory of types, the whole introduction of orders was a useless detour. However, this criticism appears to be unfounded.

Chwistek and Copi ([3], [4]) have argued that the Axiom of Reducibility reinstates the semantical paradoxes, contrary to Russell's own claims. John Myhill has, I believe, convincingly defended the Axiom of Reducibility against this attack [15]. The solutions offered to the semantical paradoxes depend crucially on the fact that they involve intensional contexts such as "Epimenides asserts that...." Of course, if we take a strictly extensional view of propositional functions and classes, then the added complexities induced by the ramification are indeed useless. However, we have to remember that *Principia Mathematica* must be considered as an abstract form of *applied* logic. With this in mind, there can be no objection to the introduction into the theory of such intensional functions. In fact, their introduction is unavoidable, given the idea of logic as universally applicable.

7. *Retrospect*

Russell's final espousal of a radical theory of types represents a considerable retreat from the view of logic as a universal science. Even the Frege/Russell definition of number, which Russell sought valiantly to preserve, had to go, at least if it is interpreted in the way it was originally intended. Instead of a single cardinal number zero, there is a number zero for each type.

The fragmentation of arithmetic which results is overcome by a simple device, typical ambiguity. Whitehead and Russell make type distinctions formally invisible by leaving them tacit. It was this device which Quine later converted into a formal principle in the system of "New Foundations", which as we saw can be considered as a particular realization of the zigzag idea.

It is an odd fact that although the theory of types is considered to be one of Russell's outstanding contributions to logic, he himself tended to think of it as an only partly satisfactory stopgap (see, for example, Russell's comments late in life as quoted by Spencer Brown [25]). It is less surprising if we consider that the adoption of the theory meant the abandonment of his most deeply held beliefs about the nature of logic.

Nevertheless, I think we would be wrong to condemn Russell for his mixed feelings on the subject. The view of logic as a universal science is still one of considerable appeal, as I stressed earlier. On the other hand, it is *not* clear whether or not this idea can be given a coherent formal expression which is adequate to the derivation of the foundations of mathematics. At the present moment, most mathematicians tend to accept Zermelo–Fraenkel set theory in a somewhat dogmatic way. Russell's example shows us the way to a more tentative, questioning attitude to foundations. However, the alternatives proposed to standard set theory have so far failed to win wide acceptance, and in fact this rejection of alternative foundations seems to have solid reasons behind it. Unless radically new and fruitful ideas are proposed it seems wise to conclude with Russell that some form of type theory is essential in the foundations of logic. The idea of a type-free universal logic for the time being at least remains for us, as it was for Russell, a beautiful but impossible dream.

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