New evidence concerning Russell's substitutional theory of classes

by Gregory Landini

I. INTRODUCTION

IT IS WELL known that Russell regarded his new theory of denoting (of 1905) as the conceptual breakthrough that "made it possible to see, in a general way, how a solution of the contradictions might be possible" (Schilpp 1944, p. 14). The solution, of course, was the non-assumption of classes as single logical subjects. The theory of denoting was an important first step because it showed the way to provide a treatment of classes *as if* they were single logical subjects. In his 1908 article, "Mathematical Logic as Based on the Theory of Types", we find the following contextual definition effecting this solution:

$$f\{x : \psi x\} = df(\exists\phi)((x)(\phi!x \equiv \psi x) \;\&\; f(\phi!\hat{z})).$$

The contextual definition appears to make the assumption of propositional functions as single logical subjects; and this has come to be the accepted view. But, according to Russell, the non-assumption of classes realized here employs only the "technical convenience" of using symbols for propositional functions in subject positions. The convenience was supposed to be eliminable by using a technique of substitution (Russell 1908, p. 89).

Just what substitutional technique Russell had in mind remained a mystery for some time, however. On 14 December 1905 Russell had read an article entitled "On Some Difficulties in the Theory of Transfinite Numbers and Order Types" before the London Mathematical

Society. (The article was subsequently published in the proceedings of the society on 7 March 1906.) In it he set out the main alternatives for avoiding the contradiction. The preferred alternative was a substitutional theory according to which neither classes nor propositional functions were assumed as single logical subjects. Because the contradiction was formulable in terms of functions, Russell felt that "the assumption of propositional functions is open to the same arguments, pro and con, as the admission of classes" (Russell 1905, p. 154). It was this early theory of substitution which was the direct result of Russell's studies on the new theory of denoting. The theory was able to treat classes as if they were single logical subjects, and it allowed what would be quantification over classes. Moreover, by assuming propositions (true or false) as single logical subjects instead of propositional functions or classes, the theory built homogeneous typing into the logical form of propositions whose grammatical form suggested that they were about classes. In this way, Russell avoided having types of logical subjects, and the univocity of being of all logical subjects was preserved.

Nonetheless, the subsequent articles in which Russell went on to elaborate a substitutional theory of classes and relations went largely unnoticed. Russell himself was partly the cause. Its first detailed public elaboration in "On the Substitutional Theory of Classes and Relations" (1906a) was read before the London Mathematical Society in May of 1906, but the article was withdrawn from publication. Russell's decision to withdraw the article seems to be related to his desire to include a solution of what are now called "semantic" paradoxes such as the Liar Paradox. In a letter to Jourdain dated 14 June Russell wrote:

> I feel more and more certain that this theory is right. In order, however, to solve the *Epimenides*, it is necessary to extend it to *general* propositions, i.e., to such as $(x) . \phi x$ and $(\exists x) . \phi x$. This I shall explain in my answer to Poincaré's article in the current *Revue de Métaphysique*. (Quoted from Grattan-Guinness 1977, p. 89)

Poincaré's article was "Les Mathématiques et la Logique". It contained criticisms of the new mathematics of the infinite and a proposed solution—namely, the Vicious Circle Principle. Russell was eager to address the criticisms and explain that his substitutional theory is what is required by adherence to the Vicious Circle Principle. In September Russell published his reply entitled "Les Paradoxes de la Logique" (1906b). (The English title is: "On 'Insolubilia' and Their Solution by Symbolic Logic".) In it he espoused the substitutional theory, and as promised in his letter to Jourdain, it was now extended to account for the Liar Paradox.

In Russell's mind, Poincaré's Vicious Circle Principle revealed the source of the paradoxes. According to the principle, "whatever involves an apparent [bound] variable must not be among the possible values of the variable" (Russell 1906b, p. 204). Since the variable is to be unrestricted, Russell's conclusion is that no single entities involve apparent variables. In the Paradox of the Liar we have the statement: "There is some proposition I am now asserting and it is false." If this statement itself expresses a proposition, then it would be a value of its own apparent variable; and the contradiction ensues. Russell's solution was to maintain that there are no generalized propositions—that is, there are no propositions which contain apparent variables. Only those statements that do not contain quantifier phrases express propositions. Thus, the statement of the Liar does not express a single proposition and so cannot be within the range of its own variable.

Russell realized there were difficulties in abandoning propositions containing apparent variables. In particular, since quantifiers could only range over propositions not containing apparent variables, he had to introduce a "reducibility axiom" so that what would amount to quantification over generalized propositions could be effected. But in so far as his article indicates, Russell thought he had reconciled the substitutional theory with the Liar Paradox.

