Was the axiom of reducibility a principle of logic?

by Bernard Linsky

THE TITLE OF this paper is in the past tense to indicate that the question it will address is whether the Axiom of Reducibility is a principle of logic according to the view of logic that Russell had when writing the first edition of *Principia Mathematica*.[1] It is often said that Logicism was a failure because when it avoided the Scylla of contradiction in Frege's system it fell into the Charybdis of requiring obviously non-logical principles at Russell's hands. The axiom of reducibility is cited along with the Axiom of Infinity as a non-logical principle which Russell had to add to his system in order to be able to develop mathematics.

I want to consider this criticism of the axiom from several points of view. Why is it thought that the axiom of reducibility is not a principle of logic? What reasons does Russell actually give for doubting its logical status? Are they good reasons?

1. OBJECTIONS TO THE AXIOM

The Ramified Theory of Types of the first edition of *Principia* goes beyond the divisions of a "simple" theory between individuals, first-order propositional functions which apply to individuals, second-order functions which apply to first-order functions, etc. It further subdivides each of those groups according to the range of the bound variables used in the definition of each propositional function. Russell often used the example of the predicate "x has all the properties of a great general" which will itself be a property of great generals, but not one within the range of that particular quantifier "all". It will

1 Page references will be to A.N. Whitehead and B. Russell, *Principia Mathematica to *56* (Cambridge: Cambridge U. P., 1962).

thus define a propositional function of a higher type than any of the variables bound by that quantifier. Similarly a difference of type would have to be marked in the theory of the real numbers between the property of "belonging to the set X" and "being the least upper bound of the set X", as the latter is defined in terms of a quantifier ranging over *all* members of X. The consequent division of numbers, and types generally, into different orders makes much ordinary reasoning seemingly invalid. The axiom of reducibility eases this difficulty. It asserts that for any propositional function, of whatever type, there is a *coextensive* propositional function of the lowest type compatible with its arguments, called a "predicative" propositional function.[2] Thus there will be a propositional function true of just those individuals with all the properties of a great general, which itself is of the same type as those properties quantified by "all". There will be a propositional function true of the least upper bound of a set X of the same type as the function "is a member of X", and so on. The ready availability of predicative propositional functions, guaranteed by the axiom of reducibility, allows them to substitute for classes in Russell's famous "no-class" theory of classes.[3] Sentences seemingly about "the class of ϕs" are to be analyzed as existential sentences asserting that some predicative propositional function coextensive with ϕ has the given properties. The axiom of reducibility thus both avoids some of the stringency of the ramified theory of types and guarantees the existence of the predicative propositional functions that replace classes. It is these very virtues that have been the source of doubt about the axiom.

2 Strictly speaking there are an infinite number of axioms of which the following applies to one-place propositional functions $*12.1 \vdash : (\exists f) : \phi x . \equiv_x . f!x$. Thus for every propositional function ϕ there is a materially equivalent *predicative* function f of the lowest order compatible with arguments of the same type. I follow the formulation of the ramified theory of types of Alonzo Church, "Comparison of Russell's Resolution of the Semantical Antinomies with That of Tarski", *J. of Symbolic Logic*, 41 (1976): 747–60. The following brief sketch will be enough for what follows. Variables and constants are assigned *(r-)types*, i for individuals and $(\beta_1, \beta_2, \ldots, \beta_m)/n$ for m-ary propositional functions of *level n* with arguments of types $\beta_1, \beta_2, \ldots, \beta_m$. (Propositional variables will have r-type $(\ )/n$.) $n \geq 1$ is the *level* of the expression (when $n=1$ it is *predicative*). $(\alpha_1, \ldots, \alpha_m)/k$ is *directly lower* than $(\beta_1, \ldots, \beta_m)/n$ if $\alpha_1=\beta_1, \ldots, \alpha_m=\beta_m$ and $k<n$. The *order* of an individual variable (type i) is 0, the *order* of a variable of type $(\beta_1, \ldots, \beta_m)/n$ is $N+n$ where N is the maximum of the orders of $\beta_1, \ldots, \beta_m$. The force of the division of types is felt in the restriction on well-formed formulas: $f(x_1, x_2, \ldots, x_m)$ is a *wff* iff f is a variable or constant of type $(\beta_1, \ldots, \beta_m)/n$, x_1 is a variable or constant of type β_1 or directly lower, ..., and x_m is a variable or constant of type β_m or directly lower. That propositional functions defined with quantifiers will have a raised type is made explicit by comprehension schemas including: $\exists f^{(\beta_1, \ldots, \beta_m)/n} f(x_1, \ldots, x_m) \equiv_{x_1, \ldots, x_m} P$ where the bound variables of P are of order *less* than the order of f and the free variables and constants are not greater. The axiom of reducibility is very similar: $(\phi^{(\beta_1, \ldots, \beta_m)/n})(\exists f^{(\beta_1, \ldots, \beta_m)/1})\phi(x_1, \ldots, x_m) \equiv_{x_1, \ldots, x_m} f(x_1, \ldots, x_m)$.

