RUSSELL TO FREGE, 24 MAY 1903: "I BELIEVE I HAVE DISCOVERED THAT CLASSES ARE ENTIRELY SUPERFLUOUS"

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It was his consideration of Cantor's proof that there is no greatest cardinal, Russell recalls in *My Philosophical Development*, that led in the spring of 1901 to the discovery of the paradox of the class of all classes not members of themselves. "Never glad confident morning again", were Whitehead's reported words (*MPD*, p. 58). Whitehead was, of course, wrong. Russell had many new confident mornings. One in particular apparently occurred on 19 May 1903. On 23 May Russell wrote in his journal: "Four days ago I solved the Contradiction—the relief of this is unspeakable" (*Papers* 12: 24). The "solution" was communicated to Whitehead, who responded by telegram: "Heartiest congratulations Aristoteles secundus. I am delighted" (Clark, p. 111). But it was not long before Russell knew his proposal was inadequate. On the form he would scrawl: "*A propos* of solving the Contradiction. [But the solution was wrong]" (*ibid.*; Russell's brackets). The episode is intriguing. What was the proposal?

Russell's correspondence with Frege sheds some light. On 20 October 1902 Frege had sent a letter suggesting a way to avoid the contradiction. Frege's suggestion, formulated in a great hurry so that it might appear in the Appendix of Vol. II of the *Grundgesetze der Arithmetik*, was to abandon his Basic Law V,

$$\acute{z}\phi z = \acute{z}\Theta z\,.\equiv.\,(x)(\phi x\,.\equiv.\,\Theta x),$$

and to replace it by

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$$\text{V}' \quad \acute{z}\phi z = \acute{z}\Theta z\,.\equiv.\,(x)(x \neq \acute{z}\phi z \;\&\; x \neq \acute{z}\Theta z :\supset: \phi x\,.\equiv.\,\Theta x).$$

The underlying idea was to accept that one concept may have the same extension as another even though the two concepts are not coextensive.

Though he thought the underlying idea "probably correct",[1] Russell expressed reservation in a reply of 12 December:

> ... I find it difficult to accept your solution even though it is probably correct. Do you deny, e.g., that all classes form a class? And if this is admitted, then it is possible that — $a \cap a$. Moreover, the class of non-humans is a non-human. Otherwise it must be admitted that not all objects fall either under *a* or not-*a*; namely, if *a* is a range of values, then *a* falls neither under *a* nor under not-*a*. This contradicts the law of excluded middle, which will be inconvenient to say the least.[2] (Frege *1980*, p. 151)

Frege's response of 28 December does not mention the contradiction, and Volume II of his *Grundgesetze* appeared early in 1903.

Russell's reservations concerning Frege's solution continue in a letter to Frege of 20 February 1903:

> What you say about my contradiction is of the greatest interest to me. Do you believe that the range of values remains unchanged if some subclass of the class is assigned to it as a new member? Extension seems to fit this view better than intension. But I feel far from clear about this question. (Frege *1980*, p. 155)

Frege replied on 21 May, explaining that in general a class does not remain unchanged when a particular subclass is added to it, but that two concepts may have the same extension (the same class) when the only difference between them is that this class falls under the first concept but not under the second (p. 157). Then on 24 May 1903 Russell excitedly wrote Frege of a solution of the paradox of classes: "I received your letter this morning, and I am replying to it at once, for

[1] The same evaluation appears in Russell's Appendix to *The Principles of Mathematics* (p. 522).

[2] Frege's "$\vdash a \cap b$" is Peano and Russell's "$a \in b$". Russell should have put "$\vdash a \cap a$".

I believe I have discovered that classes are entirely superfluous. Your designation $\hat{z}\phi(z)$ can be used for ϕ itself, and $x \in \hat{z}\phi(z)$ for $\phi(x)$. … this seems to me to avoid the contradiction" (p. 159). The proposal is also mentioned in a letter to Jourdain of 14 March 1906. Russell recalls the history of his investigations into solving the contradictions:

My book [*The Principles of Mathematics*] gives you all my ideas down to the end of 1902.… Then in 1903 I started on Frege's theory that two non-equivalent functions may determine the same class; also I considered whether one should distinguish the two definitions:

$$x \in_1 u \,.\, = \,.\, (\exists\phi) \,.\, u = \hat{z}(\phi z) \,.\, \phi x \quad \text{Df.}$$
$$x \in_2 u \,.\, = \,:\, (\exists\phi) \,.\, u = \hat{z}(\phi z) : u = \hat{z}(\psi z) \,.\, \supset_\psi \,.\, \psi x \quad \text{Df.}$$

But soon I came to the conclusion that this wouldn't do. Then, in May 1903, I thought I had solved the whole thing by denying classes altogether; I still kept propositional functions, and made ϕ do duty for $\hat{z}(\phi z)$. I treated ϕ as an entity. All went well till I came to consider the function *W*, where

$$W(\phi) \,.\, \equiv_\phi \,.\, \sim \phi(\phi).$$

This brought back the contradiction, and showed I had gained nothing by rejecting classes. (Grattan-Guinness *1977*, p. 78)

It seems, then, that it was the naive assumption that each open wff (well-formed formula) stands for a single entity, a propositional function, that brought on a new contradiction. That is, it appears that it was Russell's discovery that a contradiction analogous to that of classes—a contradiction involving the propositional function "being a propositional function not predicable of itself"—that brought the proposal to an abrupt end. At first blush, Frege himself seems to corroborate this interpretation in his late response of 13 November 1904. Although his doctrine of the "unsaturatedness" of the function makes function symbols in isolation ill-formed, he observes that a contradiction analogous to that of classes is readily forthcoming on Russell's proposal.[3]

[3] This interpretation has won many supporters. See, for example, Lackey (*1973*, p. 129) and Bell (*1984*, p. 169).

This view, however, cannot be historically correct. The contradiction of propositional functions was known to Russell in writing the *Principles*. Russell's comment that the *Principles* was completed on 23 May 1902 notwithstanding, it is now known that certain parts took much longer. Blackwell reports that the Appendix on Frege did not reach the printer until 15 November and that Russell did not read proofs until February 1903 (*1984*, p. 285). But this cannot explain the matter. The *Principles* was out in May. Jourdain acknowledged receiving a copy on the 10th (*ibid.*, p. 283). Russell's discussion of the paradox of propositional functions occurs in Chapter VII, page 88, so it has to have predated Russell's letter to Frege of May 1903. Moreover, Russell's original letter of 16 June 1902 communicating the contradiction of classes, also contains a contradiction in terms of the predicate (property/concept) "being a predicate not predicable of itself". The contradiction of propositional functions is entirely analogous and, in fact, occurs in a letter to Frege of 24 June only eight days later (Frege *1980*, p. 133). Something more is therefore involved in Russell's proposal. In what follows, I wish to suggest what it may have been.

