PART V OF *THE PRINCIPLES OF MATHEMATICS*

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I here present a collation of the printer's copy of Part V of *The Principles of Mathematics* and the printed text of the first edition. As with two earlier studies of Parts I and II of *Principles*, the aim is to establish the changes Russell made in the text between May 1902, when the manuscript was sent to Cambridge University Press, and May 1903, when the book was published.[1] Since Part V develops ideas on continuity and infinity that depart substantially from the corresponding sections of Russell's latest pre-Peano manuscript, it is reasonable to suppose that a collation of Part V should illuminate the development of Russell's thought in this important area and period of time.[2]

In fact, what we can learn from the collation of Part V is rather different from what we learn from the collation of Parts I and II. In the case of Part I, we know, from Russell's correspondence with his wife, Alys, that he engaged in a major revision of that part in May 1902. So the printer's copy presents Russell's views as of May 1902. In the case of Part II, we know that Russell wrote a version of it in June 1901. There is no clear evidence of substantial rewriting after that

[1] Kenneth Blackwell, "Part I of *The Principles of Mathematics*", *Russell*, n.s. 4 (1984): 271–88; Michael Byrd, "Part II of *The Principles of Mathematics*", *Russell*, n.s 7 (1987): 60–70.

[2] In keeping within the limits of the earlier studies in this series, my collation does not report the alterations visible on the manuscript.

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time.[3] So, for Part II, the printer's copy presents Russell's views as of June 1901, or possibly later. On the other hand, the printer's copy of Part v dates primarily from November 1900. It contains terminology, arguments, and views that conflict in various ways with the versions of Parts I and II written in 1901 and 1902. Thus, the printer's copy of Part v can give us valuable insight into Russell's logical and philosophical ideas very shortly after he began his careful study of Peano's work. In particular, the collation shows that the central doctrine of the published text, logicism, was *not* a view that Russell held in November 1900.

1. *The Manuscript Text*

The initial leaf of the printer's copy of Part v is dated "November, 1900", and the final leaf is dated more precisely, "November 24, 1900". The upper left-hand corner of each leaf bears the notation "IC", presumably for the title of Part v, "Infinity and Continuity". The leaves are numbered consecutively, 1 to 199. There are several pages that bear "a" numbers (e.g. "35a"), which were probably inserted at some point after the initial period of composition.[4] There are several pages with double numbers; these leaves were taken from the *Principles of Mathematics* manuscript of 1899–1900, written prior to Russell's study of Peano.[5]

The list of variants is given at the end of the essay. It is constructed following the models of the previous collations in this series. The list

[3] Byrd, pp. 61–2.

[4] The added leaves are 25a, 35a, 37a, 43a, 84a, 101a, 104a, 108a, 118a, 124a, 157a, and 157b. The terminology and nature of these additions suggest that they may have been composed in 1901, perhaps around the time Russell was writing the May–June 1901 versions of Parts I and II.

[5] This manuscript is Russell Archives file 230.030320. The leaves with double numbers are 74, 76, 77, 108, and 157b, and their original locations were respectively 27, 29, 30, 41, 42, and 73 of Part v of the 1899–1900 manuscript. See John King, "A Report on the Manuscripts of 'Analysis of Mathematical Reasoning', 'The Fundamental Ideas and Axioms of Mathematics', and 'The Principles of Mathematics'" (unpublished, copy in RA). In addition, while folios 42 to 49 do not have double numbers, it is clear from subject matter that they are pirated folios 12 to 19 from Part v of the same manuscript; Russell simply converted a "1" into a "4", rather than adding a new number.

is read as follows. At the left is a number such as 260: 18. This means page 260, line 18 from the top. To the right is the reading from the published text of the first impression of *Principles*. This is followed by a square bracket and then the corresponding reading from the printer's copy. Editorial brackets enclose my comments.

The alterations to the text are substantial, both in quantity and, I shall argue, significance. There are approximately 3,500 words of manuscript text which are altered in form or deleted. By way of comparison, the list of variants for Part I, which is about the same length, consists of 1,900 words of altered text. Five completely new sections were added to the printer's copy after May 1902: §§299–301 and 348–9. Other sections contain major changes or new paragraphs: §§253, 254, 285, 338, 344, 347.

As indicated above, the manuscript bears the date "November, 1900" on the first page. Russell's recollection of the composition of *Principles* bears this out. To Jourdain, he wrote,

> During September 1900 I invented my Logic of Relations; early in October I wrote the article which appeared in RdM VII 2–3; during the rest of the year I wrote Parts III–VI of my *Principles* (Part VII is largely earlier, Parts I and II wholly later, May 1902)....[6]

The *Autobiography* makes a similar claim about Parts III–VI, though leaving a different impression about Parts I and II:

> With the beginning of October I sat down to write *The Principles of Mathematics*, at which I had already made a number of unsuccessful attempts. Parts III, IV, V, and VI of the book as published were written that autumn. I wrote also Parts I, II, and VII at that time, but had to rewrite them later, so that the book was not finished in its final form until May 1902. (*Auto.* I: 192–3)

[6] Ivor Grattan-Guinness, *Dear Russell—Dear Jourdain* (London: Duckworth, 1977), p. 133. The letter is from April 1910. Recent work by Rodríguez-Consuegra certainly challenges the claim that the *RdM* paper was in any sort of complete state in October 1900. See F. Rodríguez-Consuegra, *The Mathematical Philosophy of Bertrand Russell: Origins and Development* (Basel: Birkhauser, 1991).

The terminology and doctrine of the manuscript of Part v suggest that it was composed early and not significantly revised before submission in May 1902. Here are two examples of terminology. At a number of places in Chapter XXXIII, Russell replaces the manuscript use of the quantifier word "some" with the words "a" or "a variable". The reason for the change is clear in the light of Russell's theory of denoting concepts, which is explained in Chapter v of the published text, but which had already been worked out in some detail in Chapter IV ("Conjunction and Disjunction") of the draft of Part I written in May 1901.[7] In this theory, Russell uses "some" and "a" to indicate contrasting quantifier scopes, with "some" wider and "a" narrower. In the altered passages in Chapter XXXIII, the narrow scope reading is the appropriate one; so, "some" needs to be replaced by "a".

A second example is the replacement of the manuscript word "concept" at several places by the word "term". Under the influence of Moore's "The Nature of Judgment", Russell's ontology, as he began to write *Principles*, consisted entirely of (mind-independent) entities called *concepts*. It was concepts that words designate (fol. 14, 264: 35); "term" and "concept" were used synonymously to cover every entity, including physical objects, such as tables (fol. 179, 335: 40–4). But in a considered departure from this position, Russell argues, in Chapter II of the May 1901 version of Part I, that concepts are a proper subclass of terms.[8] Russell cites as examples of terms which are not concepts Socrates, points of space, instants, bits of matter, and particular states of mind, a list which is repeated in Chapter IV of the published text.

In addition, the footnotes in Part v, citing Russell's 1901 papers on the logic of relations and the notion of position in space and time, are absent from the manuscript.[9] In the first case, Russell refers the reader to a now nonexistent appendix. In the second case, he refers ahead to Part VII.

There also two major doctrinal differences that will be discussed in some detail below. First, the manuscript nowhere proposes the familiar

[7] RA1 230.030320, fos. 24–8; *Papers* 3: 195–201.

[8] *Ibid.*, fol. 10; *Papers* 3: 189–90.

[9] "Sur la Logique des relations", *Revue de mathématiques*, 7 (1901): 115–148; "Is Position in Space and Time Absolute or Relative?", *Mind*, 10 (1901): 30–51; both now in *Papers* 3.

logicist definition of the concepts *cardinal number* and *ordinal number*. Indeed Russell explicitly denies that numbers are classes and maintains that number is philosophically indefinable. Second, as J. A. Coffa has pointed out, the manuscript of Part v holds that the various proofs of Cantor's Theorem (that 2^{α} is always greater than α) contain *errors* in their application to very large classes, such as the class of all classes.[10]

It is clear that by the middle of 1901, Russell had changed his mind on both these matters. The version of Part I written in May 1901 begins with the explicitly logicist description of pure mathematics found in the published text. The published version of "The Logic of Relations" explains, at the end of its first section, the nominal definition of a cardinal number as an equivalence class of similar classes. In regard to Cantor's Theorem, Russell writes to Couturat in October 1901 that Cantor's proof was "irrefutable"[11] We are thus led to the conclusion that while Russell wrote Part v in November 1900, he sent it to the printer in May 1902, largely unrevised, containing views at variance with those of the submitted manuscript of Part I.