His published article notwithstanding, Russell's unpublished manuscripts from the period are filled with criticisms of the theory's commitment to propositions as single logical subjects, and to all appearances quantification over propositional functions is reintroduced. It seems, therefore, that the semantic paradoxes such as that of the Liar led Russell to abandon the substitutional theory late in 1906. The accepted view has come to be that by 1908 it is propositional functions and not propositions that are the values of the variables of quantification. Indeed, this interpretation seems to be encouraged by Russell himself. For in the article "The Theory of Logical Types" published in 1910 he proclaims that "there are no propositions"; and says he is explaining the views he set out in his earlier 1908 article (Russell 1910, p. 215).

It is difficult, however, to reconcile this view with the fact that Russell explicitly endorses a substitutional technique in his 1908 article. In a recent paper I have argued that the substitutional theory played a more central role in the historical development of the mature theory of types than has been thought (Landini 1987). Behind the technical conveniences of the notation of propositional functions, Russell was espousing a modified version of the substitutional theory in 1908.

Although Russell says he is explaining the 1908 theory in "The Theory of Logical Types", it would be a mistake to conflate these two theories (Cocchiarella 1980, p. 95ff.). For in 1908 Russell does allow quantification over propositions. To deal with the semantic paradoxes such as the Liar, he introduced a hierarchy of "orders" of propositions based upon the admissible ranges of apparent variables for propositions. Moreover, the ramified hierarchy of orders of propositional functions is defined by reference to this propositional hierarchy (Russell 1908, p. 77). Thus, propositions are assumed as single logical subjects in 1908.

Further, I have discovered that from a historical standpoint the substitutional theory is inseparably linked with the 1908 hierarchy of propositions. The possibility of there being such a link came to my attention by examining Cocchiarella's observation that the substitutional theory is in conflict with Cantor's power-class theorem (Cocchiarella 1980, p. 90). The conflict yields a paradox which stems from the assumption of propositions as entities. The paradox is intensional in nature, since its formulation turns on the identity of propositions and Russell held that equivalent propositions need not be identical. In this respect, the paradox is like that of the Liar. Indeed, Russell considered it to be among the "paradoxes of propositions" which are all of the same sort as the Epimenides. But the paradox does not depend upon semantic notions such as "designation" or "truth" and thus it should not be characterized as a "semantic" paradox. Nonetheless, Russell's 1908 hierarchy of "orders" of propositions blocks this paradox no less so than the Liar, and the addition of the axiom of reducibility does not change this.

There seems to be no explicit acknowledgement of the conflict and its connection with the hierarchy of propositions in Russell's published writings on type-theory. However, I have recently discovered evidence in unpublished manuscripts that Russell was aware of the conflict. Most important is the manuscript entitled "The Paradox of the Liar". (The manuscript is dated September 1906, but seems to have been written after the article "Les Paradoxes de la Logique".) Here Russell not only points out the *syntactic* conflict, he uses a 1908-style hierarchy of "orders" of propositions to avoid it.[1] This new evidence is revealed in what follows.

[1] The manuscript raises many criticisms of the substitutional theory and responds only to some of them. But Russell added a note to the manuscript in June of 1907 which indicated that he doubted whether the remaining criticisms were telling against the theory. This shows that he then believed that a substitutional theory was still possible. In fact, he wrote in a manuscript entitled "Fundamentals" (dated 1907) that "types won't work without no-classes. Don't forget this" (p. 47).

II. CLASSES IN THE SUBSTITUTIONAL THEORY

In order to see the conflict we must briefly present a sketch of how the substitutional theory (of April–May 1906) uses matrices to represent propositional functions of individuals. Russell introduces the notation "p/a" which he calls a "matrix". (Both "p" and "a" are taken to be names of logical subjects, and Russell calls the proposition named by "p" the "prototype".) Next he introduces the notation "$p/a;b$" which abbreviates the expression "the result of substituting b for every occurrence of a in p". As with all definite descriptions, the expression "$p/a;b$" is an incomplete symbol and can never occur in isolation. It is to be contextually defined in accordance with the 1905 theory of definite descriptions as follows:

$$\psi(p/a;b) = df(\exists q)(p/a;b!q \;\&\; (r)(p/a;b!r \supset r = q) \;\&\; \psi(q)).$$

The notation "$p/a;b!q$" abbreviates "q results from substituting b for every occurrence of a in p". Thus, while "$p/a;b$" is an incomplete symbol, "$p/a;b!q$" expresses a proposition. Russell goes on to define the negation of a proposition as follows:

$$\sim p = df\ p \text{ is false.}$$

He then defines "$\sim(p/a;b)$" as "$(\exists^1 q)(p/a;b!q \;\&\; \sim q)$". Here we use "$(\exists^1)$" to abbreviate "exactly one" and define it in the usual way.