I. THE *PRINCIPLES*' PARADOX OF PROPOSITIONS

In order to make our suggestion as to nature of the proposal of 24 May 1903, we must first return to the *Principles* and its discussion of the paradoxes. Cantor's paradox of the greatest cardinal number is based on Cantor's power-class theorem that $A < P(A)$, for all classes A. Russell at first doubted the theorem, writing in "Mathematics and the Metaphysicians" that "the master had been guilty of a very subtle fallacy" (*1901*, p. 69). Accepting it was not easy. The theorem conflicted with what seemed to Russell as secure principles at the foundation of logic itself. The fundamental doctrine of *The Principles of Mathematics* is that "whatever is, is one". How could Cantor's result be reconciled with this apparently unassailable truth? Whatever is one is an entity, an individual, or what Russell calls a "logical subject"; and so the totality of entities must have the greatest cardinality. Its power-class cannot be greater.

Russell's *Principles* presents two options: (1) avoid the assumption of "classes-as-one", i.e. the assumption that classes are entities; and (2) allow classes-as-one but find principles to determine when such classes can be admitted without risk of error. Our setting out just these two options is intended to be contentious. One might object: what about the theory of logical types sketched in Appendix B of the *Principles*? Is this not a third option? The reason we have presented only (1) and (2) is to emphasize the fact that Russell knew full well that (if left unto itself) the doctrine of types violates the fundamental doctrine of the *Principles*. Russell proposed his sketch of a type theory as a purely formal dodge. The theory, he explains, would have to be allied with a theory of "plural logical subjects". "The fundamental doctrine upon which all rests", he wrote, "is the doctrine that the subject of a proposition may be plural, and that such plural [logical] subjects are what is meant by classes ..." (*PoM*, p. 517). That is, logic would have to show the way to do Logicism without assuming classes-as-one. Only the "class-as-many" would be admitted. Its being "many" notwithstanding, it would be treated as if it were a logical subject. Hence, types properly fall into option (1). The *Principles*, however, takes option (2). A "class-as-many", as Russell called it, is really just the many entities individually denoted (though collected together) by means of a denoting concept. For example, in denoting each man, the denoting concept "every man" collects a class-as-many. There is no commitment here to a single entity or "class-as-one". But Russell did not then know how denoting could account for classes of classes.[4] A class surely seems to be one when it is a member of another class. "[W]ithout a single object to represent an extension", he explains, "Mathematics crumbles" (p. 515). Accordingly, the *Principles* assumes that there are—sometimes—classes-as-one; and some among them are members of themselves. Logical principles were to be found which determine when it was safe to assume that a given open wff determines a class-as-one. "The class-as-one ...", Russell says, "is probably a genuine entity except where the class is defined by a quadratic function ..." (p. 518).

There is another paradox, moreover, that seemed to Russell to escape the theory of types. On 29 September 1902 he wrote to Frege:

[4] The new theory of denoting, viz., the 1905 theory of "incomplete symbols", proved to be much more fruitful. From it Russell invented his "substitutional theory" which enabled a solution of the paradoxes consistent with the fundamental doctrines of the *Principles*. (See Landini *1989*.)

My proposal concerning logical types now seems to me incapable of doing what I had hoped it would do. From Cantor's proposition that any class contains more subclasses than objects we can elicit constantly new contradictions. E.g.: If m is a class of propositions, "$p \in m \,.\supset_p .\, p$" represents their logical product. This proposition itself can either be a member of class m or not. Let w be the class of all propositions of the above form which are not members of the pertinent class m, i.e.,

$$w = p \ni [\exists m \ni \{p \,.=:\, q \in m \,.\supset_q.\, q :.\, p \sim\in m\}];$$

and let r be the proposition "$p \in w \,.\supset_p .\, p$". We then have

$$r \in w \,.\equiv.\, r \sim\in w.$$

The paradox is set out in the *Principles* (p. 527), and is based on Russell's conception of a proposition as an objective truth or falsehood. It is generated by the familiar Cantorian diagonal construction. Each class of propositions m is associated one-to-one with a unique proposition,[5] namely, '$(p)(p \in m \supset p)$'. Then form the class w:

$$\{p : (\exists m)(p = (q)(q \in m \supset q) \;\&\; (p \sim\in m)\}.$$

This class of propositions will itself be uniquely correlated with a proposition: '$(p)(p \in w \supset p)$'. But now we get

$$(p)(p \in w \supset p) \in w \,.\equiv.\, (p)(p \in w \supset p) \sim\in w,$$

which is a contradiction.[6]

The derivation requires the assumption that propositions are more finely individuated than by their truth-conditions; i.e., ident-

[5] Russell says this proposition represents the logical product of the class m. (It is also written as "$\wedge$'m".) In opposition to Bell (*1983*), Russell does not conflate the class, say $\{p,q\}$, with the proposition 'p & q'. The logical product of two classes A, B is the intersection of the two classes, i.e., $\{z : z \in A \;\&\; z \in b\}$. In analogy, Russell speaks the conjunction 'p & q' as the "logical product" of two propositions p, q. Since there is no infinite conjunction of propositions, the logical product of a (possibly infinite) class m of propositions would be '$(p)(p \in m \,.\supset.\, p)$'.