3 The "theory" is stated as an axiom schema very like the contextual definition of definite descriptions: $*20.1\ f\{\hat{z}(\psi z)\} . \equiv : (\exists \phi) : \phi!x . \equiv_x . \psi x : f\{\phi!\hat{z}\}$.

Objections to the axiom of reducibility often combine several related points, in particular, that it makes an existence claim that is not purely logical, that it seems *ad hoc* and so lacks the obviousness of genuine logical principles, that the whole ramified theory of types of which it is a part is itself not purely logical, and, indeed, borders on incoherence since it seems to take back with the axiom of reducibility all of the ramification of types which is its hall mark. I consider these objections in turn.

It is often suggested that the axiom of reducibility is like the axiom of infinity in making an existence claim and as such is not a principle of logic. Viewed as a comprehension scheme, or perhaps like the axiom of separation, the axiom would look like the axioms of Zermelo–Fraenkel set theory, which are seen as rivals to logicism as a foundational scheme.[4] Set theory is viewed as having given up the project of reducing mathematics to logic, and instead as having resorted to just postulating the existence of those distinctively mathematical entities that are needed to develop the rest of mathematics. The axiom of reducibility also seems to postulate the existence of peculiarly mathematical entities, predicative propositional functions, and so would seem to be of a piece with set theory.

Russell himself was suspicious of a priori existence proofs. He often claimed that logic cannot prove the existence of certain things, such as God, or how many things there are in the world.[5] But this is not a very good reason to say that logicism with the axiom of reducibility is a failure. One could have known that no logicist program could work if the problem lies in proving existence claims. In arithmetic we can prove many existence claims, for example, that there is an even number between 4 and 8. Since we could prove an existence claim, if logicism were correct, then logic could prove an existence claim, which is impossible, Q.E.D. There is no need to find the particular source of the existence claim to disqualify the logicist program, we know it must be there from the start. (One might provide an analysis of mathematical existence claims that gives them some other logical form, just as Russell's theory of descriptions analyzes descriptions as not really singular terms. For Frege and Russell, however, there were legitimately *logical* objects, whether courses of values or propositional functions, and quantification over math-

4 Gödel says that in the realist, simple, theory which ought to replace the ramified theory of types, "... the place of the axiom of reducibility is now taken by the axiom of classes, Zermelo's *Aussonderungsaxiom* ..." in "Russell's Mathematical Logic", in P.A. Schilpp, ed., *The Philosophy of Bertrand Russell*, The Library of Living Philosophers (Evanston, Ill.: Northwestern U., 1944), pp. 140–1.

5 See *Introduction to Mathematical Philosophy* (London: Allen and Unwin, 1919), p. 141.

ematical objects was to be reduced to quantification over them.) Surely then, it is no objection to the status of the axiom of reducibility as part of a logicist program, that it asserts an existence claim. The objection must rely on the nature of the existence claim which is made. One might argue that logic should make no assumptions about the number of *individuals*, or lowest level entities, that there are. While one might be allowed to avoid a free logic, and assume that there is at least one, the assumption of a countable infinity of objects, as made by the axiom of infinity, might lie outside of purely logical principles. As merely a claim about the existence of propositional functions, however, this restriction does not bar the axiom of reducibility from logic.

Another, related, objection is that the axiom of reducibility simply undoes the construction of the hierarchy of propositional functions that is the very purpose of ramifying the theory of types.[6] If the higher type propositional functions of a given order are seen as constructed from those of lower type, then adopting the axiom of reducibility would be self-defeating. If all the classes there are have been already constructed at the first level, then all the convoluted ways of producing defining conditions for classes out of simpler classes do not really accomplish anything. Doesn't this make the constructions pointless? Quine has argued that the axiom of reducibility is a platonistic existence assertion and so violates the constructivist motivation for the ramification of the theory of types. Quine's objection thus combines the two lines of criticism I am discussing. He charges that the axiom both undoes the effect of the ramification and commits the theory to a platonistic view of propositional functions (which, when its use/mention confusions are cleared up, amounts to a theory of sets).