Russell expresses quantification by means of the notation "$(x)(p/a;x$ is true)", which would be contextually defined as "$(x)((\exists^1 q)(p/a;x!q \;\&\; q$ is true))". Quantification should be interpreted objectually. Since Russell explicitly says that it is an entity which is to be substituted for the entity a in the proposition p, the values of the quantifiers must range over logical subjects (i.e., all "entities" including propositions), not over constants and sentences. Any inclination to interpret the substitutional theory of 1906 as akin to the modern substitutional interpretation of quantification should, therefore, be avoided.[2]

As we can see, the substitutional theory assumes that there are primitive object-language predicates "truth" and "falsehood" which stand for properties of propositions. His 1905 theory of definite descriptions, elaborated in "On Denoting", also reflects this view. Russell set out the basis of the view in his 1904 review of Meinong's Theory of Complexes and Assumptions. He wrote: "It may be said, and this I think is the correct view, that there is no problem at all in truth and falsehood; that some propositions are true and some are false, just as some roses are red and some white ..." (p. 75). On this early view, "truth" is not a semantic relation between a sentence (or mental entity) and a proposition, but is rather an unanalyzable property of propositions themselves.

By 1910, however, Russell came to develop a "correspondence" theory of truth, and his new view acknowledged that "truth" involves what we now call "semantic" elements. But the substitutional theory should not be thought to be committed to a semantic truth predicate in its object-language. The truth-predicate need not appear when a context is given which would complete the symbol "$p/a;b$". We can see this, for example, in the definition of the coextensivity of propositional functions of individuals:

$$p/a = p'/a' = df(x)\{(\exists^1 q)(\exists^1 r)([p/a;x!q] \;\&\; [p'/a';x!r] \;\&\; (r \equiv q))\}.$$

Moreover, where Russell wrote "($p/a;b$ is true)", which he contextually defined as "$(\exists^1)(p/a;b!q \;\&\; q$ is true)", we can simply write "$(\exists^1 q)(p/a;b!q \;\&\; q)$". There is, therefore, no essential dependency on an object-language truth-predicate in the substitutional theory.[3]

Now Russell's motivation for using matrices to represent propositional functions was quite clear. In a notation reflecting the assumption

[2] It has been argued that, while in earlier "no-class" theories Russell operated substitution and free variables simultaneously, in the substitutional theory of 1906 free variables were eliminated (see Grattan-Guinness 1977, p. 75). I disagree. Readers should not interpret *substitution* as the same as the *replacement* of a linguistic symbol in a sentence by a constant.

[3] Peter Hylton (1980) has argued that an object language truth-predicate is essential to the substitutional theory, and that *semantic* paradoxes led Russell to abandon the theory in 1906. Hylton suggests that every class of propositions will be correlated one-to-one with a unique proposition asserting that every proposition in the class is true. He formulates a matrix $q*/r$,

$$(\exists q,p)(r = (x)(q/p;x \supset x \text{ is true}) \;\&\; \sim(q/p;r))/r.$$

Then he derives a contradiction by substituting the proposition $(x)(q*/r;x \supset x$ is true) for r. However, Hylton fails to see that a semantic "truth"-predicate is not essential to the viability of the theory. But even with the assumption of a truth-predicate, Hylton neglects Russell's published solution of "semantic" (and non-extensional) paradoxes in "Les Paradoxes de la Logique". Since there are no generalized propositions, a statement expressing the identity of a proposition with what would be a generalized proposition is not well formed, and equivalence will not suffice.

of propositional functions as single logical subjects, propositional functions would be represented by the expression "$\phi\hat{x}$". But in Russell's view, the variable "x" in "ϕx" is logical or formal and thus must be wholly unrestricted; it ranges over all logical subjects. The laws of logic do not have a restricted scope. They are universally valid for all logical subjects. The unrestricted variable of logic reflects what Russell had called in *The Principles of Mathematics* the doctrine of the "univocity of being of all logical subjects", and he never gave up this fundamental view. So if propositional functions are indeed logical subjects, then there should be nothing which prevents them from applying to themselves; and yet this is what led to the paradox. The theory of homogeneous types was to avoid this. But because of the doctrine of the univocity of being of all logical subjects, a theory of types of logical subjects is philosophically impossible.