[6] See also A. Church (*1984*) for a discussion of this paradox. Church's special purposes lead him to introduce a new symbol to formulate the paradox. But Church's symbol is not needed here.

ical propositions must have identical constituents. This is essential for the right-to-left direction. For here we have

$$(\exists m)((p)(p \in w \supset p) = (p)(p \in m \supset p) \,.\, \& \,.\, (p)(p \in w \supset p) \sim\in m),$$

and the identity of the propositions must yield that $m = w$. Moreover, Russell points out that formally coextensive propositional functions might not be identical (*PoM*, p. 528). Consider a class m which is such that '$(p)(p \in m \,.\supset .\, p)$' is not one of its members. Next consider the class $m\ U\{(p)(p \in m \,.\supset .\, p)\}$, i.e. m^*. The proposition which is the logical product of m^* is '$(p)(p \in m^* \,.\supset .\, p)$', or better,

$$(p)(p \in m\ U\{(q)(q \in m \,.\supset .\, q)\} :\supset: p).$$

Now we have

$$(p)(p \in m\ U\{(q)(q \in m \,.\supset .\, q)\} :\supset: p) \,.\equiv.\, (p)(p \in m \,.\supset .\, p)$$
$$\& \ (p)(p \in m \,.\supset .\, p)$$

as a logical truth. Thus

$$(p)(p \in m\ U\{(q)(q \in m \,.\supset .\, q)\} :\supset: p) \,.\equiv.\, (p)(p \in m \,.\supset .\, p)$$

is also a logical truth. If formally coextensive functions were to be identical, we would have:

$$(p)(p \in \hat{z}\ U\{(q)(q \in \hat{z} \,.\supset .\, q)\} :\supset: p) \,.=.\, (p)(p \in \hat{y} \,.\supset .\, p).$$

This is of the form $F\hat{z} = G\hat{y}$. So then the proposition Fm is identical to the proposition Gm. That is,

$$(p)(p \in m\ U\{(q)(q \in m \,.\supset .\, q)\} :\supset: p) \,.=.\, (p)(p \in m \,.\supset .\, p),$$

or better,

$$(p)(p \in m^* \,.\supset .\, p) = (p)(p \in m \,.\supset .\, p). \text{ Yet } m^* \neq m.$$

Both of these requirements are met. The non-identity of equivalent propositions and the non-identity of logically coextensive propositional functions are a part of the theory of the *Principles*. Thus the paradox shows that the assumption of propositions as individuals conflicts with Cantor's power-class theorem. Each class m of propositions will be correlated one–one with a unique individual—a proposition.

Frege replied to Russell's letter on 20 October 1903. He expresses perplexity over Russell's conception of a proposition. Explaining that he distinguishes the sense (*Sinn*) and the reference (*Bedeutung*) of signs, Frege says that he uses the term "proposition" for a sentence which expresses a thought (*Gedanke*) as its sense and has a truth-value as its reference. A class m is never part of a thought; it is the sense of the sign "m" that is its part (Frege *1980*, p. 149).

Russell may indeed have conflated his own notion of a proposition with Frege's notion of a *Gedanke*, but this is not likely. For in his response of 12 December, he explains that the new antinomy should be formulable against Frege's theory. He writes:

> I cannot bring myself to believe that the true or the false is the meaning [reference] of a proposition [sentence] in the same sense as, e.g., a certain person is the meaning [reference] of the name Julius Caesar. But this is an incidental matter. It must be admitted that there are different senses, and it is to be supposed that classes of senses have numbers. Now the sense of "$p \in m \,.\supset_p .\, p$" stands in a one–one relation to m; consequently, there is the same number of senses as there is of classes of senses. (Pp. 150–1)

Frege's next letter, dated 28 December, endeavours at length to show that the new antinomy is merely apparent. There is, in fact, no one–one correspondence between classes of thoughts and thoughts.

Frege explains that if the expression of the identity involved in the contradiction, viz., "$(p)(p \in m \supset p) = (p)(p \in n \supset p)$", is to assert the identity of the thoughts, then the context is one of indirect speech ("*oratio obliqua*"). We have: "The thought that '$(p)(p \in m \,.\supset .\, p)$' is identical with the thought that '$(p)(p \in n \,.\supset .\, p)$'". In indirect speech, the ordinary sense of a sign becomes its reference; and the sign now has an indirect sense. Thus, for instance, the above identity is to be read: "The thought that all thoughts belonging to class M are true is identical with the thought that all thoughts belonging to class N are true." Since the context "the thought that …" is indirect, it is the indirect sense of the symbols "M" and "N" that occur. But now in

The thought that *all thoughts belonging to class M are true* belongs to class M,

the first occurrence of "M" (in the italicized clause) has its indirect reference; yet in the second occurrence it has its direct reference and refers to a class (*ibid.*, p. 153). Now in the right-to-left direction of the derivation of the paradox Russell has:

$$(\exists m)((p)(p \in w \supset p) = (p)(p \in m \supset p) \,.\, \& \,.\, (p)(p \in w \supset p) \sim\in m).$$

The first conjunct is in indirect speech and yet the second conjunct would need to be in direct speech—so that "m" refers to a class. But then the quantifier "$(\exists m)$" would have to take as a value an indirect sense in the first conjunct and a class in the second. The contradiction is not formulable.

In the end, Russell conceded the point to Frege. The paradox of propositions cannot be generated in terms of Frege's notion of a *Gedanke.* But he accepted neither Frege's sense/reference distinction nor the doctrine of "referential shift". Russell's assumption of propositions makes such manoeuvres impossible, and so he requires a new solution of the paradox of propositions.

2. THE SUPERFLUOUSNESS OF CLASSES

The contradiction of propositions was, in fact, more often the subject of the correspondence between Frege and Russell than the contradiction of classes itself. It occurs in the 24 May 1903 letter heralding the superfluousness of classes. By looking at the treatment of this paradox in the letter, we can solve our problem concerning the nature of the proposal to make classes superfluous. Russell's 1906 recollection to Jourdain shows that the time between his investigation of Frege's suggestion that identical classes may yet fail to be coextensive and his own suggestion of superfluousness is quite close. I believe the two are related.

In the 24 May letter to Frege, Russell appeals to Frege's suggestion, coupled with his new theory, to solve the paradox of propositions. He writes:

Concerning "$p \in m \,.\supset_p .\, p$", I now write "$\phi(p) \supset_p p$" instead, i.e., "The property ϕ does not belong to any objects which are not true". It seems to me that there is no difficulty in this, provided one does not require "$p \equiv q$" to express an identity. The function which gave rise to difficulties for me was $\sim\phi\{\phi(p) \supset_p p\}$. But now these difficulties have been overcome by means of the theorem in your appendix, according to which

$$\vdash : \exists(\phi, \psi) \,.\, [(\phi(p) \supset_p p)\, \mathrm{I}'(\psi(p) \supset_p p) \,.\, \sim\phi\{\phi(p) \supset_p p\} \,.\, \psi\{\psi(p) \supset_p p\}].$$