Criticism of the axiom of reducibility is sometimes more indirect. Following Ramsey, it is often charged that the ramified theory of types involves an unnecessary complication of the simple theory of types, one introduced in order to deal with semantic paradoxes that are not properly logical paradoxes.[7] The axiom of reducibility then inherits the non-logical character of the system to which it belongs. Accordingly, a defence of the axiom will require a defence of the ramified theory of types itself as a system of logic. The answer to these questions comes in seeing the ramified theory of types as a system of intensional logic which includes the "no-class" account of sets, and indeed the

6 Quine presents this objection in his *Set Theory and Its Logic* (Cambridge, Mass.: Harvard U. P., 1969), pp. 249–58. See also Myhill cited in note 11.

7 F.P. Ramsey, "The Foundations of Mathematics (1925)", in his *The Foundations of Mathematics and Other Logical Essays* (London: Routledge & Kegan Paul, 1931), pp. 1–61. Quine makes this objection as well. On his account, the set-theoretic nature of the axiom is hidden by its quantification over propositional functions which are creatures of the confusion of use and mention, of semantics and ontology.

whole development of mathematics, as just a part. A defence of the axiom of reducibility, then, leads to a defence of the whole ramified theory of types and the logicist project to which it belongs.

2. THE ORIGIN OF THE AXIOM

Although this paper is concerned with the justification of the axiom of reducibility within Russell's views at the time of writing *PM*, a look at the earlier history of the principle will help to explain its role in his thinking. One stereotype of the evolution of Russell's thought is that he first had a simple theory of types, designed to handle his original paradox of sets, later adding the "ramification" in order to handle the semantic paradoxes, and then, realizing that the ramification made impossible the project of reducing mathematics to logic, introduced the axiom of reducibility, undoing the effect of the ramification.

This account is quite wrong. It may follow a natural ordering of topics in a presentation of the theory of types, which Russell himself uses, but it does not present any historical development of the theory. To begin with, type theory was effectively ramified from its earliest formulations around 1905–06. The distinctive feature of ramification is to distinguish propositional functions which take arguments of the same type by the ranges of the bound or "apparent" variables that occur in them. Russell's early attempts at solving the paradoxes deliberately avoided any division of types. This was in part due to his desire to see all quantifiers as unrestricted, which was in turn due to belief in the universal character of logic.[8] But once Russell accepted Poincaré's analysis of the paradoxes as due to a vicious circle, he immediately saw that the ranges of universal quantifiers in propositional functions needs to be restricted to a specific totality, in other words, the need for ramification.

Just before accepting the need for types Russell held his "substitutional" theory according to which all quantifiers are unrestricted but the (seeming) quantifiers over propositional functions *are* restricted. This is not a real restriction of quantifiers, however, because expressions for propositional functions are "incomplete symbols" which can be eliminated by contextual definitions. The real range of quantifiers is all objects and all propositions. The next stage of development for Russell was to see the need for type distinctions among propositions. In the paper "On 'Insolubilia' and Their Solution by Symbolic Logic",[9] Russell adopts the vicious-circle principle as the analysis of the paradoxes and the consequent need for at least type distinctions among proposi-

8 See Peter Hylton, "Russell's Substitutional Theory", *Synthese*, 45 (1980): 1–31.

9 Reprinted in *Essays in Analysis*, ed. Douglas Lackey (New York: Braziller, 1972), pp. 190–214.

tions, while still denying the reality of propositional functions with his "substitutional" view. Yet he immediately acknowledges that "for every statement containing x and an apparent variable is *equivalent*, for all values of x, to some statement ϕx containing no apparent variable" (p. 212). Thus for example, propositions about *all* propositions of a given sort, say all those asserted by Epimenides, must be (materially) equivalent to some proposition which does not include such quantification. This claim amounts to an axiom of reducibility.

What reason did Russell give for believing such a claim? This passage appears in response to the criticisms of the logicists' account of induction that Poincaré based on the vicious-circle principle. For a logicist, the principle of induction says that *all* properties possessed by 0 and hereditary with respect to the successor relation are possessed by all numbers. The vicious-circle principle requires that the quantification over "all" properties must be restricted to avoid reference to any impredicative properties defined with apparent variables ranging over numbers.[10] As this is required in so many uses of the induction principle, adopting the vicious-circle principle seems to "destroy many pieces of ordinary mathematical reasoning" (p. 211). Thus Russell saw that the adoption of a type theory with the consequent restriction of bound variables to ranges of significance undercuts those principles which seem to rely on unrestricted quantification (even when the real quantifiers are only those ranging over objects and propositions). The definition of identity, that x and y are identical if and only if they share *all* properties, is another such example, which will be discussed more below.

