Discussion

What became of Russell's "relation-arithmetic"?

by Graham Solomon

IN *MY PHILOSOPHICAL DEVELOPMENT* (1959) Russell lamented the general lack of interest in his relation-arithmetic. He remarked (p. 72) that "from the mathematical point of view" relation-arithmetic was his "most important contribution" to *Principia Mathematica*. It also has philosophical importance:

> I think relation-arithmetic important, not only as an interesting generalisation, but because it supplies a symbolic technique required for dealing with structure. It has seemed to me that those who are not familiar with mathematical logic find great difficulty in understanding what is meant by 'structure', and, owing to this difficulty, are apt to go astray in attempting to understand the empirical world. For this reason, if for no other, I am sorry that the theory of relation-arithmetic has been largely unnoticed. (P. 76)

A *structure* (in *Principia Mathematica*, a *relation-number*) is the isomorphism type of a system of relations. A *system of relations*, consisting of one or more relations, is essentially the sort of object contemporary model theorists mean when they refer to "structure".[1] The principal differences between a model-theoretic structure and a Russellian system of relations are that the domain of a system of relations consists only of the fields of the

[1] Rudolf Carnap was one of the few logicians to use the term "structure" in Russell's sense, and to continue using it in that sense after the emergence of model theory around 1950. See, for example, *Introduction to Symbolic Logic and Its Applications* (1958, §34).

relations, and these fields need not be sets. *Relation-arithmetic*, developed in Part IV of *Principia Mathematica*, is the arithmetic of relation-numbers. Addition, multiplication, and exponentiation were defined for arbitrary relation-numbers. Ordinal numbers are a species of relation-numbers, and Russell found it possible to adapt the definitions of arithmetical operations appropriate for ordinals, as developed by Cantor and other set-theorists, to arbitrary relation-numbers.

Here is one definition from the theory of relation-arithmetic. The *ordinal sum* of the relation-number of the relation P and the relation-number of the relation Q is the relation-number of the ordinal sum of P and Q, provided that the field of P and the field of Q have no common terms. The ordinal sum of P and Q is the relation holding between x and y when x and y stand in P or in Q, or when x is a member of the field of P and y is a member of the field of Q. For example: if P and Q are relations that generate series, the ordinal sum of P and Q is the series that is obtained by attaching the Q-series to the end of the P-series.

Russell also defined the notion of an *infinite ordinal sum*. If one wishes to sum an infinite number of relations, the relations have to be indexed as the field of some ordering relation. The sum of an infinite system of relation-numbers over an indexing relation R is the sum over R of a system of relations with disjoint fields.

According to Russell, relation-arithmetic did receive some attention:

> That it was not wholly unnoticed I learnt, to my surprise, through a letter, received in 1956, from Professor Jürgen Schmidt of the Humboldt University in Berlin. Some parts of the theory, as he informed me, were used in what is called the 'lexicographical problem', which consists in defining alphabetical order among words in a language of which the alphabet is infinite. (1959, p. 76)[2]

Actually, however, the arithmetic of relation-numbers received more significant attention from Alfred Tarski and some of his students during the 1940s and 50s. Russell was certainly aware of Tarski. In his *Autobiography* he reprints a 1939 letter to Quine: "I quite agree with your estimate of Tarski; no other logician of his generation (unless it were yourself) seems to me his equal" (1978,

[2] See Schmidt (1955).

p. 467).

Tarski studied, quite thoroughly, central aspects of the arithmetic of relation-numbers (which he called "relation types") in *Ordinal Algebras* (1956). His intention to do so was announced in *Cardinal Algebras* (1949), a study of the arithmetic of cardinal numbers. There seems to me no reason why Russell could not have followed Tarski's algebraic approach. No doubt Russell's lament is primarily a reflection of his own lack of interest in most developments in mathematical logic after publication of the second edition of *Principia Mathematica*.

Tarski provided a general algebraic setting for discussing relation-arithmetic. He constructed an algebra consisting of a set A of arbitrary elements, the operation Σ on finite and countably infinite sequences of members of A, the operation + on couples of members of A, the operation * on single members of A, and a distinguished member O of A. He set out some postulates for the elements and operations and showed that they are satisfied by taking A to be the set of relation-numbers of binary reflexive relations, Σ and + to be ordinal summation operations, * to be conversion and O to be the relation-number of the empty relation. Partially ordering and simply ordering relations are species of binary reflexive relations, and thus the theorems that he proved for binary reflexive relations in general hold for these special cases. Here is one of the theorems he proved: Let P and Q be binary reflexive relations. Say that P is an *initial segment* of Q when there is a binary reflexive relation R disjoint from P such that $P+R=Q$, and say that P is a *final segment* of Q when $R+P=Q$. Then P and Q are isomorphic if P is isomorphic to an initial segment of Q and Q is isomorphic to a final segment of P.