Russell's letters to Alys in May 1902 suggests this interpretation as well. Throughout early May, Russell is working on the revision of Part I; he gives sufficient information in the letters that we can trace reasonably accurately what chapter he is working on.[12] Russell finished this revision on 13 May. On 16 May, he writes that he expects to have the book finished in another two months. Then only a week later, he is finished with the manuscript; he writes to Alys:

> Thee will be amused surprised and amused, after all my talk of 2 months, to hear that I finished my book yesterday. I found a pile of old ms, which I had expected to have to rewrite, required only a few additions and corrections.[13]

[10] J. Alberto Coffa, "The Humble Origins of Russell's Paradox", *Russell*, nos. 33–34 (spring–summer 1979): 31–8. Russell's paper "Recent Work on the Foundations of Mathematics" (in *International Monthly*, 4 [1901]: 83–101; reprinted as "Mathematics and the Metaphysicians", *Mysticism and Logic* [New York: Doubleday, 1957]; in *Papers* 3), written in January 1901, contains a similar remark about errors in Cantor, but with no analysis of the proofs.

[11] This letter is quoted in Coffa, p. 38.

[12] Blackwell, pp. 278–9.

[13] Quoted in Blackwell, p. 280.

What is "the pile of old ms." that "required only a few additions and corrections"? Given the time constraints involved here, it is reasonable to think that "the pile of old ms." was the fall 1900 version of Parts III–VI.

2. *The Missing(?) Parts* I *and* II

Russell's comments in the *Autobiography* account of the composition of *Principles* imply that Russell wrote versions of Parts I and II in the fall of 1900, and subsequently rewrote them. If so, they have disappeared without a trace. In his recent book *Bertrand Russell and the Origins of the Set-Theoretic Paradoxes*, Garciadiego advances the view that Parts I and II were not written until May and June of 1901.[14] I think that a careful study of the manuscript of Part V lends support to Garciadiego's position.

There are two kinds of relevant manuscript evidence from Part V.[15] I assume throughout that the manuscript of Part V was written in the fall of 1900. First, there is a striking contrast between the back references to Parts I and II on the one hand and Parts III and IV on the other. There are only two explicit back references to Parts I and II in the manuscript; all others were added after May 1902. The first occurs at 278: 6, and has been discussed by Garciadiego; it reads: "Arithmetical theories of irrationals could not be treated in Parts I or II, since they depend essentially on the notion of order". In the printed text, the words "Parts I or II" are replaced by "Part II". This back reference is of the most general sort; it requires only that the Parts on Logic and Number not utilize the concept of order. This is a substantive claim, especially as regards the Part on Number, but

[14] Alejandro Garciadiego, *Bertrand Russell and the Origins of the Set-Theoretic Paradoxes* (Basel: Birkhauser, 1992), pp. 88–92.

[15] It is also significant that double- and triple-numbered leaves from the manuscripts of Parts I and II contain only numbers from extant manuscripts. Some triple-numbered leaves in Part I contain numbers from the 1899–1900 version, from the June 1901 version, and the May 1902 version. If there were a fall 1900 version, it would be surprising that these pages would find no place in it, since, even as late as 1902, Russell was satisfied with the accounts provided in these leaves. Two such leaves are fos. 137 and 138 of the manuscript of Part I. These were originally fos. 4 and 5 of the Part on Number in the 1899–1900 version of *Principles*, and then appeared as fos. 48 and 49 of the June 1901 version of Part II.

Russell could certainly have written this with only an outline of Parts I and II at hand.

The second back reference occurs at 282: 41; the manuscript reads:

> Another objection to the above theory is that it supposes rationals and irrationals to form part of one and the same series generated by relations of greater and less. This raises the same kind of difficulties as we found to result, in Part I, from the notion that the integers and rationals belong to the same series, or that some rationals are integers.

The printed text changes the back reference to Part II.[16] The earlier view alluded to here is the one set forth in Chapter XVIII of Part II. In §145, Russell writes:

> From the fact that ratios are relations it results that no ratios are to be identified with integers: the ratio of 2 to 1, for example, is a wholly different entity from 2.

This is virtually the same position as the one Russell maintained in the 1899–1900 version of the Part on Number. There he denies that fractions and ratios are numbers on the grounds of their relational character (fos. 13, 15). So, this back reference is certainly consistent with the absence of a newly written version of Parts I and II.

The absence of substantive, detailed back reference to Parts I and II is in sharp contrast to an array of fairly detailed back references to Parts III and IV.[17] Several of these are worthy of special note. At

[16] The manuscript's back reference may simply be an error on Russell's part. It is also possible that, at an early stage in the composition of *Principles*, Russell might have planned to begin the book with the Part on Number. This is the order of the 1899–1900 version of *Principles*. If, as I argue later, logicism was not initially the powerful organizing theme of the fall manuscript, the inversion of Parts I and II in the early planning would not be so surprising. Against this possibility is the note to be found on the initial leaf of Part II (Whole and Part): "I must preface Arithmetic, as Peano does, by the true Logical Calculus, to be called Book I, *The Individual*."

[17] The printer's copy of Parts III and IV each bear the date "October 1900" on their first leaf. Back references to Parts III and IV in the manuscript occur at the following places in the published text: 269: 12, 269: 30, 287: 10, 288: 39, 297: 44, 320: 18, 320: 20, 341: 33, 345: 10, 353: 38, 353: 45, and 354: 3. In contrast to Parts I and II, no new back references were added after May 1902.

345: 10 and 353: 45, Russell refers back to an earlier discussion of mathematical induction in Part III. The first of these reads: "And as for the notion that in every series there must be consecutive terms, that was shown, in the last Chapter of Part III, to involve an illegitimate use of mathematical induction." This clearly and accurately cites a particular chapter (Chapter XXIII) of Part III, and the issue in question is one on which Russell had changed his mind subsequent to the 1899–1900 version of *Principles*. At 269: 30, Russell refers back to the discussion of progressions in Part IV. This concept is not used in the pre-Peano version of *Principles*, although it is the topic of §3 of "The Logic of Relations".

Another kind of evidence supporting Garciadiego's view rests on the discussion of *logical* matters in Part V. A variety of topics of this sort are considered both in the manuscript and the published text: the meaning of "any" and "some", formal implication, the nature of relations, intensional and extensional treatments of classes, the notion of the variable. In the published text, these discussions are invariably accompanied by reference to specific chapters or sections of Part I, often with explicit indication that the material from Part I is being recapitulated. In the manuscript, *none* of these discussions refer back to Part I. In most of these cases, the published text substantially alters the manuscript. Moreover, the passages in the manuscript often suggest that the matters under consideration have *not* been previously discussed. Here are four examples:

(1) At 263: 9, Russell introduces the term *referent* and *relatum* as a way of referring to the domain and range of a binary relation. The published text precedes this with the remark: "For the present purpose, it will be well to recall two technical terms, which were defined in Part I." The back reference is to Part I, page 24. The manuscript reads instead: "For the present purpose, it will be well to introduce two technical terms, to which I shall adhere in future."

(2) There is a lengthy altered passage at 264: 7–32, in which Russell's views on propositional functions and the variable are stated.[18] This discussion ends in the published text with the remark: "But the

[18] In the manuscript, propositional functions are called "propositions containing variables".

investigation of these points has already been undertaken in Part I, and enough has been said to illustrate how a proposition may be a function of a variable." The manuscript contains a less sophisticated account of the relevant issues and concludes with the comment: "But the investigation of these points would carry us too far from our purpose, and enough has been said to illustrate how a proposition may be a function of a variable."

(3) The concepts of "any" and the variable are again discussed at 351: 6–12. The published text concludes: "On the logical difficulties of this conception I need not now enlarge: enough has been said on the subject in Part I." The manuscript reads: "On the logical difficulties of this conception I need not now enlarge: enough has been said to show what is meant by a variable."

(4) In the discussion of the definition of cardinal number at 305: 8, the published text reads: "By the principle of abstraction, we can give as we saw in Part II, a formal definition of cardinal number." In the manuscript, we find instead: "By means of the axiom of abstraction, which as we have seen, always takes us from the logically subsequent to the logically prior, we can give what, for formal purposes, may be regarded as a definition of cardinal numbers." This is a passage to which I will return later. For now, I note that the manuscript passage certainly suggests that the "formal" definition of cardinal numbers is here being introduced for the first time.