The simultaneous appearance of types and the axiom of reducibility is all the more remarkable for the fact that at the time Russell did not even see the hierarchy of types as one of properties or propositional functions, but rather simply of propositions. In the "Insolubilia ..." paper he was still in the grip of the "substitutional" theory which attempted to define away propositional functions with contextual definitions and the notion of replacing one object by another in a proposition. The axiom of reducibility, then, did not make its appearance as a view about the existence of propositional functions, but rather as a necessity given the need to restrict quantifiers to types. Given that restriction, generalizations over all properties must be replaceable by quantifiers ranging over only a certain type of properties. The axiom of reducibility guarantees that restricting attention to properties of only one type will not invalidate standard patterns of reasoning about *all* properties because if any property does not apply to an object one of that chosen type will not. Thus induc-

10 Poincaré would seemingly ban all impredicative properties, while Russell would restrict them to a distinct type.

tion says that any property possessed by 0 and hereditary with respect to the successor relation will be possessed by all numbers. The axiom of reducibility says that any property of numbers will be coextensive with a predicative property so that if a number lacks a property it will lack a coextensive predicative property and if it has a property it will have a coextensive predicative property. Consequently Russell is able to use as his definition of identity the weaker claim that $x = y$ if and only if x and y have all the same predicative properties ($*13.01$ $x{=}y\ .\ =\ :\ (\phi):\ \phi!x\ .\supset .\ \phi!y$ Df). The axiom of reducibility allows one to restrict attention to the predicative properties of x and y. A justification of the axiom of reducibility, then, must consist in a reason to believe that one can so restrict attention to the properties of one preferred type, the predicative functions.

In what follows I wish to explain the role of the axiom of reducibility in Russell's thinking at the time of *Principia Mathematica* when his ontology was considerably different. In fact, I believe, the ontology he had in the background in *PM* is what provided the justification for the axiom at that time. What can be learned from its earlier appearance, however, is that the axiom was an integral part of the notion of a theory of types, not some afterthought used to patch up a defect resulting from the addition of ramification to an earlier, simpler theory of types with reducibility that needs justification, not the axiom on its own.

3. RUSSELL'S DOUBTS ABOUT THE AXIOM

Let us turn, then, to Russell's views in *Principia Mathematica*. To evaluate those it is instructive to look at his reasons for later *abandoning* the axiom of reducibility which marks one of the characteristic differences between the systems of the first and second editions. What reasons did Russell himself give for doubting that the axiom is logical? He says very little about it in the Introduction to the second edition of *PM*. All he says is:

> One point in regard to which improvement is obviously desirable is the axiom of reducibility ($*12.1.11$). This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can rest content. (P. xiv)

Russell's objection is hardly explicit. We know what the objection doesn't amount to by looking at the argument *for* the axiom in the Introduction to the first edition:

> That the axiom of reducibility is self-evident is a proposition which can hardly be

> maintained. But in fact self-evidence is never more than a part of the reason for accepting an axiom, and is never indispensable. The reason for accepting an axiom, as for accepting any other proposition, is always largely inductive, namely that many propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false,and nothing which is probably false can be deduced from it. (Pp. 59–60)

He goes on to say that the inductive evidence for the axiom of reducibility is good so the real problem is that:

> ... although is seems very improbable that the axiom should turn out to be false, it is by no means improbable that it should be found to be deducible from some other more fundamental and more evident axiom. It is possible that the use of the vicious-circle principle, as embodied in the above hierarchy of types, is more drastic than it need be, and that by a less drastic use the necessity for the axiom might be avoided. (P. 59)

In the second edition Russell says that by following Wittgenstein's example in making the logic extensional one can avoid the axiom of reducibility but still prove many useful theorems.[11] So this objection to the axiom of reducibility was that it should be proved from more self-evident axioms, or avoided in the proof of the desired theorems by adopting a different axiom. Russell then does not require that the axiom be self-evident, and does not express any doubt about its truth; rather he thinks it redundant. (It is important to note that Russell includes *logical* axioms among those that need not be self-evident, as long as they have the right deductive strength.) At the time of the first edition he thought that the axiom might be redundant because an excessively strong form of the vicious-circle principle had introduced too many types which then had to be integrated with the axiom of reducibility. The vicious-circle principle forbids the existence of any entity such as a totality, or propositional function, which depends on itself in the wrong way. If a propositional function depends on a totality, then it cannot be a member of that totality, and hence belongs to a new type, thus ramifying the type theory. But with the principle of extensionality anything true of one propositional function will be true of every coextensive one, so the *only* thing on which a