However, if the symbols for propositional functions are taken to be incomplete, and to be contextually defined, then it is possible to build the theory of types into the logical form of their contextual definitions. Russell's matrices do just this. Matrices of the form "p/a" go proxy for propositional functions of individuals. Dyadic relations of individuals are represented in terms of matrices of the form "$q/(a,b)$"; triadic relations are represented in terms of matrices of the form "$q/(a,b,c)$", and so on. Propositional functions of functions of individuals are next constructed in terms of matrices of the form "$q/(r/c)$", or alternatively "$q/(r,c)$", where "r/c" is understood to form a function of individuals. To express what would be the predication of the one function of the other Russell writes "$\{q/(r/c);p/a\}$ is true" or alternatively "$\{q/(r,c);p,a\}$ is true". The notation "$q/(r,c);p,a$" abbreviates "the result of simultaneously substituting p for every occurrence of r and a for every occurrence of c in q".

Higher types of functions are constructed in accordance with this pattern. Relations between individuals and functions of individuals can also be represented. The matrices for such relations are of the form "$q/\{b, p/a\}$". A relation between two functions of individuals would be of the form "$q/\{p/a, r/c\}$", etc. In this way, homogeneous stratification is built into the very logical form; no function can meaningfully apply to itself.

Classes are still to be understood as the extensions of propositional functions insofar as a class is what is common to coextensive propositional functions. Where $p/a = r/c$, for instance, the functions define the same class. Thus, a matrix of the form "$q/(p,a)$" defines a class of classes (of individuals) if the following holds:

$$(r,c)(r',c')([r/c = r'/c'] \supset [q/(p,a);r,c \equiv q/(p,a);r'c']).$$

This applies for higher types as well, so that we are assured that where $q/(p,a) = q'/(p',a')$ the propositional functions define the same class. (Note that Russell used "(r,c)" to abbreviate "$(r)(c)$", and "$(\exists r,c)$" to abbreviate "$(\exists r)(\exists c)$".) To illustrate how the theory works consider the definition of the number 0_1. Recall that in Russell's type theory, 0_1 is the class of all empty classes of individuals. In the substitutional theory the matrix "$(x) \sim (p/a;x)/p,a$" goes proxy for 0_1. The matrix "$(c \neq c)/c$" goes proxy for the empty class Λ (of individuals). Thus, $\Lambda \varepsilon 0_1$ is represented by asserting that $\{(x) \sim (p/a;x)/p,a;(c \neq c), c\}$ is true. That is, it is represented by asserting that the result of simultaneously substituting the proposition $(c \neq c)$ for p and c for a is true. Since $(x)(x = x)$, we know that this holds. What would be quantification over classes is then straightforwardly effected by means of quantifying over propositions. For example, $(\exists y)(y \varepsilon 0_1)$ is represented by asserting that $(\exists p',a')(\exists^1 t)(\{(x) \sim (p/a;x)/p,a;p',a'\}$ is true). By using matrices in this way, mathematics was to be constructed without the assumption of classes or propositional functions as logical subjects.

III. THE SUBSTITUTIONAL THEORY WITHOUT GENERALIZED PROPOSITIONS

Russell never published the April–May version of the substitutional theory. As his letter to Jourdain indicates, it was his desire to apply the theory to the Paradox of the Liar that played a role in his withdrawing the article from publication. Russell thought that the solution of the Liar requires that there are no "generalized propositions"—i.e., no single logical subject (proposition) is expressed by a statement containing a quantifier phrase. Statements without quantifier phrases, on the other hand, do express single propositions, whether true or false. And only propositions are in the range of the quantifiers. The non-assumption of generalized propositions, however, imposes serious difficulties for the substitutional theory. But by September of 1906 Russell thought that these difficulties could be overcome. His article "Les Paradoxes de la Logique", which espoused the substitutional theory without assuming generalized propositions, was to explain this.

Russell tells us that a statement containing a quantifier phrase, such as "$(x)(x = x)$", is to be interpreted as asserting indeterminately all the propositions $x_1 = x_1$, $x_2 = x_2$, $x_3 = x_3$, etc. Similarly, a statement such as "$(\exists x)(x = x)$" is interpreted as asserting an ambiguous proposition from among these.

"Truth" and "falsehood" are still primitive properties of propositions, but "truth" or "falsehood" applied to a statement has a different meaning. A generalized statement is "true" just when the proposition(s) it asserts are true. Russell interprets the Liar as making the statement "There is some proposition I am now asserting and it is false". Since statements are not propositions, the statement of the Liar cannot apply to itself and is therefore "false".