To summarize, the nearly complete absence of back references to Parts I and II and the manner in which logical issues are introduced in the manuscript of Part V make quite credible Garciadiego's contention that Parts I and II were not written in the fall of 1900. The evidence for nonexistence is better, I think, in the case of Part I than in the case of Part II. Given the rudimentary nature of the logical discussions of Part V, it is not clear what the Part on Logic would have contained in the fall of 1900. The manuscript of Part V leaves one with the strong impression that Russell is working on logical matters as he writes Part V. On the other hand, much of the material needed for Part II was already available to Russell in the fall of 1900. Sections 2, 3, and 4 of "The Logic of Relations" would have provided a satisfactory basis for some of the technical parts of Chapters XI through XIV, although the explicit logicist reduction of Chapter XI would have been supplanted by appeals to the "axiom" of abstraction, The later chapters of Part II (Chaps. XV through XVIII) are recognizably older; the influence of Peano in them is less pervasive. For example, the discussion of infinite wholes in Chapter XVIII is quite similar in structure to the chapter of the same title in 1899–1900 version of *Principles*. Chapter XVIII of the printer's copy of *Principles* argues the need for the concept of *magnitude of divisibility* in much the same way as is done in the Chapter "Ratio in connection with Whole and Part", the last chapter of Part II of the 1899–1900 version.

3. *The Status of Logicism in November 1900*

It has been noted before that several pre-*Principles* manuscripts maintain the indefinability of number and numbers.[19] Because of its later date, the discussion in Part I, Chapter II, "Pure Numbers and their Relations", of the 1899–1900 version of *Principles* is most relevant to the present discussion. There Russell holds that "numbers are indefinable" (fol. 9; *Papers* 3: 18). Russell supports this claim by arguing that the individual numbers (e.g. 2) are *not* complex concepts. If the numbers were complex concepts, then they would *imply* their constituents, but, Russell notes, there seem to be no such implications (fol. 9; *Papers* 3: 18–19). Moreover, he criticizes the attempt to define particular numbers by addition; for example, 2 = 1 + 1 (fol. 6; *Papers* 3: 17–18). Russell's argument against this proposal is approximately the one presented in the published text at 135: 20–9.

Demonstrating the truth of logicism is presented in the Preface to *Principles* as one of the two main objects of the book. Russell aims to "prove that all pure mathematics deals with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a small number of fundamental logical principles ..." (p. xv). This might well lead one to think that the familiar form of logicism found in the published text of *Principles* somehow emerged directly and immediately from Russell's initial acquaintance with Peano's logical techniques.

Recent work by Rodríguez-Consuegra questions this assumption. He notes that "the logicist claim did not appear in his first writings

[19] Rodríguez-Consuegra, p. 79.

after the Congress."[20] The writings considered are an incomplete draft of "The Logic of Relations", dated October 1900, and an essay Russell left unpublished, "Recent Italian Work on the Foundations of Mathematics".[21] In the first case, he notes that the explicit logicist reduction of the concept of *cardinal number* of a class, proposed at the end of §1 of the published text, is absent from the October 1900 manuscript. In the second case, he points out the absence of explicit logicist reductions, and the apparent endorsement at folio 3 (*Papers* 3: 353) of the idea that specific branches of mathematics have their own indefinable ideas and axioms. I note additionally that at folio 7, Russell *denies* that the number 2 is a class. There Russell is pointing out that the epsilon relation is not transitive; he writes: "For example, 2 is a number, number is a class, but 2 is not a class" (*Papers* 3: 354).[22] Certainly an inapt example, if you hold that 2 is the class of all 2-membered classes.

The collation of Part V reveals logicism in a complex, transitional state. First, it shows that all explicit logicist definitions of *cardinal number*, *ordinal number*, and *relation number* were added after May 1902. These changes occur in §§253, 284, 295, and 299. These famous definitions in terms of equivalence classes are simply *absent* from the November 1900, manuscript. On the other hand, Russell is already prepared to offer an explicit identification of the real numbers with segments of rationals. Thus, in the manuscript of Chapter XXXIII ("Real Numbers"), Russell writes: "A real number, so I shall contend, is nothing but a certain class of rationals" (fol. 270). This chapter of the manuscript appears in the published text with only minor changes.

What then were Russell's views about the cardinal numbers and the concept *cardinal number* in November 1900? To answer this, we should consult two passages removed from the published text; these occur at §§284 and 338. At the beginning of §284, Russell briefly criticizes Cantor's definition of a cardinal number as the general idea

[20] *The Mathematical Philosophy of Bertrand Russell*, p. 186.

[21] See *ibid.*, p. 142, for a brief discussion of the dating of this manuscript as from October 1900. It is now published in *Papers* 3.

[22] Russell makes the same claim about the number 2 at folio 177 of the manuscript of Part V. This passage is removed from the published text; see the collation for 356: 6–43.

obtained from a collection by abstracting from the nature and order of its elements. Russell holds that this is not a "true" definition. He *then* makes the point that the definition presupposes that there is *some* property that a collection has that is independent of the nature and order of its elements. In the published text, it appears that the latter claims is the reason why Cantor's proposal is not a "true" definition. But this is not a correct description of the argument that appears in the manuscript. There the paragraph continues with the following surprising assertion:

> In fact, number is a primitive idea, and it is a primitive proposition that every collection has a number. It is therefore philosophically correct that a specification of number should not be a formal definition.

This position resembles closely the view of number advocated in the 1899–1900 version of *Principles*. Russell simply declares that number is primitive; there is no argument presented for this claim. Certain facts about the relation between numbers and classes are simply to be postulated. Moreover, one paragraph later, Russell attempts to show the *philosophical* inadequacy of a proposed formal definition of cardinal number. Thus, as in the 1899–1900 version of *Principles*, he accompanies his declaration of the primitiveness of number with arguments against proposed definitions of number. What *has* changed between the earlier version and the printer's copy of *Principles* is that in the latter, Russell introduces the qualifier "philosophical"—he denies that there is a *philosophically* correct definition of number, whereas in the earlier version, he simply denies that there is a definition.

The qualifier is important, because in the printer's copy, Russell is prepared to admit that one can give a *mathematically*, or *formally*, adequate definition of cardinal number or of particular cardinal numbers. Now Russell does *not* present the famous definition of the cardinal number of the class M as the class of all classes similar to M. Rather he offers a definition "by abstraction", using the equivalence relation of similarity between classes. Classes M and N are similar if and only if there is a one-one correspondence from M onto N. Similarity is reflexive, symmetric, and transitive. By what Russell calls the principle, or axiom, of abstraction, the fact that a relation has these properties entails that it can be "analyzed as the product of a many-one relation and its converse" (*PoM*, p. 305).[23] According to Russell, in the case of similarity, this analysis "indicates a common property of similar classes", a property which "we call their cardinal number" (*PoM*, p. 305). This definition, Russell claims, is "the best way to introduce the cardinal integers mathematically and to make it plain that there are such entities" (fol. 97).

By the middle of 1901, Russell regarded this sort of definition as unsatisfactory on the grounds that "it does not show that only one object satisfies the definition" (*PoM*, p. 114). This criticism of definition by abstraction also occurs at the end of §1 of "The Logic of Relations". As Rodríguez-Consuegra has pointed out, this claim does not occur in the October 1900 version of the paper.[24]

In the manuscript version of Part v, Russell continues by saying that the *mathematically* acceptable definition by abstraction of cardinal number is not *philosophically* acceptable. He makes two points. First, Russell observes that "the relation of similarity is complex, and presupposes the cardinal integers, which are therefore not, in the philosophical sense of the word, *defined* by means of similarity" (fol. 97, Russell's emphasis). Of course, even granted that similarity is complex, that alone does not preclude its use in a definition of cardinal number. The crucial claim is that the notion of similarity "presupposes" the cardinal integers. In the period before *Principles*, Russell's use of "presupposition" is often explicable as one-way implication: A presupposes B if and only if A implies B but not conversely.[25] In the case of concepts, this should be read as saying that the being of A implies the being of B, but not conversely. The notion of implication at issue here is clearly not material implication, since Russell holds that both the relation of similarity and the cardinal numbers have being. The exact character of this notion need not concern us here.

Russell is thus claiming that (1) the being of similarity implies the

[23] In the printer's copy, the claim is called the *axiom* of abstraction, but this is changed to *principle* in the published text. By 1901, Russell came to regard the claim as provable, hence the change in terminology. It is Theorem 6.2 in the published text of "The Logic of Relations".

[24] "Russell's Logicist Definitions of Numbers, 1898–1913: Chronology and Significance", *History and Philosophy of Logic*, 8 (1987): 141–169 (at 149–50).