11 Russell even claims this about the ramified theory of types in the first edition. This claim is shown incorrect in John Myhill,"The Undefinability of the Set of Natural Numbers in the Ramified *Principia*", in George Nakhnikian, ed., *Bertrand Russell's Philosophy* (London: Duckworth, 1974).

propositional function can depend is its members, and so the type theory cannot be ramified. Extensionality thus weakens the force of the vicious-circle principle by limiting what a propositional function can depend on, and thus what it could depend on viciously. Other principles limiting the dependence of propositional functions by identifying some which are distinguished in the full ramified theory would have the same effect. It is clear that even at the time of writing the Introduction to the second edition Russell thought that the principle of extensionality was too strong. Thus this doubt about the axiom of reducibility was a doubt about the vicious-circle principle and the number of type distinctions it introduces, rather than a doubt about the existence claim made by the axiom.

A richer guide to Russell's thinking about the axiom of reducibility, however, is in the *Introduction to Mathematical Philosophy*, written in 1918 between the two editions of *PM*. There he expresses doubts about whether the axiom has the character of a regular principle of logic. One worry is that the axiom is not general or widely enough applicable. Thus: "This axiom, like the multiplicative axiom and the axiom of infinity, is necessary for certain results, but not for the bare existence of logical reasoning" (p. 191). He goes on to explain that it does not have the universal applicability of, for example, the quantifier laws and so *could* be just added as a special hypothesis whenever it is used. Here, obviously, Russell is concentrating on the use of the axiom of reducibility in the construction of the natural and real numbers. It is not so clear that the axiom would seldom be used outside of the theory of classes and mathematics, as can be seen from its role in proofs about identity. The need for impredicative definitions in mathematics, in particular for the notion of the *least upper bound* of a set of real numbers, has led many to see the "certain results" to which Russell alludes, to be only a limited part of higher mathematics. Indeed, the appeal of developing a "predicative" analysis has suggested that the axiom is in fact debatable. But of course Russell's project was to develop mathematics and so his attention was precisely on the role of impredicative definitions in mathematics. Attention to the identity of indiscernibles should remind us of the frequency of talk of all properties of a thing within metaphysics. Since without the axiom of reducibility such talk is banned, we see that the axiom is not just needed for the development of higher mathematics.

Russell goes on to question the *necessary* truth of the axiom of reducibility, demanding that a logical truth be true in "all possible worlds" and not just in this "higgledy-piggledy job-lot of a world in which chance has imprisoned us" (p. 192). So Russell has qualms about both the generality of the axiom and its necessary truth, features he took to be characteristic of logical truths. It is these remarks that are most likely the source of many of the claims that the axiom of reducibility is not a principle of logic because it is not as intuitively obvious enough, or a general enough principle of reasoning.

Russell's objections here are a mixed lot. He has a theoretician's concern that the axiom is *ad hoc* and could be replaced by more basic principles. He is also concerned that the axiom is not a *necessary* truth, another feature which does not distinguish logic from a very general metaphysical theory of the world. Russell's concern that the axiom is only of use in mathematics, and not a general principle of reasoning, shares this character. Given Russell's earlier remarks that axioms need not be directly evident, it is hard to attribute to him a view of the nature of logic which marks it off from a more substantive, metaphysical theory other than by differences of degree.

These, then, were Russell's various qualms about the axiom of reducibility. What can be said in defence of the axiom of reducibility? Should Russell have had such qualms about it given its role in his logic?

4. *PRINCIPIA MATHEMATICA* AS INTENSIONAL LOGIC

I wish to argue that in fact the axiom plays a crucial role in the logic of *PM* because of the nature of propositional functions, in particular, the distinctive role of predicative propositional functions. It is a realist view about propositional functions in particular, a view about predictive propositional functions as encapsulating the real features of objects, which serves as a justification for the axiom within the philosophical system of *PM*.