Russell uses "I'" (which he calls "*Identität*", or "identity"), and he defines it as follows:

$$x\mathrm{I}'y \,.\, =_{df} .\, (\phi)(\phi x \supset \phi y).$$

That is, I is indiscernibility (*ibid.*, p. 159). The theorem in the Appendix to Vol. II of the *Grundgesetze*, to which Russell refers, allows that identical classes may be such that their defining properties are not coextensive. This shows that Russell accepts the theorem, suitably reinterpreted so that class symbols are replaced by function symbols. Indeed, since Russell holds that propositions are more finely individuated than sameness of truth-conditions, the indiscernibility of '$\phi(p) \supset_p p$' and '$\Theta(p) \supset_p p$' requires that a relationship analogous to Frege's "identity" between classes holds between the functions ϕ and Θ. It must be analogous, that is, in allowing that propositional functions are "identical" and yet not coextensive. From the context of Russell's letter, the relationship can only be indiscernibility. Russell, it seems, is proposing that there are indiscernible propositional functions that are not coextensive. Accordingly, the superfluousness of classes was not itself to be the whole of Russell's solution of the contradiction of classes—Frege's plan (though not his specific replacement for Basic Law V) was also to be adopted in a form applicable to propositional functions.

Recall that Frege tracks down the source of Russell's paradox of classes to his Basic Law V*b*. That is, consideration of Russell's class of all classes not members of themselves leads to the following result:

$$(\exists f)(\exists g)(\acute{z}fz = \acute{z}gz \,.\, \& \,.\, \sim(x)(fx \equiv gx)).$$

In Frege's view, this showed that Basic Law V*b*, viz.,

$$\hat{z}fz = \hat{z}gz \,.\supset.\, (x)(fx \equiv gx),$$

was false. There will be some concepts that apply to different objects and yet have the same extensions. Russell, as we have seen in his correspondence, found this result difficult. Indeed, he found it inconsistent with the very conception of a class as the extension of a concept. Frege admitted as much, writing that "... this simply does away with the extensions of concepts in the received sense of the term" (*1903*, p. 260). In Russell's view, if classes exist as the extensions of concepts, Frege's Basic Law V must stand. The analogous situation with propositional functions, however, does not do violence to our natural conceptions. The principle

$$\text{(Ind)} \quad (\phi, \Theta)(\phi I \Theta \supset (x)(\phi x \equiv \Theta x))$$

may fail to hold. It is quite plausible (so we shall argue) that for a short time after the *Principles* Russell came to believe that this is, indeed, what happens.

We have already had occasion to observe that the fundamental doctrine of the *Principles* is that "whatever is, is a logical subject". The pure or formal variable of logic respects this doctrine. Its range is wholly unrestricted. It is not often appreciated, however, that the notion of a "logical subject" (or "term") and what Russell calls a "term of a proposition" go together. Russell distinguishes what he calls "concepts" (properties and relations) and "things". Concepts have both a predicable and an individual nature. They are capable of what Russell calls a "curious [and indefinable] two-fold occurrence" in propositions. For instance, *humanity* occurs predicatively ("as concept") in the proposition "Socrates is human". The very same concept occurs as a term of the proposition "Humanity belongs to Socrates" (*PoM*, p. 44). Things are logical subjects which can only occur in propositions as a "term of the proposition". For instance, Socrates occurs as term of the proposition "Socrates is human". He is both a constituent of the proposition and what the proposition is about (p. 45). "Every term ...", writes Russell, "is a logical subject; it is, for example, the [logical] subject of the proposition that itself is one" (p. 44). The notion of a "logical subject" is born from the notion of "occurring as term of a proposition". Accordingly, the doctrine that "whatever is, is one" is

just the doctrine that "whatever is, can occur as term of a proposition".

Russell came to see[7] that this doctrine blocks any attempt to formulate the paradox of propositional functions in terms of the formula "$\sim \hat{\phi}(\hat{\phi})$". Formulae comprehending propositional functions must be characterized for *all* entities, and "$x(x)$" is ill-formed. However, using Russell's symbol "I" for indiscernibility, we can formulate the paradox using the wff "$(\phi)(\hat{z}I\phi \,.\supset.\, \sim \phi\hat{z})$", or alternatively using the wff "$(\exists\phi)(\hat{z}I\phi \,.\,\&\,.\, \sim \phi(\hat{z}))$". But now, rather than a contradiction, we arrive at the theorem:

$$(\exists\phi)(\exists\Theta)(\phi I \Theta \;\&\; \sim (x)(\phi x \equiv \Theta x)).$$

Indiscernible propositional functions may yet fail to be coextensive. This is exactly analogous to the situation for classes that Frege was led to embrace.

It is important to observe, however, that on pain of contradiction, there is no primitive relation of "identity" in the system. That is, if we add a primitive symbol "=" and regard it as a relation which allows full substitutivity, the paradox returns. Accordingly, the identity symbol must be neither admissible in wffs comprehending properties and relations, nor a substituend of a variable of any type. In light of this, it may seem that the special status of identity in the system may well have been what eluded Russell. This would be so if the function W that Russell mentions in his recollection to Jourdain was:

$$(\phi)(\hat{z} = \phi \,.\supset.\, \sim \phi\hat{z}).$$

But this formulation of the contradiction was also discussed in the *Principles* (p. 97). We do not yet have a complete answer to our question.

[7] In the *Principles*, p. 87, Russell observes that the variation of a property or relation occurring as concept is ruled out by his account of quantification and the doctrine of the unrestricted variable. Variation, he observes, requires that the property or relation occur as term. This point recurs in many unpublished manuscripts—e.g., "On Meaning and Denotation", fol. 77, and "On Fundamentals", fol. 30.

3. RUSSELL ON FREGE ON THE CONTRADICTION

We are, however, on the right trail. We can see this by consulting unpublished manuscripts. Of importance are Russell's notes on Frege and "General Theory of Classes".[8] Among the notes on Frege is a manuscript with the file number 230.0303420–F2. Here Russell discusses Frege's derivation that identical classes may be defined by concepts that are not coextensive. In a segment entitled "Frege on the Contradiction", he also discusses his paradox of propositions and even deduces the very theorem concerning propositions that would appear in the 24 May 1903 letter to Frege (fol. 18). The manuscript "General Theory of Classes" is undated, but it too was likely composed close to the time of the 24 May letter to Frege. It contains, for instance, the definition of "$u \,|||\, v$" appearing in the letter (fol. 135b); and, the paradox of propositions appears (fol. 145) using the form "$\phi p \,.\supset_p .\, p$" as well as the form "$p \in m \,.\supset_p .\, p$". Finally, there is the manuscript "On Meaning and Denotation", and we shall primarily focus on it. This manuscript is also undated. Some of its content allies it with manuscripts of 1904 such as "Points about Denoting". The dating is probably accurate, but there is a section (fos. 75–98) which may well be (at least) a return to earlier ideas. We again find the two distinct definitions of membership given in "General Theory of Classes", as well as a return to the earlier manuscript's work on the notion of a "simple" (i.e., non-quadratic) function (fol. 88).