While this avoids the Liar Paradox it poses difficulties for the substitutional theory. In particular, Russell had defined "$p/a;b!q$" to be "q results from substituting b for every occurrence of a in p". But if "p" is a statement containing a quantifier phrase, then it does not express a single proposition and so there is no single proposition q which would result from the substitution. Thus, the contextual definition of "$p/a;b$" requires amendment.

But how are we to understand the notation "$\phi/a;b$", where ϕ is a statement containing a quantifier phrase? One suggestion, which I adopted in my earlier article, is to begin by reading the notation "$p/a;b$" as an assertion of the proposition q such that $p/a;b!q$. Then we could read notation such as "$(y)(Fy \supset Fa)/a;b$" as an assertion of all the propositions got by substituting b for a at each of its occurrences (if any) in the propositions asserted by the statement "$(y)(Fy \supset Fa)$", i.e., in $(Fy_1 \supset Fa)$, $(Fy_2 \supset Fa)$, $(Fy_3 \supset Fa)$, and so on. This interpretation enables us to state Russell's *1906* reducibility axiom (in the monadic case and for functions of individuals), as follows:

$$(\exists p,a)(x)(p/a;x \equiv \phi/a;x),$$

where "ϕ" is a metalinguistic variable ranging over sentences containing quantifier phrases. This formulation is attractive since it parallels Russell's notes in manuscript. The 1906 axiom says that for any matrix whose prototype is a statement there is a coextensive matrix whose prototype stands for a proposition. This enables Russell to capture all that would have been captured if there were generalized propositions within the range of the quantifiers. But the interpretation strays from Russell's claim that "$p/a;b$" is an incomplete symbol to be contextually defined. Here I wish to make a different suggestion which remains closer to Russell's own original reading.

Russell's 1906 axiom of reducibility assures (in the monadic case) that for any statement "ϕ" which contains a quantifier phrase, there is a single proposition p containing the entity a which is such that $(x)(p/a;x \equiv \phi x)$. By contextual definition, the axiom is,

$$(\exists p,a)(x)\{(\exists^1 q)(p/a;x!q \ \& \ (q \equiv \phi x))\}.$$

Again "ϕ" is to be understood as a metalinguistic variable for sentences containing quantifier phrases.

The axiom of reducibility assures, for example, that $(\exists p,a)(x)(\exists^1 q)(p/a;x!q \ \& \ [q \equiv (y)(Fy \supset Fx)])$. That is, for any entity x there is a single proposition q such that $p/a;x!q$ and q is true if and only if all the propositions $(Fy_1 \supset Fx)$, $(Fy_2 \supset Fx)$, $(Fy_3 \supset Fx)$, ... and so on, are true. Thus, instead of the matrix "$(y)(Fy \supset Fa)/a$", which is problematic since "$(y)(Fy \supset Fa)$" does not name a single proposition, Russell can use "p/a" . Of course, Russell will also have to deal with cases which would have allowed the substitution of a generalized proposition. Without generalized propositions as single entities Russell cannot allow, "$\{q/(r,c);(y)(Fy \supset Fa),\ a\}$ is true". Reducibility is again called into play. The axiom assures that "$\{q/(r,c);p,a\}$ is true" will suffice as a replacement. Finally, Russell must handle cases which would have involved the substitution of a generalized proposition singly, such as in "$\{(p = p)/p;(y)(y = y)\}$ is true". The axiom of reducibility assures that there is a proposition which is equivalent to any statement, so that we have $(\exists r)(r \equiv (y)(y = y))$. Hence, "$\{(p = p)/p;r\}$ is true" is the closest replacement.

As we can see, the denial of generalized propositions greatly increases the complexity of the substitutional theory. But given the reducibility axiom is formulable and extendable to higher types, the elimination of generalized propositions seems well underway. If this is a plausible account, then we have a reconstruction of Russell's reasons for claiming that his solution of the Liar Paradox does not undermine the construction of classes and of mathematics in the substitutional theory. Moreover, the reducibility axiom does not reintroduce the Paradox of the Liar. For the actual statement made in the Liar is relevant, and although reducibility assures that there is an equivalent proposition within the range of its quantifier, this will not suffice for the paradox (p. 212).

IV. RUSSELL'S NEW PARADOX OF p_0/a_0

Russell's doctrine of the univocity of being of all logical subjects was realized in both the April–May 1906 and the September 1906 substitutional theories by allowing the substitution of a proposition for an individual. Propositions were individuals at this time insofar as Russell used the term "individual" to mean "logical subject" or "entity". Because all logical subjects were on a par, it was possible to form a matrix representing a universal class of all entities, propositions or otherwise. The matrix "$(a = a)/a$" goes proxy for the class of all entities, for the result of substituting any entity for the entity a in the proposition $a = a$ is clearly a true proposition. Moreover, the substitutional theory involves an intensional logic of propositions insofar as the equivalence of propositions does not assure their identity. This feature allowed Russell to prove that the universal class is infinite (Russell 1906b, p. 203). Unfortunately, allowing all propositions to be classed together in this way leads to a conflict between the technique of substitution and Cantor's power-class theorem.