[25] See Part II of the 1899–1900 version of *Principles*, fos. 1–3, for discussion of the notion of logical priority (*Papers* 3: 35–6).

being of the cardinal numbers, and (2) the being of the cardinal numbers does not imply the being of similarity.

With respect to the second point, Russell simply states that the "cardinal integers, finite and transfinite alike, are logically independent of classes, which have to them the same kind of relation as quantities have to magnitudes" (fol. 97). This view of the independence of classes and numbers had been endorsed by Russell in the initial chapter of Part I in the 1899 version of *Principles* (fol. 1; *Papers* 3: 15). This independence is relevant to the second point because similarity is a relation between classes. If the being of the cardinal numbers implied the being of the relation of similarity between classes, then it would certainly appear to follow that the being of the cardinal numbers implied the being of classes, so that classes and numbers would not be logically independent.

To explain the independence claim, Russell offers the comparison with quantities and magnitudes. Quantities have magnitudes; different quantities may have the same magnitude; a given quantity can only have one magnitude of a kind.[26] Classes have cardinal numbers; distinct classes may have the same cardinal number; a given class can only have one number. More significantly for the independence claim, it is coherent to suppose that there are some magnitudes such that there is no quantity that has that magnitude. A quantity, according to Russell, is a magnitude "particularized by temporal, spatial, or spatiotemporal position" (*PoM*, p. 167). But a magnitude does not have to be particularized in this way in order to have being. The corresponding claim in the case of classes would be that it is coherent to suppose that there are numbers such that no class has that number. And indeed, it is to this question that Russell turns in the passage under consideration.

Russell has still to argue that the being of similarity implies the being of numbers. To defend this claim, Russell brings forward what he regards as a serious weakness in the method of definition by abstraction:

[26] For Russell's use of "quantity" and "magnitude", see *Principles*, p. 159. This contrast had already been worked out in the 1899 version of *Principles*; see *Papers* 3: 54–9.

> For the above method will only define such numbers as are the numbers of some class; if there be others, they remain indefinable. (Fol. 97)

The comparison with quantities and magnitudes is, I think, quite clear: there may well be magnitudes of certain types (temperature, for example) such that no quantity has that magnitude. So, to define magnitudes of this sort by abstraction from an equivalence relation on quantities would leave uncharacterized those unrealized magnitudes. It would be erroneous to define a magnitude of that type as the magnitude of some quantity of that sort.

In the case of cardinal number, a similar point holds. The definition by abstraction will "define only such numbers as are the numbers of some class ..." (fol. 97). Thus the adequacy of a definition by abstraction of cardinal number requires that every number be the number of some class. Now, Russell does not think that there are in fact numbers which are not the number of some class. But he does think that this can, and ought to, be proved. However, according to Russell, proof of this claim will involve appeal to classes of *numbers*. Thus Russell writes:

> It can be proved, it is true, that there is no number, finite or transfinite, which is not the number of some class.... But the proof that every number is the number of some class is only obtainable by considering classes of numbers, and therefore presupposes the being of all the numbers there are. (Fol. 97)[27]

So, to establish the sufficiency of the definition by abstraction using similarity, we must assume that there are numbers. In this sense, the definition presupposes numbers. Russell therefore concludes that definition by abstraction is "a method for indicating a class of entities, but does not show that these are complex, or in any philosophical sense definable" (fol. 97).

Russell's views on the definability of cardinal number in November 1900 thus differ from the published text in two major respects:

[27] Russell does not here explain why the proof in question would require the consideration of numbers. What he may have had in mind is the kind of proof he gives in §339 for the existence of infinite classes.

(1) Russell offers a "formally adequate" definition by abstraction, based on similarity between classes, but does not propose the familiar nominal definition using classes of equivalence classes;

(2) He holds that although cardinal number is formally, or mathematically, definable, it is not philosophically definable.

In the published text of *Principles*, the *philosophical* sense of "definition", which is characterized as the analysis of a complex concept into its constituents, is largely shunted aside. It is declared "inconvenient" and "useless" (p. 111). The definitions presented are said to serve all *mathematical* purposes of cardinal numbers (p. 116). Whether there is some other set of entities which can be seen, by inspection, to be the *real* cardinal numbers is "irrelevant to Mathematics" (p. 116). The benchmark for the adequacy of the definitions is that they fulfil all mathematical purposes, nothing more.

Russell's views in November 1900 on the definition of *cardinal number* stand in sharp contrast to his view on the definition of *real number*. Russell offers a straightforward nominal definition of real numbers: a real number is a segment of rational numbers.[28] Russell defends this definition, contrasting it with a definition by abstraction of the reals. Cantor had defined two infinite classes u, v of rationals as *coherent* if and only if u and v have no maximum, for every element of u, there is a greater element of v, and conversely, for every element of v, there is a greater element of u. Coherence is an equivalence relation on classes of rationals. So, according to the principle of abstraction, this indicates a third term, some common property, shared by coherent sets of rationals. These properties might be regarded as, or as determining, the real numbers.

The manuscript of Part V contains two somewhat different reactions to the proposed definition by abstraction. At the end of Chapter XXXIII, Russell writes, "The third term, as we see from the preceding discussion, is the segment which both define"[29] (fol. 35). What the

[28] A (lower) segment L of rational numbers is a non-null class of rationals, not including all rationals, with the property that a rational r is a member of L if and only if there is rational s in L such that r is less than s (*PoM*, p. 271).

[29] The published text reads, "The third term, as we see from the preceding discussion, *may be taken to be* the segment which both define." The change reflects, I believe, Russell's realization of the presence of equally viable alternatives; he could have used upper segments instead of lower, or equivalence classes of coherent sets of rationals.

preceding discussion in Chapter XXXIII has shown is that the class of segments of rationals form a *perfect* series; that is, it contains all and only its limit points. More generally, Russell has argued that segments have all the mathematical properties "commonly assigned to the real numbers" (p. 270). This he takes to be sufficient grounds for inferring that the segments of the rationals are the reals; the segments are the common properties shared by coherent series of rationals.

At the end of Chapter XXXIV, Russell proceeds differently. He first lodges a *criticism* of the principle of abstraction: "But the principle leaves us in doubt as to what the real numbers really are." To my knowledge, this criticism is not lodged elsewhere against the principle of abstraction. In response to this, Russell points out that segments have all the relevant mathematical properties of the reals. He adds that positing an additional set of entities with these same properties is a "wholly unnecessary complication" and so "an irrational actually *is* a segment of rationals which does not have a limit" (p. 286, Russell's emphasis). To an apparently philosophical objection to the principle of abstraction, Russell responds by pointing out that segments suffice for mathematical purposes. To distinguish the reals from the segments on the basis of "some immediate intuition" or by an axiom asserting the existence of limits of all series of rationals would be "fatal to the uniform development of Arithmetic and Analysis from the five premisses which Peano has found sufficient" (p. 286).

Despite the different treatments accorded definition by abstraction, both chapters concur in dispensing with the contrast between philosophically and mathematically adequate definitions. Russell argues that segments are the reals, precisely because they have the relevant mathematical properties. There is no further philosophical question raised about the adequacy of the definition. The question of whether the real numbers are philosophically definable is either not discussed, or else not distinguished from the issue of whether they are mathematically definable. Why the treatment of cardinal number and real number is so different at this stage of Russell's work is an interesting one that lies beyond the scope of the present paper.

In the light of the foregoing, it is clear that the logicist project stated in the Preface to *Principles* was not the unifying theme of the book when Russell began to write in the fall of 1900. We also know that it had become a major organizing theme by May 1901. The version of Part I written in that month begins with a chapter entitled "The Definition of Pure Mathematics". The first sentence of the chapter characterizes pure mathematics in almost exactly the way the published text does:

> Pure Mathematics is the class of all propositions of the form "*a* implies *b*", where *a* and *b* are propositions each containing at least one variable, and containing no constants except logical constants or such as can be defined in terms of logical constants. (RA1 230.030320, fol. 1; *Papers* 3: 185)

To justify this definition, Russell claims, requires a detailed analysis of mathematical propositions. This analysis would occupy "the subsequent parts of the present work" (fol. 2; *Papers* 3: 185). We ought then to ask: (1) What, if anything, can we learn about the process between November 1900 and May 1901, whereby logicism came to be a unifying theme of *Principles*? (2) When Russell began to write *Principles* in the fall of 1900, did he have some unifying theme or set of themes in mind? If so, what were they?