In "On Meaning and Denotation" Russell observes that it would be a fallacy to attempt to derive the contradiction (of predicates) by means of the following formula comprehending the function F:

$$(\phi) \,.\, F(\phi\xi) \equiv \phi\{f(\phi\xi)\},$$

where "f" is a function constant. To be sure, one could then get

$$F(-F\xi) \equiv -F\{f(-F\xi)\},$$

[8] There are two manuscripts with this title. We intend the manuscript with file number 230.030940.

by instantiating to "$-F$" (fol. 77). But the formula introducing the function F, according to Russell, is "second-order" since it is meaningfully defined only for arguments which are the "meanings of propositional functions" (i.e., the meaning of propositional function symbols). But now the quantifier ranges over all "first-order" propositional functions—i.e., functions introduced so that their arguments are all entities, functions or otherwise. Hence, instantiating the quantifier to "$-F$" is illegitimate. The value of f taken at $-F\xi$, that is, $f(-F\xi)$, may not be a propositional function at all. Russell goes on to observe, however, that this problem is avoided when f is just identity (fol. 77). Thus, he puts

$$(\phi) \,.\, F(\phi\xi) \equiv \phi(\phi\xi)$$

and derives a contradiction by instantiating to "$-F$". The fallacy here, Russell explains, is that F is second order and yet the function ϕ is first order (fol. 78). Thus, "$-F$" is not admissible as a substituend for "ϕ" (fol. 78). Russell's conclusion is that so long as propositional function symbols can be introduced which are meaningful only with function symbols as arguments, "types of functions seem unavoidable" (fol. 78). But his general plan is not to justify a theory of restricted quantifiers, e.g., those for functions applicable to all entities, those for functions applicable to only first-order functions, etc. On the contrary, the point seems to be that propositional functions should not be introduced in terms of conditions which meaningfully apply only with function symbols as arguments. That is, if there really is one unequivocal mode of being and if propositional functions are single logical subjects, then wffs comprehending propositional functions should be given in terms of individual variables ranging over all entities. This blocks a formulation of the paradox in virtue of the formula "$\sim \hat{\phi}(\hat{\phi})$".

Russell goes on to say that he can introduce classes, using Frege's notation "$\acute{z}\phi z$", provided he can discover what a class is; and provided that, like Frege's *Werthverlauf* (range of values), the notion can be extended to double and triple functions (fol. 79). With regard to classes, as opposed to ranges generally, Russell notes these properties:

(1). They have to do with propositional functions exclusively.
(2). Two formally equivalent propositional functions define the same class.

(3). A class may have no members.
(4). If a class has one member, it is not identical with that one member.
(5). It would seem that two propositional functions which are not equivalent may determine the same class.

Russell goes on to say that the argument concerning "second-order" functions—viz., that they must be properly introduced in virtue of conditions that apply to all entities, functions or otherwise—applies against any similar attempt to get classes as members of themselves (fol. 79). Nonetheless, he observes that in

$$(x)(F(x) \,.\equiv.\, (\phi)(x = \hat{z}\phi z \,.\supset.\, \sim\phi x)),$$

the function F is introduced by a condition defined for all entities, functions or otherwise. Thus, "if we put $\hat{x}(Fx)$ as argument to F, we get the contradiction; or at least, the Frege modification, showing the membership of a class to be not determinate when the class is given" (*ibid.*).

Russell next formulates what he calls "Frege's paradox". He writes (fol. 80):

If we take Frege's paradox, it is

$$f^2|x \,.=.\, (\exists\phi)\,.\,\{x = f|(\phi\xi)\,.\sim\phi x\} \quad \text{Df}$$
$$\vdash : f^2|(f|f^2\xi) \,.=.\, (\exists\phi)\,.\,\{f|(f^2\xi) = f|(\phi\xi)\,.\sim\phi|(f|f^2\xi)\}$$
$$\vdash :.\, -f^2|(f|f^2\xi) \,.\equiv:\, f|f^2\xi = f|\phi\xi \,.\supset_{\phi}.\, \phi/\{f|f^2\xi\} :\supset: f^2\,|(f|f^2\xi)$$
$$\vdash f^2|(f|f^2\xi) \qquad \vdash : (\exists\phi)\,.\,\{f|(f^2\xi) = f|\phi\xi\,.\sim\phi|(f|f^2\xi)\}.$$

Here the function $f^2|x$ is properly introduced for it is applicable to all entities, propositional functions or otherwise.[9] Now if we existentially instantiate

$$(\exists\phi)\,.\,\{f|f^2\xi = f|\phi\xi\,.\sim\phi|(f|f^2\xi)\},$$

[9] In "Points about Denoting", Russell uses the vertical slash notation "$\phi|x$" to mark the fact that ϕ can be varied separably from x in "ϕx". He explains that "the plan is now to use $\hat{f}x$ or ϕx for an unanalyzed complex, and to write $f|x$ or $\phi|x$ in the case where we can detach a constant element to be called the *function*" (fol. 15). But I do not know whether the slash carries this meaning here.

and if we know that $f|x$ is a one–one function so that

$$(x)(y)(fx = fy \,.\supset.\, x = y),$$

we shall derive that $f^2\xi = \phi\xi$. If identity implies coextensivity, as surely it must, then we get $\vdash : -f^2|(f|f^2\xi)$ and the contradiction.

This version reveals that Russell was aware that his paradox of classes can be generalized. Generalization is achieved by defining f^2 in terms of an arbitrary constant function f which is one–one. The contradiction follows taking $f|f^2\xi$ as argument to $f^2|x$. Obviously, one candidate for f is a function mapping coextensive propositional functions one–one onto their extensions. For Russell the expression "$\hat{z}\phi z$" abbreviates "the class defined by ϕz", and this is a function of ϕz. Here we have $f(\phi\xi) = \hat{z}\phi z$.[10] Thus $f|f^2\xi$, which Russell takes as argument to f^2, would be just $\hat{z}f^2z$. In this case the paradox shows, as Frege concluded, that there is no such one–one function; non-coextensive propositional functions may yet have the same extension.