Tarski showed how to make modifications in order to generalize his results to arbitrary relation-numbers (and, also, to abstract relation algebras). The operations + and Σ were both essentially defined by Russell, following Cantor's definitions for the special case of ordinal numbers. Tarski's Σ is Russell's infinite ordinal summation restricted to countably infinite sequences. In the context of uncountably infinite sequences the axiom of choice is required. Tarski wished to avoid use of the axiom of choice, implicated as it is in paradoxical decompositions of the sort he studied with Stefan Banach in the 1920s.[3]

[3] The Banach–Tarski paradox: any two bounded sets with non-empty interiors in R^n, $n \geq 3$, are equidecomposable.

Russell's sales pitch for relation-arithmetic in *My Philosophical Development* casts it as "required for dealing with structure". And we need this concept of structure, he thinks, for understanding the empirical world. We can grant that a study of the arithmetical properties of structures has mathematical value. Does it have philosophical value? Russell developed the view in *The Analysis of Matter* (1927) that science gives us a posteriori (and purely) structural knowledge of unobservables. Scientific knowledge is knowledge of the structural properties of the external world. We discover, through science, what structures the external world satisfies.

This view was fatally criticized by M.H.A. Newman in an article on Russell's book in *Mind*.[4] The problem is that so long as the external world is of the right cardinality, any structure exists on it trivially. The only nontrivial constraint on the existence of any relation on a domain is the cardinality of the domain. So if scientific knowledge is of structure only, it must be a priori.

Russell acknowledged the force of this criticism in a letter to Newman:[5]

> Dear Newman,
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> Many thanks for sending me the off-print of your article about me in *Mind*. I read it with great interest and some dismay. You make it entirely obvious that my statements to the effect that nothing is known about the physical world except its structure are either false or trivial, and I am somewhat ashamed at not having noticed the point for myself.
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> It is of course obvious, as you point out, that the only effective assertion about the physical world involved in saying that it is susceptible to such and such a structure is an assertion about its cardinal number. (This by the way is not quite so trivial an assertion as it would seem to be, if, as is not improbable, the cardinal number involved is finite. This, however, is not a point upon which I wish to lay stress.) It was quite clear to me, as I read your article, that I had not really intended to say what in fact I did say, that *nothing* is known about the physical world

[4] "Mr. Russell's 'Causal Theory of Perception'" (1928).

[5] Printed in the *Autobiography*, p. 493. Russell kept two technical letters from Newman following his reply and designated them as "Shop", i.e. as candidates for a published selection of his philosophical correspondence.

> except its structure. I had always assumed spacio-temporal continuity with the world of percepts, that is to say, I had assumed that there might be co-punctuality between percepts and non-percepts, and even that one could pass by a finite number of steps from one event to another compresent with it, from one end of the universe to the other. And co-punctuality I regarded as a relation which might exist among percepts and is itself perceptible.
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> I have not yet had time to think out how far the admission of co-punctuality alone in addition to structure would protect me from your criticisms, nor yet how far it would weaken the plausibility of my metaphysic. What I did realise was that spacio-temporal continuity of percepts and non-percepts was so axiomatic in my thought that I failed to notice that my statements appeared to deny it.... (24 April 1928)

Newman was a prominent mathematician (aspects of his career are discussed in Andrew Hodges' biography of Alan Turing). Russell acknowledged his "criticisms and suggestions" in the chapter on the construction of points in *The Analysis of Matter*. Russell does not mention Newman in *My Philosophical Development*. I think it likely that Russell believed his later epistemological theories avoided Newman's criticism—they allow knowledge by acquaintance of non-structural features of the empirical world—and preferred to avoid discussion of the mistaken theory of *The Analysis of Matter*. Whether appeal to "acquaintance" makes the post-Newman theories more plausible seems to me to be an open question. It is still an important and unsolved problem to determine the role played by the concept of structure in explaining our knowledge of the empirical world.[6]

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[6] For a discussion of Newman's critique and its bearing on the structuralist epistemologies of Russell, Schlick and Carnap, see W. Demopoulos and M. Friedman, "Critical Notice: Bertrand Russell's *The Analysis of Matter*: Its Historical Context and Contemporary Interest" (1985). For a discussion of Richard Braithwaite's use of Newman's critique against Eddington, see my "Addendum to Demopoulos and Friedman (1985)" (1989).

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