The conflict with Cantor's power-class theorem arises in the following way. Supposing the class of all logical subjects (propositions or otherwise) is denumerable, we can assign every logical subject x a natural number $\#x$. Since every subclass of the class of logical subjects is represented in terms of a matrix of the form p/a where p and a are logical subjects, we can assign each a rational number $\#p/\#a$. As is well known, the natural numbers have the same cardinality as the rational numbers. Thus, the class of all logical subjects will have the same cardinality as its power-class. (A similar argument can be formed no matter what infinite cardinality the class of all logical subjects has.) But Cantor's power-class theorem says that there is no function from any class onto its power-class; the power-class of any class always has greater cardinality!

We are now in a position to see that Russell was aware of the conflict. In a manuscript entitled "Logic in which Propositions are not Entities", dated April–May 1906, Russell wrote:

> The above theory comes to this: that we can substitute any entity for any entity, and any proposition for any proposition, but never an entity for a proposition or a proposition for an entity.
>
> But if we don't have a hierarchy of propositions, it looks as if we should get into difficulties from the fact that there are more classes of propositions than propositions, and that, to all appearances, we can establish a 1–1 function from all classes of propositions to some propositions. (P. 15)

In the first paragraph we see that originally Russell thought it was only necessary to prevent propositions from being substitutable for entities—i.e., those entities which are on a par with concrete individuals. But then he realizes that this will not do by itself; if we can class all propositions together, then the conflict with Cantor's power-class theorem will just repeat on a higher level.

In a manuscript entitled "On Substitution", also dated April–May 1906, Russell devoted his attention almost exclusively to this contradiction in the substitutional theory. It is to this contradiction that he refers in his manuscript "Paradox of the Liar" of September 1906:

> A second-order proposition is one in which either "all values" or "any value" of p occurs, or a complex $p/a;x!q$ occurs. I think the latter above is sufficient: all second-order propositions that arise contain $p/a;x!q$.
>
> We shall need a notation, say p^2, for any second-order proposition. Then we have $p^2/a;x!q^2$ and also $p^2/p;q!q^2$. Both are significant. The former substitution will affect only origin and argument in $p/a;x!q$; the latter will affect only prototype and resultant. Both these substitutions are third-order propositions. Thus $p/\alpha;\beta!q$ is always of higher order than any of its constituents; this disposes of the fallacy which led to the abandonment of substitution before, i.e.,
>
> $$p_0 . = : (\exists p,a) : a_0 . = . p/a;b!q : \sim(p/a;a_0) :. \\ \supset : p_0/a_0;(p_0/a_0;b!q) : \sim(p_0/a_0;[p_0/a_0;b!q])$$
>
> for here we substitute for a_0 the proposition $p_0/a_0;b!q$, which is necessarily of higher grade than a_0. (P. 72)

By translating the notation, we see that Russell has formulated a matrix p_0/a_0—i.e.,

$$(\exists p,a)(a_0 . = . p/a;b!q \ \& \ \sim(p/a;a_0))/a_0$$

Next Russell considers the proposition $p_0/a_0;b!q$, and derives

$$p_0/a_0;[p_0/a_0;b!q] \equiv \sim(p_0/a_0;[p_0/a_0;b!q]),$$

which is a contradiction.

Of course, in the September 1906 substitutional theory of "Les Paradoxes de la Logique", there are no generalized propositions. Since p_0 is generalized, there is no proposition $p_0/a_0;b!q$. Thus, it cannot be substituted for a_0. Nonetheless, Russell's early Axiom of Reducibility will reintroduce the p_0/a_0 paradox. So the denial of generalized propositions which blocks semantic paradoxes will not block this intensional paradox.[4] This shows that Russell was aware in 1906 that the substitutional theory conflicts with Cantor's power-class theorem. Moreover, he was aware that the denial of generalized propositions, coupled with a reducibility axiom collapsing statements to propositions, causes grave difficulties for the substitutional theory.

If classes of logical subjects are to be represented in terms of matrices, then classing together all logical subjects must form an "illegitimate totality". In particular, Russell's formulation of the matrix p_0/a_0 reveals that if classes of propositions are represented in terms of matrices, then classing all propositions together has to be ruled out. (As we saw, this is so even if there are no generalized propositions—given the 1906 axiom of reducibility.) "All propositions" must, therefore, be an illegitimate totality. Thus, he concludes that propositions must be divided into "orders".