But generalization shows that there may be other candidates for f. And some may be defined on classes and such that $f(x) \neq x$. That is, Frege's replacement for V*b*, viz.,

$$\text{V}'b \qquad \hat{z}\phi z = \hat{z}\Theta z \,.\supset.\, (x)(x \neq \hat{z}\phi z :\supset: \phi x \,.\equiv.\, \Theta x),$$

is inadequate. Leśniewski found the inadequacy in 1938 (Sobociński *1949*, p. 220ff.). Geach (*1956*) brought Leśniewski's argument to light in the context of a Fregean theory of classes. He assumes there are at least two objects and begins from the principle

$$(x)(x \in \hat{z}\phi z \,.\equiv.\, x \neq \hat{z}\phi z \;\&\; \phi x),$$

which is implicit in some of Frege's remarks.[11] Quine (*1955*) came

[10] See below, p. 18, and also Russell's discussion in "General Theory of Classes", fol. 128.
[11] In Frege's letter to Russell of 20 October 1902 he puts V′,

$$\hat{z}\phi z = \hat{z}\Theta z \,.\equiv.\, (x)(x \neq \hat{z}\phi z \;\&\; x \neq \hat{z}\Theta z :\supset: \phi x \,.\equiv.\, \Theta x),$$

upon the inadequacy independently of Geach, deducing a contradiction from Frege's replacement $V'b$ itself. Russell does not seem to have come upon these proofs.[12] However, Russell observed on (fol. 18) of his "Frege on the Contradiction", that one such candidate for the function f is a function such that $f(\phi) \,.=.\, \phi x \supset_x x$. Similarly, let f be a function defined on classes such that $f(m) = (p)(p \in m \,.\supset.\, p)$. Given Russell's theory of propositions, f is not only one–one but also such that $(x)(fx \neq x)$. But now Russell could not have accepted Frege's replacement for Vb. The replacement fails to avoid the paradox of propositions. Of course, this by itself would not have impressed Frege since the paradox of propositions is not formulable in his system. Nonetheless, it does provide Russell with a rejection of Frege's replacement.

Now the manuscript "On Meaning and Denotation" goes on with Russell attempting to define membership in a system which allows Frege's basic idea. Russell adopts "the" ("ι") as a primitive notion from which all denoting functions are derived. In a passage that now

as the replacement of Basic Law V, and goes on to write:

$$\text{Then: } \vdash\!\!\!\top\; a \cap a.$$

Now Frege may have intended to add this as an axiom. The word "then" cannot mean derivability for V′ does not assure that $(x) \sim (x \in x)$. It does, however, follow from Geach's principle. Or better, it follows by replacing Frege's original definition of "membership", viz., $x \in y =_{df} (\exists\phi)(y = \acute{z}\phi z \;\&\; \phi x)$, with the new definition:

$$x \in y =_{df} (\exists\phi)(y = \acute{z}\phi z \;\&\; x \neq \acute{z}\phi z \;\&\; \phi x).$$

(The new definition and $V'b$ yield Geach's principle.) Linsky and Schumm (*1974*) pair Geach's principle with V′ and show the weakness of Frege's modifications by proving that o = 1. But this may well be uncharitable since Frege may not have intended to have both V′ *and* a new definition of membership.

[12] Resnik (*1980*, p. 215) offers a generalization of the contradiction of classes. He puts:

$$R_f x =_{df} (\exists\phi)(x = f(\acute{z}\phi z) \;\&\; \sim\phi x)$$

and then considers the situation when $\acute{z}R_f z$ is taken as argument to R_f. Resnik shows that (assuming two objects) Frege is committed to there being a one–one function f on classes which is such that $(x)(fx \neq x)$. Of course, the failure of Frege's way out does not require that $(x)(fx \neq x)$, but only that in particular that $f(\acute{z}R_f z) \neq \acute{z}R_f z$. Russell is surely quite close to this.

can only evoke a smile, he explains: "It is plain that ι is a fundamental logical notion and that it would be merely shirking to invent a dodge for getting on without it" (fol. 84). Russell then puts:

$$\iota(f\xi) =_{df} \text{the term satisfying } f\xi.$$

The symbol "ι", he says, must be flanked by a propositional function symbol else we get a meaningless (denotationless) expression;[13] and if more than one term has the property f, then it is to be given a conventional denotation—perhaps the denoting concept "nothing" (fos. 86 and 89). He also has (fol. 84):

$$u \text{ Cls } f\xi \;.=.\; u \text{ is a class determined by } f\xi.$$
$$\vdash \therefore (x)(fx \equiv gx) : u \text{ Cls } f\xi \,.\, v \text{ Cls } g\xi : \supset .\, u = v. \quad \text{Pp}$$

Thus, "the class of all z such that $f\xi$" is the definite description "the term satisfying the function 'y is a class determined by $f\xi$'. That is,

$$\acute{\xi}(f\xi) = \iota(y \text{ Cls } f\xi) \quad \text{Df.}$$

Now Russell adopts (fol. 87) Frege's Basic Law Va:

$$(x)(\phi x \equiv \Theta x) \,.\supset.\, \acute{z}\phi z = \acute{z}\Theta z.$$

But since he is operating under Frege's basic idea, a definition of membership such as

$$x \in_{1} u =_{df} (\exists\phi)(u = \acute{z}\phi z \,.\&.\, \phi x),$$

[13] Though now applied to propositional functions, this is similar to Peano's treatment of "ι". For Peano, "ι" stands for an operator on classes such that where A is a singleton class, "ιA" stands for the single member of A. Thus "ι" (or alternatively "℩") expresses an operation which is the inverse of the singleton operation "ι", i.e., where "℩a" $=_{df}$ "$\{a\}$". See *Formulaire* (1901), p. 31. In giving a conditional definition, Peano regarded the expression "ιA" as undefined when "A" does not stand for a class or when it does not stand for a singleton class (*Formulaire* [1895], p. 50). Russell's 1905 treatment, of course, entirely abandons the operator view in favour of the theory of incomplete symbols.

would not do by itself. Russell examines a second membership relation,

$$x \in_2 u =_{df} (\exists\phi)(u = \hat{z}\phi z \,.\, \& \,.\, (\Theta)(u = \hat{z}\Theta z \,.\supset.\, \Theta x)),$$

which avoids this problem (fol. 85). According to $\in_2$, if u is determined by ϕ and Θ, then ϕ and Θ are coextensive.