4. *Errors in Cantor*

Whatever may have been the unifying themes Russell had in mind in the fall of 1900, it is very clear that a principal objective of the November 1900 version of Part V of *Principles* was to argue that the fundamental notions of continuity and infinity were free from contradiction. At the end of Part V, Russell concludes:

> ... we found that all the usual arguments, both as to infinity and as to continuity, are fallacious, and that no definite contradiction can be proved concerning either. (Fol. 199)[30]

[30] In the published text, this sentence continues with an acknowledgement of the difficulties raised by the newly discovered paradoxes: "although certain special infinite classes do give rise to hitherto unsolved contradictions" (p. 368).

This consistency thesis represents a major change from the view maintained in the 1899–1900 version of *Principles*. There Russell admitted the necessity, and apparently the consistency, of infinite classes, but denied the coherence of infinite number (Part V, fos. 50, 51). Chapter V of Part V criticized Cantor's views on transfinite number on the grounds that they lead to what Russell calls "the antinomy of infinite number" (fol. 40; *Papers* 3: 119). I will not discuss Russell's argument in any detail here. He attempts to extract a contradiction from the idea that there is a number of all numbers. The reasoning bears some similarity to the argument for the Burali-Forti paradox, but it shows that Russell had not yet attained an adequate grasp of the distinctive properties of the transfinite ordinal and cardinal numbers.

Thus it is a *central* contention of the November 1900 version of Part V that no such contradictions can be established. So, for Russell, the conflict between Cantor's power set theorem and its apparent failure in the case of large classes, such as the universal class, *required* resolution. This is undertaken in the final chapter of Part V, "The Philosophy of the Infinite". As is now well known, Russell argues that there are "errors" in Cantor's proofs.[31] In so doing, reasoning similar to that involved in Russell's Paradox makes its first appearance. However, in its original appearance, Russell does not argue that he has *discovered* a contradiction; rather he claims, using this reasoning, to have shown how a contradiction *can be avoided*.

Coffa has provided an excellent discussion of the crucial passage in Chapter XLIII. What follows is, I think, supplementary to his account. In this passage, Russell argues that Cantor's Theorem is false in the case of *Class*, the class of all classes (fos. 197, 198). Cantor's famous diagonalization argument for the Power Set Theorem purports to show that for any many-one relation k from a class C to its power class $P(C)$, there is a member of $P(C)$ not in the image of k. In the

[31] The first published discussion of this portion of the manuscript of Part V is in Coffa. The only published claim alleging errors in Cantor is in "Recent Work on the Principles of Mathematics", *Papers* 3.

It should also be noted that the presentation of the Burali-Forti paradox at §301 was inserted after May 1902. This fact was first pointed out by G. H. Moore and A. Garciadiego in their paper, "Burali-Forti's Paradox: a Reappraisal of Its Origins", *Historia Mathematica*, 8 (1981): 319–50.

case of *Class*, Russell considers the many-one relation k defined by:

$$k(x) = \begin{cases} \{x\}, \text{ if } x \text{ is a class, but not a class of classes} \\ x, \text{ if } x \text{ is a class of classes} \end{cases}$$

Cantor's proof claims that when the diagonal procedure is applied to such a relation, "a new term", to use Russell's words, a term not in the image of k, is obtained. Contrary to Cantor, Russell claims that:

(*C*) In the case of *Class* and the relation k, Cantor's method has not given a new term.

In support of (*C*), Russell offered two strands of argument. It is fairly clear that Russell had not clearly distinguished these strands; one of the strands is sandwiched between statements of the other strand. While both strands of argument yield (*C*) as conclusion, they are incompatible with each other.

Given the class and the relation k, let u' be the class (purportedly) defined by Cantor's procedure:

x is a member of u' if and only if x is not a member of $k(x)$.

The first strand of argument in support of (*C*) is the following:

But u' is a class of classes, and is therefore identical to $k(u')$.[32]

Here Russell accepts that u' is a class of classes; but given the definition of k, it is not "a new term", since $k(u') = u'$. In this case, Russell attempts to support (*C*) by maintaining that:

(*A*) In the case of *Class* and the relation k, Cantor's method yields a term, but not a new term.

[32] I have transposed Russell's notation to the familiar functional form. Instead of "$k(u')$", Russell uses "$k_{u'}$".

A second strand of argument, with quite different implications, immediately follows:

In fact, the procedure is, in this case, impossible; for if we apply it to u' itself, we find that u' is a $k(u')$, and therefore not a u'; but from the definition, u' should be a u'. (Fol. 197)

If the procedure is in this case impossible, then the conclusion to be drawn is that there is no such class as u'. Even when his ontology included nonexistent objects, Russell resolutely held that impossible specifications are true of nothing at all.[33] So the conclusion that one would expect Russell to have drawn is that in this case, the relevant procedure specified nothing at all. So, the second strand of Russell's argument holds:

(*B*) In the case of *Class* and the relation k, the diagonal procedure yields nothing at all.

Both (*A*) and (*B*) imply (*C*), but they are incompatible with each other.[34]

The first strand of argument should not, and apparently did not, satisfy Russell for very long; for it does not identify any error in Cantor's argumentation. It is clear, of course, that since $k(u') = u'$, u' is in the image of k. But Cantor had an *argument* that it is not, and the real problem is to locate where that argument fails. Otherwise, the threatened contradiction remains. Moreover, if it is granted that the diagonal procedure does indeed specify a class, then the remaining part of Cantor's argument is elementary and classically valid. By the fall of 1901, this is Russell's view; he writes Couturat: "I thought I could refute Cantor; now I see that he is irrefutable."[35]

Russell's argument supporting conclusion (*B*) is similar in many respects to the argumentation of Russell's paradox. Since u' is a class of classes, $k(u') = u'$. The class u' is characterized by the condition: x

[33] This is clear from Russell's discussion of null class-concepts in *PoM*, pp. 73–6.

[34] That the specification in the diagonalization argument fails to specify a class is the conclusion drawn in set theories admitting the universal set. See T. E. Forster, *Set Theory with a Universal Set* (Oxford, 1992).

[35] Letter to Couturat, 2 Oct. 1901; quoted by Coffa, p. 37.

is in u' if and only if x is not in $k(x)$. Hence, u' is in u' if and only if u' is not in $k(u')$. Then applying the identity, u' is in u' if and only if u' is not in u'.

The argument, as actually presented by Russell in the November 1900 manuscript, is not formulated in a sure-handed way. In the relevant passage, we are told to apply the procedure to u' itself. The only procedure suggested in the passage that can be meaningfully applied to u' is the condition: x is in u' if and only if x is not in $k(x')$. This yields "u' is in u' if and only if u' is not in $k(u')$." Russell then says that upon this application, we "find" that u' is in $k(u')$. We could find this out from the biconditional cited if we knew that u' is not in u'. But that is not a premiss or assumption in the argument as Russell presents it; rather it is the *conclusion* which Russell draws from the claim that u' is in $k(u')$. So, it is unclear how application of the procedure enables to "find" that u' is in $k(u')$. From this, it follows, as Russell says, that u' is not in u'.

The basis for Russell's next claim is also unclear; he claims that we may infer "from the definition", that u' should be in u'. The definition in question could be either the definition of the relation k or the definition of the class u'. If the reference is to the latter, the problem of interpretation is same as for the claim, just considered, that u' is in $k(u')$. If the reference is to the former, the relevance of the definition is not evident. Certainly, it immediately follows from the definition that $k(u') = u'$. So we could infer that u' is in u' if we knew that u' is in $k(u')$. But this just returns us once again to the problem of the preceding paragraph. While Russell's Paradox is certainly near at hand in this passage from Part v, Russell account of the logic of the situation is not yet perspicuous.