Russell suggests that the hierarchy of "orders" of propositions is based upon the legitimate range of their bound variables. The lowest order is that of individuals which are now said to be without complexity so that no proposition is an individual. Next are the "first-order propositions", which are those propositions with no quantifiers together with those whose quantifiers range over individuals. Then there are "second-order propositions", which contain quantifiers ranging over first-order propositions; and so on.

With the new hierarchy of propositions, a new Axiom of Reducibility is needed. The new axiom assures that for any matrix we can always find a coextensive matrix that is "predicative"—i.e., its prototype is a proposition whose order is next above the highest order of its argument(s). The new axiom does not reintroduce the p_0/a_0 paradox. It does not collapse the orders of propositions to one order. But it does recapture, in extensional contexts, all that is lost by limiting quantification to propositions of a given order.

It should be noted that the orders are to be reflected in what is to count as a proper substitution. The class represented by the matrix "$(a = a)/a$" is now confined to all "individuals", where no proposition is to be regarded as an individual; the class represented by the matrix "$(p^1 = p^1)/p^1$" contains all "first-order" propositions; the class represented by the matrix "$(p^2 = p^2)/p^2$" contains all second-order propositions, and so on. No proposition can be substituted for an "individual", and no second-order proposition can be substituted for a first-order proposition, etc. Thus, there can be no class of all propositions irrespective of their order.

Russell's actual restrictions on what counts as a proper substitution are, however, even more severe than is necessary. As we can see in the above quote, he says that "$p/\alpha;\beta!q$ is always of a higher order than any of its constituents". But why not allow both the prototype and the argument to be of the same order? Perhaps Russell worried that a contradiction would arise if it is possible to represent every subclass of the class of (say) first-order propositions by means of matrices q/p , where both q and p are first-order propositions. To be safe, he seems to think that the prototype must always be of a higher order than the argument.

But there is no difficulty in allowing the proposition that results from a substitution to have constituents that are of the same order as it is.[5] That is, we can allow matrices of the form q/p, where both q and p are of the same order. Hence, we can take the order of $p/\alpha;\beta!q$ to be the order of q. All that seems essential is that the order of the entity substituted should be the *same* as the order of the argument for which it is being substituted. Let the propositional quantifier in p_0 range over propositions of order n. The order of p_0 will then be $n+1$, and the order of $p_0/a_0;b!q$ is the same as the order of p_0. But for $p_0/a_0;b!q$ to be admissibly substituted for a_0 in p_0 it would have to be of order n. This follows from the fact that whatever is substituted for a_0 in p_0 have the same order as a proposition $p/a;b!q$ which is obtained from the existential instantiation of the quantifiers in p_0. Thus, a substitution such as $p_0/a_0;[p_0/a_0;b!q]$ is improper.

In any case, Russell is clearly using the hierarchy of "orders" of propositions to avoid the syntactic conflict with Cantor's theorem. In Russell's view, the introduction of a 1908-style hierarchy of "orders" of propositions is essential to the consistency of the substitutional the-

[4] The contradiction follows from contextual definitions and Russell's 1906 Axiom of Reducibility. By the 1906 Reducibility Axiom we have:

$$(\exists t,a_0)\{(x)(t/a_0;x \equiv (\exists p,a)(x = p/a;b!q \;\&\; \sim(p/a;x)))\}$$

Then assume, for the left-to-right direction of the contradiction, that $(\exists^1 r)(t/a_0;[t/a_0;b!q]!r \;\&\; r)$. The right-to-left direction is equally straightforward.

[5] There would be a contradiction if every class in the power-class of the class of first-order propositions could be represented by a matrix q/p, where q and p are both first-order propositions. (A similar contradiction would arise for the power-class of the class of second-order propositions, and so on.) But these contradictions only show that one cannot represent every class in the power-class by matrices q/p, where q and p are of the same order. Moreover, a matrix q/p , where q is second-order and p is first-order, is "predicative". (A matrix is "predicative" when its prototype is next above or equal to the highest order of its arguments.) Thus, Russell's axiom of reducibility does not reintroduce the paradox.

ory.[6] The ramified theory of types which this hierarchy imposes would, therefore, seem to be an inescapable consequence of the framework of the substitutional theory.