Russell realizes that $\in_1$ and $\in_2$ do not agree even where u is a class.[14] At first he wonders whether this lack of agreement is really a difficulty, but then recognizes the extent of the trouble. He writes:

> … it is not necessary that they should, since we never infer acquaintance of propositional functions from identity of their classes, i.e. we never use
>
> $$(\alpha) \quad u \operatorname{Cls} f\xi \,.\, u \operatorname{Cls} g\xi \,.\supset:\, (x) \,.\, fx \equiv gx.$$
>
> This proposition is not one which is called for in practice; hence it may be denied. But we still hold that every function defines a class, though not that all functions defining a given class are equivalent.
>
> The only way in which the above proposition (α) is required is in counting classes; here its converse is used, i.e.
>
> $$\sim\{(x) \,.\, fx \equiv gx\} \,.\supset.\, \hat{\xi}(f\xi) \sim= \hat{\xi}(g\xi).$$
>
> Certainly Arithmetic is difficult without some restricted form of this proposition. (Fol. 85)

A replacement is needed for Frege's problematic Basic Law V*b*. The converse plays an important role in counting classes.

Russell has no replacement, and says only that he requires "a complicated primitive proposition … with suitable limitations for quadratic functions" (fol. 87). Just as in the earlier manuscript "The General Theory of Classes", there is an acknowledgement that $\in_2$ applies only to some among classes. The plan is to mark these as the classes

[14] Unlike the 14 March 1906 letter to Jourdain, "On Meaning and Denotation" has (fol. 88):

$$x \in_2 u \,.=:\, u = \hat{\xi}(f\xi) \,.\supset_{f\xi}.\, fx).$$

The result is that "$x \in_2 u$" is true when u is not a class.

determined by certain "simple" propositional functions. That is, Russell has (fol. 88):

$$\text{Simple}(\phi\xi) \,.=:\, \hat{\xi}(\phi\xi) = \hat{\xi}(f\xi) \,.\supset_f.\, (x)(\phi x \equiv fx) \quad \text{Df.}$$

Accordingly, principles need to be found (perhaps by excluding quadratic forms as was already suggested in the *Principles*) to decide which propositional functions are "simple".[15] Unfortunately, the brute reality foisting itself upon him was that the task of finding purely logical axioms for weeding out the problematic cases was insuperable. The distinction of $\in_1$ and $\in_2$ "would not work", as Russell recalls in his 1906 letter to Jourdain. The 24 May tactic of making classes superfluous, on the contrary, may have appeared (at least at first blush) to offer a means of altogether avoiding the necessity of a replacement for Basic Law V*b*. This would explain Russell's recollection to Jourdain that "I thought I had solved the whole thing by denying classes altogether."

This, in fact, appears to be the case. From the context of Russell's letter to Frege, it seems that he thought that propositional functions could be counted and arithmetic could proceed without principles for finding the "simple" propositional functions. In Russell's 24 May letter, there is no primitive identity relation. The letter sets out a definition of *Gleichheit* (or "equality") which, according to Russell, "can be used in counting just like identity" (fol. 159). Russell defines "*Gleichheit*" (which we shall write as "≈") as follows:

$$u \approx v =_{df} \{\text{Indiv}(u) \supset [\text{Indiv}(v) \;\&\; uIv]\}$$
$$\& \; \{\sim\text{Indiv}(u) \supset [\sim\text{Indiv}(v) \;\&\; u \,|||\, v]\}.$$

And he explains, "'Indiv (x)' means 'x is an object, i.e., not a function'" (fol. 159). He also puts:

$$\phi \,|||\, \Theta =_{df} (x)(\phi x \equiv \Theta x).$$

[15] See "General Theory of Classes", fol. 131ff., for a detailed discussion setting out a definition of a "quadratic" function and developing a theory of "simple" classes.

Then he goes on to give an example of how counting can proceed. He uses *Gleichheit* in defining the notion of "similar propositional functions", which replaces the notion of "similar classes". The notion of a one–one function is given by the definitions:

$$(Nc\rightarrow 1)(f) =_{df} (x)(y)(z)(fxy \;\&\; fxz :\supset .\, y \approx z)$$
$$(1\rightarrow Nc)(f) =_{df} (x)(y)(z)(fxz \;\&\; fyz :\supset .\, x \approx y)$$
$$(1\rightarrow 1)(f) =_{df} (Nc\rightarrow 1)(f) \;\&\; (1\rightarrow Nc)(f)$$

Then "ϕ sim Θ" is defined as:

$$(\exists f)\{(1\rightarrow 1)(f) \;\&\; (x)(\phi x \supset (\exists y)(fxy \;\&\; \Theta y)) \;\&\; (y)(\Theta y \supset (\exists x)(fxy \;\&\; \phi x))\}.$$

The cardinal number of a class $\hat{z}\phi z$ is next defined as the class of all classes similar to it. Since "ϕ" has now replaced "$\hat{z}\phi z$", Russell has:

$$Nc(\phi) =_{df} \dot{\Theta}\ (\phi \text{ sim } \Theta).$$

Russell then proclaims that "In this way we can do arithmetic without classes. And this seems to me to avoid the contradiction" (fol. 159).

4. COCCHIARELLA'S T**

Unfortunately, if we look closely, we see that Russell used *spiritus lenis* in his definition of "$Nc(\phi)$". This is a part of Frege's notation for a class and so Russell's expression must be transformed into one appropriate for a propositional function. Now the formula "ϕ sim $\dot{\Theta}$" is ill-formed, since formulas comprehending functions must be applicable to all entities, propositional functions or otherwise. Since Russell says that *Gleichheit* can be used in counting just like identity, he apparently hoped to be able to put "$(\exists\Theta)(\hat{z} \approx \Theta \;\&\; \phi \text{ sim } \Theta)$". But allowing "$\approx$" to occur in wffs comprehending propositional functions opens the way for the return of the paradox. We have only to put:

$$W =_{df} (\exists\Theta)(\hat{z} \approx \Theta \;\&\; {\sim}\Theta(\hat{z})).$$

Then $W(W) \equiv {\sim}W(W)$. The symbol "$\approx$" , as Russell defined it, cannot occur in a wff comprehending a propositional function without reintroducing the contradiction.