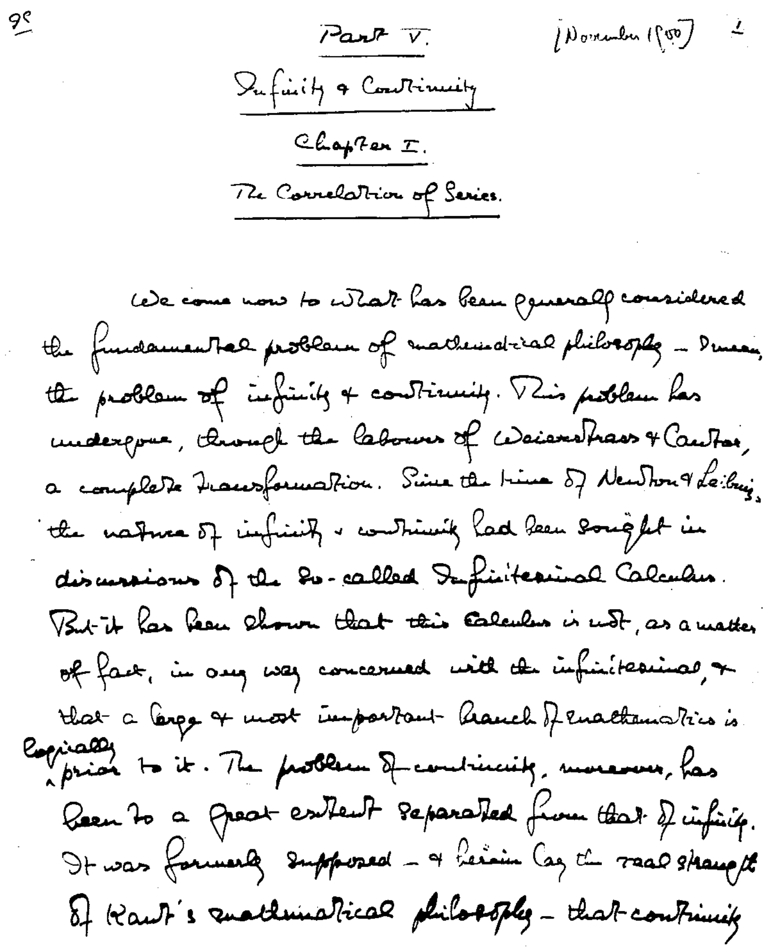
By May 1901, Russell's handling is much surer. In Chapter III, "Classes and Relations", of the penultimate version of Part I, Russell presents "a contradiction" derived from the complex relation of not being self-predicable.[36] He argues that there is no predicate predicable of all and only those predicates not predicable of themselves. In his argument, he invokes the Law of Excluded Middle. Such

[36] This passage occurs on folios 22 and 23 of the manuscript (*Papers* 3: 195) and reappears heavily edited at pages 97–8 of the published text.

a predicate P would be predicable of itself or not. He then argues for the *conditional* claims that if P is predicable of itself, then it is not, and if it is not predicable of itself, then it is. He then correctly identifies the result as a "contradiction".

When Russell read the proofs of Part v in late 1902 or early 1903, he naturally removed the discussion of "errors" in Cantor. The connection between Cantor's argument and the argument for Russell's Paradox is made explicit at §349. Russell's suggestion is "to accept the conclusion that there is no greatest number and the doctrine of types and to deny that there are any true propositions concerning all objects or all propositions" (p. 368). Yet this last claim is one that Russell regarded as unsatisfactory in 1903; for all propositions are either true or false (Appendix B, p. 526).[37]

[37] I benefitted from the help of Gregory Moore and Kenneth Blackwell in the preparation of the list of variants.



Part V. [November 1900] 1

Infinity & Continuity

Chapter I.

The Correlation of Series.

We come now to what has been generally considered the fundamental problem of mathematical philosophy – I mean, the problem of infinity & continuity. This problem has undergone, through the labours of Weierstrass & Cantor, a complete transformation. Since the time of Newton & Leibniz, the nature of infinity & continuity had been sought in discussions of the so-called Infinitesimal Calculus. But it has been shown that this Calculus is not, as a matter of fact, in any way concerned with the infinitesimal, & that a large & most important branch of mathematics is logically prior to it. The problem of continuity, moreover, has been to a great extent separated from that of infinity. It was formerly supposed – & herein lay the real strength of Kant's mathematical philosophy – that continuity

Folio 1 of the MS. of Part v of *The Principles of Mathematics*

VARIANTS BETWEEN *The Principles of Mathematics*, PART V, AND ITS MS.

CHAPTER XXXII. THE CORRELATION OF SERIES.

259: 23–8 The theory of … alike.] What follows, philosophically, is not, as some mathematicians have maintained, that these problems are *specially* concerned with numbers, but rather that they are presented wherever we find series of certain types—that they are, in fact, purely ordinal problems, which arise in Arithmetic and Geometry alike, because, in both, we find series of the types in question.

260: 18 The first, which applies to both cardinals and ordinals, is] The first is

260: 20 series of numbers] series

260: 20–2 0, proceeding in order of magnitude … and obeying] 0, and obeying

261: 5 a dependent variable] a function

261: 11 coupling every term of s …, and *vice versâ*,] between every term of S and every term of S',

261: 18 ordinal type] ordinal number

261: 24–6 *ordinally similar*; and … *likeness*.] *ordinally similar*.

261: 28 may be generated.] may often be generated.

261: 39–42 the original series. It can also … $P' = \breve{R}PR$.] the original series.

261: 43 *See my article in RdM, Vol. VIII, No. 2.] See Appendix

262: 1 a distinction] a philosophical distinction

262: 3 series by correlation. In the case] series by correlation. This distinction has no mathematical relevance, and is not even capable of any ordinal definition, but for the *philosophy* of order it is quite essential. In the case

262: 7–263: 3 But if it should happen … $Q\breve{S} = \breve{S}P^*$.] But philosophically, relations, like other terms, are simple or complex, and in the case of relations, the distinction is quite as important as elsewhere. A simple relation is one which, when it holds between two terms, does not, except in virtue of some further proposition, imply any terms or relations except the two terms and itself. A complex relation, on the other hand, may be complex in either or both of two ways. It may be, in itself and without further premisses, equivalent to two relations between the two terms, or it may be equivalent to the proposition that there is some third term to which each of our two terms is related in some way different from that in which the two original terms are related. Both these classes of complexity may be illustrated from human relationships, if we allow ourselves for the moment to regard *father* as a simple relation. "Beloved father" has complexity of the first kind, since it asserts both love and paternity. "Grandfather" has complexity of the second kind: for "*A* is my grandfather" means "There is (or was) a human being *B* who is (or was) my parent, and whose father is *A*." *Parent* illustrates a third way in which relations may be combined, namely by a disjunction or logical addition, since "parent" means "father or mother". But this method of combination does not necessarily give a relation which is more complex than its constituents, and is therefore here irrelevant.* [*The same applies to what Professors Peirce and Schroder call relative addition, a method of combination which I have found no occasion to employ.] ¶A relation which presupposes no others is philosophically simple, primitive, and indefinable; mathematically, it might be called a prime, since presupposed relations are always factors, whose product (either relative or of the ordinary logical kind) is the relation which presupposes them. The product is of the ordinary logical kind when the complexity of the relation is of our first kind (*i.e.* like *beloved father*), and relative when the complexity is of the second kind (*i.e.* like *grandfather*). The second kind of complexity presupposes one or more other terms, as well as other relations. With certain reservations, it is more or less arbitrary, mathematically, what relations we regard as simple, but philosophically there must always be only one correct answer to the question whether a relation is simple. Thus some series are generated by simple relations, others by relations which are not simple. It is, I think, capable of mathematical demonstration that a transitive asymmetrical relation, when it is complex, and one of its factors is transitive and asymmetrical, is of the form $\breve{R}PR$, where P is transitive and asymmetrical, and R is many-one; or if not already in this form, is at least capable of being brought into it.* [*See Appendix.] Series in which the generating relation is simple, or in which the constituents are not of the kind from which series are generated (*i.e.* not transitive or not asymmetrical), I shall call self-sufficient or independent series; it then results, from the above proposition, that all other series are generated by correlation, which must ultimately be correlation with an independent series.

263: 8 recall] introduce

263: 8 which were defined in Part I.] to which I shall adhere in future.

263: 12 domain] extension <*Also at 266: 31.*>

263: 21 an independent variable] a variable

263: 44 <*fn. added*>

264: 4 a propositional function] a proposition

264: 7–32: A proposition ... a function of a variable.] In this way, every proposition in which the word *any* occurs may be exhibited as a function of a variable, though it should be observed that some rather interesting logical changes are made when this is done. Let the proposition containing "any a" be called P_a, and let that containing x be called P_x. Then P_x simply, where nothing is known concerning x, cannot be true unless the proposition is one which is true of all entities. Otherwise, if P_a be true, and x be an a, then P_x is true; while if P_x follows from the proposition "x is an a", then P_a is true, but not otherwise. The peculiarity here is that, though "x is an a" seems to be a proposition, and can imply other propositions containing x, such as P_x, yet "x is an a" is, *per se*, not capable of truth or falsehood. And here we may illustrate the differences between *any* and *some*. P_a, which contains *any*, is equivalent to "if x is an a, P_x is true". But if for *any* we substitute *some* in P_a, the equivalent proposition is "if x is an a, it does not follow that P_x is false". Thus *some* seems to have essential reference to negation. But the investigation of these points would carry us too far from our purpose, and enough has been said to illustrate how a proposition may be a function of a variable.