First it is important to get clear about the role of the axiom of reducibility in Russell's theory of classes. It does not just undo the effects of the ramification as the "constructivist" reading suggests. Here I refer to the recent work of Alonzo Church and Leonard Linsky on what might be called the "intensional interpretation" of *Principia Mathematica*.[12] They have argued that despite the central project in *PM* of developing mathematics, which is a thoroughly extensional subject, the logic of *PM* is fundamentally intensional. The intensional nature of the logic explains many otherwise puzzling features of its presentation. One example is the role of scope in the theory of definite descriptions. In extensional contexts the scope differences do not have any logical effect, as long as the descriptions are proper. Why then, are they introduced with such care? Likewise several features of the "no-class" theory also

12 See Alonzo Church, *op. cit.*, note 2, and Leonard Linsky, *Oblique Contexts* (Chicago: U. of Chicago P., 1983), Appendix, as well as Warren Goldfarb, "Russell's Reasons for Ramification", in C.W. Savage and C.A. Anderson, eds., *Rereading Russell: Essays on Bertrand Russell's Metaphysics and Epistemology*, Minnesota Studies in the Philosophy of Science, Vol. XII (Minneapolis: U. of Minnesota P., 1989).

depend on the intensionality of propositional functions. In particular the significance of the axiom of reducibility depends on the logic being intensional. Linsky's argument goes like this. Russell's contextual analysis of classes, the "no-class" theory, is very similar to the analysis of definite descriptions, including the possibility of scope distinctions. Just as it is true, and *proved* in *PM* (∗14.3) that scope distinctions for definite descriptions make no difference in extensional contexts, it is true that scope distinctions make no difference in extensional contexts for sets. That this is not proved, Linsky argues, shows that Russell had in mind the application of *PM* to mathematical contexts where extensionality rules. That it *could be* expressed and proved shows that the logic was set up to handle intensional contexts. Furthermore, there are two conditions to be met for descriptions such as "the *F*" to behave like names with regard to scope and substitution. It is not only necessary that one restrict oneself to extensional contexts, but that the description be proper (i.e. that there be one and only one *F*). A similar requirement that class abstracts behave like names is that, in addition to occurring in extensional contexts, the requisite predicative propositional function must exist. But that is precisely what the axiom of reducibility says. It is, as Gödel remarked, a comprehension principle, but this is in the context of an intensional logic, one capable of expressing much more than just the extensional sentences of mathematics. So, Linsky's argument goes, Russell's ignoring of scope indicators for class abstracts, unlike his use of them with descriptions, shows that he saw himself as restricting his talk of classes to talk of *extensional* mathematical contexts, but not so restricting the logic of *PM*.

My interest is not in establishing the intensional character of the logic of *PM* but rather the logical character of the axiom of reducibility. The axiom clearly plays a crucial role in the theory of classes, given Russell's *particular* contextual definition of classes. While having the force of a comprehension principle, it does not assert the existence of some new, non-logical category of entity, but rather just of a *predicative* propositional function coextensive with an arbitrary propositional function. Because the extensional theory of classes is only part of the whole of logic, the axiom of reducibility does not just undo the ramification of the theory of types. It is crucial in the reduction of classes to logic, and unless one assumes that logicism is false and so automatically any talk of classes is not part of logic, it seems to be a quite legitimate logical notion for Russell—provided, of course, that a claim about the existence of a *predicative* propositional function with a given extension can be seen as a logical principle.

What of the charge that the axiom of reducibility undoes the whole point of the hierarchy of types of propositional functions? That assumes that the only point of the hierarchy of types is to represent the possible constructions of propositional functions and hence of classes. What view could hold, rather, that all the classes have been constructed at the level of predicative functions? The answer is that the propositional functions of higher types are needed to capture intensional phenomena. The predicative functions are needed to reconstruct the extensional part of logic, that part which deals with the extensions of predicates, or classes. The higher types are needed for purely intensional phenomena, cases where the same class is picked out by distinct intensions, i.e. propositional functions. This view requires seeing the ramified hierarchy of *PM* not as a constructivist theory of classes but rather as a theory of propositional functions which includes as a part the theory of classes, but which does much more. Of course predicative propositional functions are not extensional. They can be distinct, yet coextensive. Rather, all extensional talk about classes is analyzed as a general (existential) claim about predicative propositional functions.

What then is so special about predicative propositional functions that one can adopt a "comprehension" principle asserting the existence of a coextensive predicative propositional function for every arbitrary propositional function?