I believe that this evidence makes a very strong argument for interpreting the 1908 theory of logical types as embodying the substitutional theory. The interpretation explains Russell's continued endorsement of the technique of substitution, and his return in 1908 to propositions as single logical subjects. (After all, even assuming Russell could make the view workable, the non-assumption of generalized propositions did not succeed in solving the "paradoxes of propositions".) In addition, it reveals a deep source for Russell's well-known conviction that the paradoxes of propositions (including that of the Liar) stem from a source in common with syntactic paradoxes and require a solution if Logicism is to proceed.

If the 1908 theory assumes propositional functions as single logical subjects outright, then, as Quine argues, ramification is entirely out of place (Quine 1941). Poincaré's Vicious Circle Principle can only apply if propositional functions are conceived as dependent upon the constructive powers of the mind, or upon their linguistic representation. The whole predicative/impredicative distinction is ill-motivated when functions are assumed outright as subsisting independently of thought and linguistic representation. As Quine says, Russell must have confused use with mention—a confusion of propositional functions with the linguistic symbols representing them. For propositional functions are not "defined" at all, much less is one ever "defined in terms of a totality to which it belongs".

In the context of the substitutional theory, however, we can see that this interpretation is mistaken. Propositional functions are not in any sense "defined", but they are "dependent" upon propositions (in so far as functions are treated as if they are single logical subjects by means of the theory of incomplete symbols). And as we have just seen, the orders of propositions and the ramification that this imposes on type theory became an essential part of this dependency.

[6] It might be objected that with the introduction of "orders" of propositions the doctrine of the univocity of being of all logical subjects is lost. But it is not clear that Russell thought so. The "limitations" on the range of the variables of quantification are built into the meaning of a proper substitution. The order of the argument is given with the matrix, and a proper substitution requires that the entity substituted is to be of the same order as the argument for which it is being substituted. Russell held that only *external restrictions* on the quantifiers, and not "limitations" based on admissible range of significance, violate the doctrine of the univocity of being (Russell 1908, p. 72).

Department of Philosophy
University of Iowa[7]

REFERENCES

Cocchiarella, Nino B. (1980). "The Development of the Theory of Logical Types and the Notion of a Logical Subject in Russell's Early Philosophy", *Synthese*, 45: 71–115. Reprinted in his *Logical Studies in Early Analytic Philosophy* (Columbus: Ohio State University Press, 1987).

Grattan-Guinness, I. (1977). *Dear Russell—Dear Jourdain* (London: Duckworth).

Hylton, Peter. (1980). "Russell's Substitutional Theory", *Synthese*, 45: 1–31.

Landini, Gregory. (1987). "Russell's Substitutional Theory of Classes and Relations", *History and Philosophy of Logic*, 8: 171–200.

Quine, W.V.O. (1941). "Whitehead and the Rise of Modern Logic", in *The Philosophy of Alfred North Whitehead*, ed. P.A. Schilpp (LaSalle, Il.: Open Court), pp. 125–63.

Russell, Bertrand. (1905). "On Some Difficulties in the Theory of Transfinite Numbers and Order Types", in Russell 1973, pp. 135–64. First published in *Proceedings of the London Mathematical Society*, 4 (March 1906): 29–53. Read before the Society on 14 December 1905.

———. (1906a). "The Substitutional Theory of Classes and Relations", in Russell 1973, pp. 165–89. The paper was read before the London Mathematical Society on 10 May 1906.

———. (1906b). "On 'Insolubilia' and Their Solution by Symbolic Logic", in Russell 1973, pp. 190–214. First published as "Les Paradoxes de la Logique", *Revue de Métaphysique et de Morale*, 14 (Sept. 1906): 627–50.

———. "Logic in which Propositions are not Entities", unpublished manuscript in the Russell Archives dated April–May 1906 (McMaster University). File RA 230.030720.

———. "On Substitution", unpublished manuscript in the Russell Archives, dated April–May 1906 (McMaster University). File RA 220.101950–F1-3.

———. "The Paradox of the Liar", unpublished manuscript in the Russell Archives, dated September 1906 (McMaster University). File RA 220.010930–F1-3.

———. "Fundamentals", unpublished manuscript in the Russell Archives, dated 1907 (McMaster University). File RA 230.030760–F1-2.

[7] Research for this paper was supported by a grant from The American Philosophical Society.

———. (1908). "Mathematical Logic as Based on the Theory of Types", in R. C. Marsh, ed., *Logic and Knowledge: Essays 1901–1950* (London: Allen and Unwin, 1956), pp. 59–102.

———. (1910). "The Theory of Logical Types", in Russell 1973, pp. 215–52.

———. (1973). *Essays in Analysis*, ed. D.P. Lackey (New York: Braziller).

Schilpp, P.A. (1944). *The Philosophy of Bertrand Russell* (Evanston and Chicago: Northwestern University).