264: 35 the term which both designate.] the concept which both denote.

265: 2–3 form, as a rule, a geometrical series] form a geometrical series

265: 14–15 This was a case considered ... to dependent series.] This was a case not considered in our general account of correlation.

265: 16 two correlated independent series] two correlated series

265: 20 $R\breve{R}$ contained in 1',] $R\breve{R}$ = 1',

265: 23 is contained in identity.] is identity.

265: 25–6 This is an important point, which is absolutely fatal to the relational theory of time.*] I shall return to this important theorem in Part VII, where we shall find that it is absolutely fatal to the relational theory of time.

265: 38 in the series.] in the series. Neglecting this case, a path which merely has a denumerable collection of multiple points of a finite order, though it cannot be made a self-sufficient series, can be exhibited geometrically, as the correlation of a self-sufficient series of x's with a dependent series of y's; and this is just what the equation of the curve effects. Thus although, in such cases, there must be some series which is dependent, the choice of the independent series may be more or less arbitrary.

265: 42–3 <*fn. added*>

266: 4 elliptic functions of a real variable,] elliptic functions,

266: 12 coincide; in others, again] coincide; in others, the two orders are exactly opposite; in others, again,

266: 32 Thus] Again

266: 35 In such cases,] In either of these cases,

267: 3 as we proceed.] as we proceed, and we shall find that it has an important connection with the doctrine of limits.

267: 4–5 class is related by a one-valued function to the finite integers,] collection is a one-valued function of the integers,

267: 6 a class becomes a series having the type] a series has the type

267: 9 the definition of the transfinite ordinals.] his definition of the transfinite ordinals and cardinals.

267: 25–6 a proposition, or more properly a propositional function, containing] a proposition containing

268: 4 a certain degree of intensional simplicity] a certain constancy

268: 25 not quite identical with it.] not quite identical.

268: 25–6 there is some relation] it may be set up as an axiom that there is some relation

268: 28–9 to at least one term] to one and only one term

268: 8–9 notion of a complete series.] notion of a *complete* series.

269: 10 when there is a term] when it is connected and there is a term

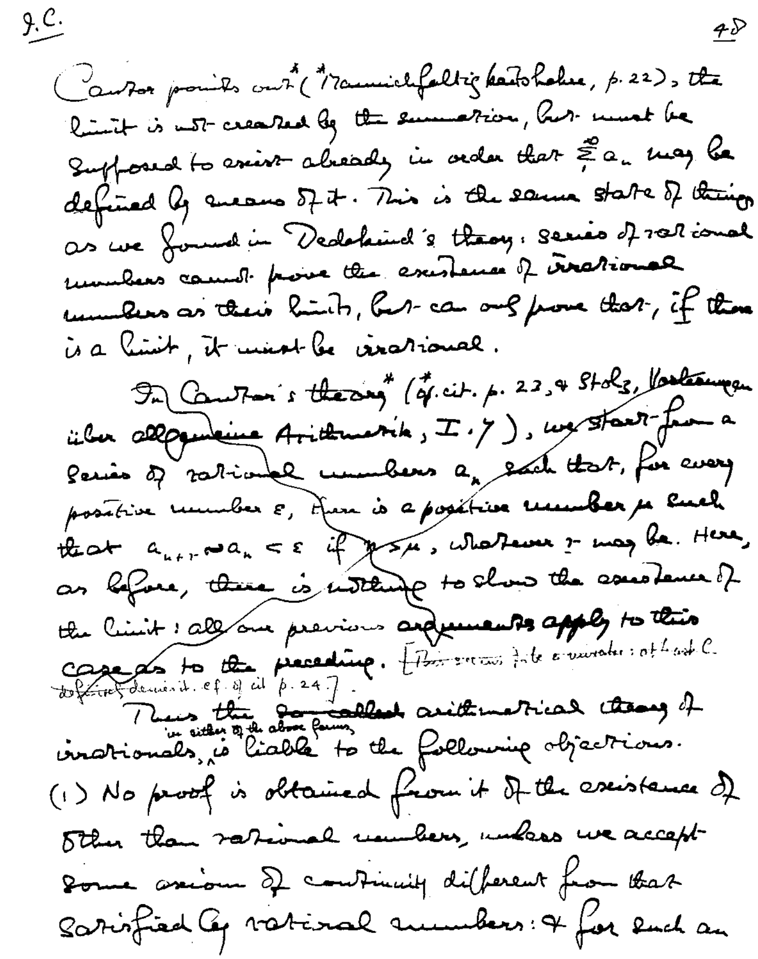
269: 15: the generating relation or its converse] the generating relation
269: 22–3 is incomplete with respect to the generating relations of the above complete series.] is incomplete.
269: 29 defined by powers] defined by multiples
269: 35–9 incomplete series. But it can be shown … generating relation.] incomplete series.

CHAPTER XXXIII. REAL NUMBERS.

270: 12 irrational numbers in the above sense;] irrational numbers;
270: 13: cannot be greater or less than rational numbers.] cannot belong to any series which contains rational numbers.
270: 17 finite cardinals] natural numbers
271: 20–1 a variable term] some variable term
271: 23 than in (1)] than (1)
271: 27 a (variable) term] some (variable) term
271: 28–9 of itself, *i.e.* with the class of rationals x such that there is a rational y of the said class such that x is less than y.] of itself.
271: 36 a variable u] some u <*Also at 272: 9.*>
271: 37–8: a variable u, *i.e.* those such that … smaller than it.] some u.
271: 45 § 61 (Turin, 1899).] (Turin, 1899), § 61.
272: 18–19 a variable term] some term *Also at 272: 29; 272: 36.*>
272: 45 <*fn. added*>
273: 23 the class of rationals] the rationals <*Also at 273: 24–5.*>
273: 41 a term] some term <*Also at 274: 1.*>
274: 15 principle of abstraction] axiom of abstraction
274: 18 may be taken to be] is
275: 9 assertion] equation

CHAPTER XXXIV. LIMITS AND IRRATIONAL NUMBERS.

276: 5 mathematics has] mathematics have
277: 2–7 <*Throughout these passages, BR uses the notation "$\pi_u x$" and "$\check{\pi}_u x$", where in the MS. he uses "u_p" and "$u_{\check{p}}$". Also at 277: 23.*>
277: 31 the whole compact series] the whole series
278: 6 Part II,] Part I or II,
278: 34, 34–5 denumerable compact series] compact series <*This change occurs twice.*>
279: 3–4 the two classes, while yet … last term.] the two classes.
279: 8 Continuity *seems*] Continuity seems
279: 26 If *all* be] If *all* is
280: 29 $x - 2$;] $x^2 - 2$; <*This is a misprint; it should read "$x^2 - 2$".*>
281: 17 a_n and b_m, for suitable values of p and q, either is a_{n+p}] a_n and b_m is a_{n+p}
281: 17 b_{m+q+1}] b_{m+q-1}
281: 33 2+ <*The text is not printed in a manner which clearly displays the fraction.*>
281: 33 and $x - 1$] and $\therefore x - 1$
282: 40–1 the same series generated by relations of greater and less.] the same series.
282: 42 in Part II,] in Part I,
282: 42–283: 1 integers are greater or less than rationals,] integers and rationals belong to the same series,
282: 43–4 p. 22. I quote … *Arithmetik, I.*] p. 22.
283: 9–10 can have relations of greater


J.C. 48

Cantor points out* (*Mannichfaltigkeitslehre, p. 22), the limit is not created by the summation, but must be supposed to exist already in order that $\sum_1^\infty a_n$ may be defined by means of it. This is the same state of things as we found in Dedekind's theory: series of rational numbers cannot prove the existence of irrational numbers as their limit, but can only prove that, if there is a limit, it must be irrational.