5. THE AXIOM OF REDUCIBILITY AND THE IDENTITY OF INDISCERNIBLES

The distinctive character of predicative propositional functions can be seen in the details of Russell's charge that the axiom of reducibility is not a necessary truth. Russell says that

> The axiom, we may observe, is a generalised form of Leibniz's identity of indiscernibles. Leibniz assumed, as a logical principle, that two different subjects must differ as to predicates. Now predicates are only some among what we called "predicative functions," which will include also relations to given terms, and various properties not to be reckoned as predicates. Thus Leibniz's assumption is a much stricter and narrower one than ours. (*Introduction to Mathematical Philosophy*, p. 192)

Russell goes on to say that the axiom *seems* to hold of the actual world, for

> ... there seems to be no way of doubting its empirical truth as regards particulars, owing to spatio-temporal differentiations: no two particulars have exactly the same spatial and temporal relations to all other particulars. But this is, as it were, an accident, a fact about the world in which we happen to find ourselves. (*Ibid.*)

How is the axiom of reducibility a "form" of the identity of indiscernibles? The identity of indiscernibles says that $x=y$ just in case all the same predicates (propositional functions) apply to x and y. The axiom of reducibility allows

one to restrict the principle to only requiring the sharing of all the same *predicative* propositional functions. For suppose that ψx but not ψy where ψ is not predicative. Then there is a predicative function ϕ coextensive with ψ, and thus true of x, which distinguishes it from y with predicative functions alone. Accordingly, Russell's definition of identity at ∗13.01 is this "restricted" form of the identity of indiscernibles, and the above reasoning is equivalent to the proof of theorem ∗13.101 ($\vdash: x=y. \supset .\psi x \supset \psi y$), one "half" of the more familiar identity of indiscernibles. (The more controversial half of the principle, that if x and y have all the same properties, then $x=y$, follows immediately from the fact that if they share *all* properties, they share all predicative properties). The axiom of reducibility is possibly a stronger principle than is needed to prove the identity of indiscernibles from the definition of identity, for the proof only requires that objects sharing all predicative properties share all properties of any type, whereas the axiom of reducibility accomplishes this by providing a predicative property which is coextensive with each arbitrary property. This may have been one of the points where Russell suspected that the axiom of reducibility might be replaced by a simpler principle. Still, however, he accepts the axiom to the extent of calling it a "generalized form" of the identity of indiscernibles. Reasons for accepting that generalized principle would, for Russell at least, provide a justification for the axiom of reducibility.

What then is the reason for accepting the axiom of reducibility and its accompanying definition of identity? I believe that for Russell it was not just a matter of stipulation that made classes coincide with predicative propositional functions. Rather, he thought that predicative propositional functions really characterize the genuine properties which individuate things in the world. Objects don't always differ in their monadic qualities, as Russell had argued against Leibniz. It was a distinctive feature of Russell's philosophy that he argued for the reality of relations and hence the irreducibility of some relational properties. Thus "being two miles from x" is a perfectly good relational property, not reducible to any monadic properties of x. It is the original stock of one-place properties, then added to it all the possible relational properties, and boolean combinations of them, which constitute the predicative propositional functions. It is because Russell saw predicative propositional functions as expressing more than just monadic qualities that he speaks of the axiom as a "generalized" version of Leibniz's principle. Leibniz, according to Russell's account, would presumably endorse an even stronger "axiom of reducibility" to the effect that every propositional function is equivalent (by analysis and not just coextensiveness) with some conjunction of monadic qualities.

Russell feared that it might be a matter of arbitrary postulation, or at least contingent, that the predicative propositional functions should suffice to distinguish all objects. Why shouldn't some higher-type property allow us to distinguish objects? Russell considers examples like "having all the qualities of a great general" and "being a typical *F*" (where the latter seems to mean something like having all the properties shared by *most Fs*). The answer can be seen in the very nature of these examples. Higher-type propositional functions do not really introduce new properties of things. They may characterize new ways of thinking of things or of classifying them, but they don't introduce any new real properties. Russell himself did not keep clear enough the distinction between propositional functions and these real properties of things, universals. That he sometimes made such a distinction is clear. It is certainly necessary to make such a distinction for him to be able to argue, as he did in the Introduction to the first edition of *PM*, both that propositional functions depend on their values, propositions, in a way that makes the vicious-circle principle applicable, and that propositions are to be analyzed according to the "multiple relation" theory into universals and particulars which are the real furniture of the world. Thus propositional functions and universals are separated by propositions in the hierarchy of dependence which the vicious-circle principle enjoins us to observe. A simpler way of seeing that Russell was committed to such a distinction is to observe that it is of the essence of propositional functions that they allow compounding by logical connectives; thus "being red or blue" is a perfectly acceptable propositional function.[13] Yet universals are only discovered as the end result of analysis—they can be objects of acquaintance, but are simple. They correspond with the primitive predicates of a fully analyzed language, not with the arbitrarily complex propositional functions. This distinction is not very clear in *PM*, however, especially as Russell did not ever explicitly mark it or even observe it at all times.[14]