Russell hoped that "$\approx$" should be allowable in wffs comprehending functions; and at the same time he thought the paradox would not be reintroduced. This would, in fact, be so if "$\approx$" were properly defined. But when paired with the definition of the vertical triple slash, Russell's definition of *Gleichheit* is flawed. It violates the very foundational principle of the system. There is an error in the second conjunct of the definiens. We saw that the vertical triple slash, which Russell uses to abbreviate coextensivity, cannot be viewed as expressing a relation. All propositional functions must be comprehended through wffs which apply to all entities. The wff "$u \parallel\!| v$" is, therefore, just as ill-formed as "$x(x)$". The flaw is corrected by replacing "$u \parallel\!| v$" in the second conjunct with "$(\exists\phi)(\exists\Theta)(uI\phi \;\&\; vI\Theta \,.\, \&\, .\, \phi \parallel\!| \Theta)$".[16] No contradiction can now come from allowing "$\approx$" (as newly defined) to appear in wffs comprehending propositional functions. That u and v are indiscernible from coextensive propositional functions does not assure that they are themselves coextensive.

Once this patch is made, Russell's 24 May 1903 letter to Frege can be seen to be quite similar to Cocchiarella's T**. T** was formulated and proposed as characterizing the original logistic background of Russell's paradox of predication (Cocchiarella *1973* and *1975*). Here standard second-order logic is conservatively extended to allow the occurrence of predicate variables in subject positions. But in accordance with the doctrine of the *Principles* that "whatever is, is a logical subject", the system requires that wffs comprehending properties or

[16] Alternatively, one could leave Russell's definition of "$\approx$" as it is and redefine the triple slash. That is, put:

$$u \parallel\!| v =_{df} (\exists\phi)(\exists\Theta)(uI\phi \,.\, \&\, .\, vI\Theta \,.\, \&\, .\, (x)(\phi x \equiv \Theta x)).$$

The slip probably occurs because Russell's definition of "Indiv(x)" is naturally rendered as:

$$\text{Indiv}(x) =_{df} (\phi) \sim (xI\phi).$$

Thus he may have thought that the presence of "Indiv(u)" and "Indiv(v)" in the second conjunct of his definition of "$\approx$" was all he needed.

relations must meaningfully be applied to all entity terms. Since not all entities are concepts (properties or relations), entity variables are not meaningfully allowed to occupy predicate positions. This, together with the fact that identity does not express a primitive relation (allowing full substitutivity) in the system, solves Russell's paradox of predicates. The system is demonstrably consistent. All that results is the failure of (Ind) in T**.

The parallel to Russell's system becomes striking when we turn to developing a theory proxying a theory of classes in T**. Cocchiarella does this by first assuming the following extensionality principle (*1975*, p. 44):

$$(\text{Ext}^*) \quad (\phi)(\Theta)((x)(\phi x \equiv \Theta x) \,.\supset.\, \phi I \Theta).$$

He then observes that a definition of "membership" in T** such as

$$x \in_1 u =_{df} (\exists \phi)(uI\phi \ \&\ \phi x)$$

does not amount to a genuine membership relation because of the failure of (Ind). A more promising approach, he explains, would be to first define the notion of a "set" as a "class" for which (Ind) holds, i.e.,

$$\text{Set}(x) =_{df} (\exists \phi)(xI\phi \ \&\ (\Theta)(\phi I \Theta \,.\supset.\, (x)(\phi x \equiv \Theta x))).$$

A genuine membership relation can then be defined as:

$$x \in_2 u =_{df} (\exists \phi)(uI\phi \ \&\ \text{Set}(\phi) \ \&\ \phi x).$$

Cocchiarella's two definitions are exactly Russell's own two membership relations discussed above. That is, they are equivalent to Russell's two definitions when the latter are set in the context of his idea of the "superfluousness of classes". Membership$_1$ is clearly the same in both cases. The superfluousness doctrine transforms Russell's membership$_2$ into:

$$x \in_2 u =_{df} (\exists \phi)(uI\phi \ \&\ (\Theta)(uI\Theta \,.\supset.\, \Theta x)).$$

This is derivable from Cocchiarella's definition.[17]

Now the central issue confronting T** is to find principles to decide when the second membership relation may be said to hold. When does indiscernibility assure coextensivity? The analog of Frege's replacement for V*b*, of course, will not work. That is,

$$\phi I \Theta \,.\supset.\, (x)(\sim(xI\phi) :\supset: \phi x \,.\equiv.\, \Theta x)$$

fails in T** + (Ext*) by the analog of the argument against Frege's V'*b* given by Quine. Logicism notwithstanding, there are some interesting (non-logical) assumptions that could be made so as to develop set theory in T**. Cocchiarella observes that one possible assumption is a variant of von Neumann's *maximization principle* (for sethood):

$$\text{MPS}^* \quad (\phi)[\text{Set}(\phi) \,.\equiv.\, \sim(\exists R)(x)(y)(z)(xRy \ \&\ xRz \,.\supset.\, yIz) \ \&\ (y)(\exists x)(\phi x \ \&\ xRy)]$$

This expresses the intuitive content of the "limitation of size" doctrine—viz., the principle that a class is a set if and only if it cannot be mapped onto the totality of individuals (p. 47). In T**, Cocchiarella explains, this principle entails Frankel's Replacement Axiom and (consequently) Zermelo's *Aussonderungsaxiom*.[18]

Russell, of course, required purely logical principles and would not have been satisfied with such an approach to developing a theory of sets in this way. Indeed, we argued that Russell's attraction to the superfluousness doctrine was precisely his hope of avoiding such special principles. Counting could proceed by means of *Gleichheit*. But as we saw, the definition of *Gleichheit* (which we wrote as "≈") contained an error. Once patched, however, *Gleichheit* is not useful for counting. For instance, Russell cannot put "$(\exists \Theta)(\hat{x} I \Theta \,.\&.\, \phi \text{ sim } \Theta)$" as the definiens for "$x$ is the cardinal number of ϕ". For two propositional functions might then have the same cardinal number even

[17] Cocchiarella's "$x \in_2 u$" is derivable as a theorem from Russell's "$x \in_2 u$" as well if we assume that $(\phi)(u)(x)(x \in_2 u \ \&\ uI\phi \,.\supset.\, (z)(\phi z \supset z \in_2 u))$.

[18] Moreover, Cocchiarella observes that given an appropriate characterization of the ordinal numbers, we get the well-ordering theorem and the axiom of choice in T** + (MPS*).

though they are not similar! What Russell hoped to gain by the superfluousness doctrine is lost. The idea of the "superfluousness" of classes was, therefore, no advance over theories requiring that the "simple" propositional functions be found.

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