If one grants that Russell had in mind some distinction between propositional functions and universals that have a metaphysically important role as what underlies the real qualities of things, then it is clear that predicative propositional functions inherit some of that character. One need not argue that any two objects will be distinguished by some universal that one has and the other lacks. That would be to claim that all objects have a unique *nature*, an implausible metaphysical assumption. But one might hold that *something* accounts for the particularity of objects, if not their qualities or natures, then perhaps their locations. If one holds a relational view of space, then the view that it is spatio-temporal location which individuates particulars is one which

13 See *Principia Mathematica*, p. 56.

14 See my "Propositional Functions and Universals in *Principia Mathematica*", *Australasian Journal of Philosophy*, 66 (Dec. 1988): 447–60, for a discussion of this point. See Nino Cocchiarella, "Russell's Theory of Logical Types and the Atomistic Hierarchy of Sentences" in *Rereading Russell*, pp. 41–62, for an explicit argument that Russell *identified* propositional functions and universals.

allows a relation to individuate.[15] If one were only interested in individuating objects two at a time then universals or relational properties might be sufficient. One could say of objects with different natures that one has *F* and the other does not, where *F* is one universal in the nature of the object. For objects with the same nature one could use relational properties as one does with spatio-temporal relations with concrete particulars. When whole classes of objects are involved one may require boolean combinations of universals as well as relational properties. Some of the objects may be distinguished by being *F*, others by being *G*, others by *not* being *H*, and so on, so that the *predicative* propositional function "being *F* and *G* but not *H* ..." is needed to mark off the class. Russell's qualms about the axiom of reducibility being contingent amount, then, to the worry that such a scheme of spatio-temporal relations is merely contingent.

As I have presented it, the axiom of reducibility marks out the special role of predicative propositional functions, which coincide with classes, and indicates the properties which individuate things in the world. One still needs the whole hierarchy of ramified propositional functions to handle all the things that can be said of the world, or thought of it, the whole realm of intensional phenomena. This makes the axiom of reducibility out to be a metaphysical principle. It is one of great generality, however, certainly unlike any principle about numbers as abstract entities or of the sort that might occur in any special science. Still, this accounts for Russell's qualms about it. While it is a very general principle of metaphysics does it have the necessity required of a principle of logic? Is it *logically* necessary? Russell was not sure. Ultimately it was the lure of doing without the ramified theory by adopting the principle of extensionality that made him give up the principle. But giving up on intensional phenomena was an extreme solution. It undid the whole relation between classes and propositional functions that was at the heart of the first edition of *Principia Mathematica*.

The axiom of reducibility was an integral part of the theory of types. From the beginning it was clear that if a theory of types requires restricting the ranges of bound variables to a given range of significance, or type, then those principles which seem to require quantifying over *all* properties must be stronger than necessary. Identity, while seeming to require the sharing of all properties, really only requires the sharing of properties of the lowest, predicative order; and the existence of sets, which should allow a set for all predicates, really only requires sets for all predicative properties. As well, all numbers, and properties of numbers, to which the induction principle will apply,

15 See D.M. Armstrong's *Universals and Scientific Realism* (Cambridge: Cambridge U. P., 1980), Chap. 11, for a discussion of spatio-temporal location and particularity.

are already represented at the lowest type. Some criticisms of the axiom and its role seem to require forgetting the intensional nature of the logic, and hence the use for all those additional, non-predicative propositional functions. Russell was aware of the need for an independent justification for the principle, one that showed how conclusions about numbers, classes and identity could be settled by only considering predicative propositional functions. That predicative functions are all that is needed for the theory of classes and numbers follows directly from the "no-class" theory and the definition of numbers as classes. The adequacy of predicative propositional functions for the definition of identity and other more "metaphysical" or non-mathematical applications of the logic of *PM*, comes from the distinction between propositional functions and universals, and the proximity of predicative propositional functions to universals. Predicative propositional functions mark the real kinds in the world. That this is so, is a fairly substantive metaphysical claim about the world, and so not obviously of same generality or as clearly "necessary" as the other principles of Russell's logic. It was, however, also not obviously out of place in the logic of *Principia Mathematica.*[16]

Department of Philosophy
University of Alberta, and
Center for the Study of Language and Information
Stanford University

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