In Cantor's theory* (*op. cit. p. 23, & Stolz, Vorlesungen über allgemeine Arithmetik, I. 7), we start from a series of rational numbers a_n such that, for every positive number ε, there is a positive number μ such that $a_{n+r} \sim a_n < \varepsilon$ if $n > \mu$, whatever r may be. Here, as before, there is nothing to show the existence of the limit: all our previous arguments apply to this case as to the preceding. [illegible]

Thus the arithmetical theory of irrationals, in either of the above forms, is liable to the following objections.
(1) No proof is obtained from it of the existence of other than rational numbers, unless we accept some axiom of continuity different from that satisfied by rational numbers: & for such an

Folio 48 of Part V of *The Principles of Mathematics*, p. 282: 18–28 and less,] can belong to one and the same series,

283: 21–2 contains as a proper part the segment] contains the segment

284: 30 $a_v \times a_{v'}$] $a_v \,.\, a_{v'}$ <*Also at 284: 34.*>

284: 33, 35 <*The operation of multiplication is denoted by the infixing of the sign* "×", as in "$b \times b'$", *whereas in the manuscript it is denoted by concatenation, as in* "*bb′*".>

285: 28 defined by a fundamental series all of whose terms are equal to a_v,] corresponding to a_v,

285: 39 principle of abstraction] axiom of abstraction <*Also at 285: 44; 286: 10; 294: 3.*>

285: 41 This term, when] This, when

286: 18 The above theory] My theory

286: 38 Chapter XXXVI] Chapter V

CHAPTER XXXV. CANTOR'S FIRST DEFINITION OF CONTINUITY.

287: 12 This definition usually satisfied Leibniz,] This definition satisfied Leibniz,

287: 16 a discovery of which is the proof set forth] a discovery of which the proof is set forth

287: 28 this view is mistaken,] this is not the case,

287: 30–1 p. 515. But cf. Cassirer, … p. 183.] p. 515.

288: 11 this] his

288: 15 cohesive] connexive <*Also at 288: 20, 35; 289: 6, 11, 20, 22, 29, 32, 35; 290: 2, 13.*>

288: 18 cohesion] connexity <*Also at 288: 20; 289: 8, 10, 22, 27; 290: 15, 16, 17, 18.*>

288: 30 field] extension

288: 34 complete series] independent series

288: 43 omitted by Vivanti:] omitted by Peano:

289: 7 $n\delta$ is less than *d*.] $n\delta < d$.

290: 44 No. 3 (1899),] No. 3,

291: 14–15 certain conditions, which may be called the conditions of convergency, must have a limit.] certain conditions must have a limit.

292: 23 Chapter XXXIII] Chapter II

292: 33–4 which its various terms define] which it defines

292: 45 a series] a compact series <*Also at 293: 10, 15.*>

293: 19 perfection requires] perfection is,

293: 25 the class of series] the class of cases

293: 28–294: 3 <*MS. folio 74 is missing.*>

294: 14–15 relation or its converse to either,] relation to either,

294: 36 denying our axiom in the case of a series of numbers.] denying our axiom.

294: 38–9 is greater than] > <*Also at 295: 4–5, 19; 328: 8.*>

294: 41 is less than] < <*Also at 295: 17; 328: 8.*>

295: 5–6 in the case of series of rational numbers which have no rational limit.] in the case of rational numbers.

CHAPTER XXXVI. ORDINAL CONTINUITY.

296: 5 construct] attempt

296: 6 is free] shall be free

296: 20–1 the finite integers.] the natural numbers.

296: 33 *Math. Annalen*, XLVI] *Math. Annalen*, XLVI, XLIX.

296: 35 found in RdM, VII, 3.] found in the Appendix.

297: 24 terms] elements

298: 14 from there being a series] from their being a series <*MS. spelling error*>

298: 20 <*The subscripts of* "*x*" *in the text are elevated by 1 in the MS. Also at 298: 21, 25.*>

298: 31 denumerable endless series] denumerable series

299: 18 with the word *continuity*,] with the word,

299: 26 A lower segment,] A segment,

299: 28–31 such that *v* has no last term … an upper segment.] such that the class of terms having to some term of *v* the relation *P* (or $\breve{P}$, as the case may be) is identical with class *v* itself. When every term of *v* has to some term of *v* the relation *P*, *v* is called a lower segment; when $\breve{P}$, an upper segment.

299: 33 a variable term] some term

299: 35 whole or part] whole and part

299: 37–8 so have *U* and *V*, though … definition.] so have *U* and *V*.

299: 39–44 <*fn. added*>

300: 37–9 <*In the MS., Russell uses a different system of notation for denoting the four classes under discussion here and subsequently on pp. 301 and 302. The correspondences are the following:*

VERBAL FORMULATION	TEXT	MS.
the class of terms before every *w*:	$w\pi$	π^w
the class of terms after every *w*:	$w\breve{\pi}$	w_π
the class of terms before some *w*:	πw	πw
the class of terms after some *w*:	$\breve{\pi} w$	$w\pi$

Russell consistently changes the notation in conformity with this table except 301: 8, where he permutes conjuncts, and at 301: 26, where the text reads "the class $v\pi$, *whose terms are* $u\pi$", *rather than as the table and logic would require:* the class $\breve{\pi}v$, whose terms are $\breve{\pi}u$.>

300: 45 Part 1 (1897), No. 461.] Part 1, No. 461.

301: 17–18 an instance of a compact series] an instance

301: 24–5 *progressions u*;] a *progression u*;

301: 33 ever arises in the given series *v*.] ever arises.

303: 13 It might be suggested] I think

303: 15–16 contains a certain term together with all the terms having to the given term] contains all terms having to some given term

303: 18–19 not complete with regard to … generated,] not complete,

303: 22–6 But every series … generating relation.] But if we insist on completeness in our series before we call it continuous, then, it would seem, no series derived from Arithmetic can be continuous. This opens a tiny door for those who desire to derive continuity from space and time, since it suggests that possibly these alone afford instances of truly continuous series. But instances are here logically irrelevant, and the *definition* of the continuum, even with the suggested addition, remains wholly independent of space and time. But this question introduces topics of general philosophy, which I wish to reserve for a later stage.

CHAPTER XXXVII. TRANSFINITE CARDINALS.

305: 4–7 In fact, … formal definition.] In fact, number is a primitive idea, and it is a primitive proposition that every collection has a number. It is therefore philosophically correct that a specification of number should not be a formal definition.

305: 8–9 By means, however, ... cardinal numbers.] By means, however, of the axiom of abstraction—an axiom, which as we have seen, always takes us from the logically subsequent to the logically prior, we can give what, for formal purposes, may be regarded as a definition of cardinal numbers.

305: 10–11 the above informal definition.] the above definition.

305: 12–13 a one-one relation ... term of the other,] a one-one relation between all the terms of the one and all the terms of the other,

305: 22 indicates at least one common property] indicates a common property

305: 22–4 This property, or, if there be several, a certain one of these properties, we may call the cardinal number of similar classes, and] This property we call their cardinal number, and

305: 25–31 terms. In order to fix upon ... set forth in Part I, Chapter X.] terms.

305: 45 <*fn. added*>

306: 9 property of similarity.] property of similarity. ¶The above is, I think, the best way to introduce the cardinal integers mathematically, and to make it plain that there are such entities. But philosophically we must remark that the relation of similarity is complex, and presupposes the cardinal integers, which are therefore not, in the philosophical sense of the word, *defined* by means of similarity. The cardinal integers, finite and transfinite alike, are logically independent of classes, which have to them the same kind of relation as quantities have to magnitudes. It can be proved, it is true, that there is no number, finite or transfinite, which is not the number of some class; but this result is obtained from other premisses, and is not an immediate consequence of the fact that numbers may be defined by abstraction, as above. And this point illustrates the weakness of definitions by abstraction. For the above method will only define such numbers as are the numbers of some class: if there be others, they remain indefinable. But the proof that every number is the number of some class is only obtainable by considering classes of numbers, and therefore presupposes the being of all the numbers there are. Definition by abstraction, in short, is a method of indicating a class of entities, but does not show that these are complex, or in any philosophical sense definable.

306: 20–2 of *u*. It can be proved ... are similar.* Thus equal, greater,] of *u*. Thus equal, greater,

306: 29–30 not sufficient. For the] not sufficient. Conversely, among transfinite cardinals, it is sufficient that no proper part of *v* should be similar to *u*; but among finite cardinals, this condition does not exclude equality. For the

306: 33–4 cardinals results from the defining difference ..., namely that when the number] cardinals is connected with one of the most interesting points in the relation of the latter to classes, namely the following. When the number

306: 36–8 <*fn. added*>

306: 39–41 Cantor's grounds for holding ... some well-ordered relation. See Cantor] See Cantor

307: 5–6 the proposition, when the finite cardinals are defined by means of mathematical induction, as well as the demonstration] the proposition, as well as the demonstration

307: 20–1: if no two have any common term,] if they have no common terms,

307: 26 of two numbers] of numbers

307: 38–9 § 3; Whitehead, ... No. 4.] § 3.

308: 6 domain] extension

308: 24–6 however, leads to difficulties when *b* is the number ... single terms of *b**.] however, demands that *b* should not be the number of all classes, or more generally, that there should be some collection of *b* terms in which some of the sets chosen out of the *b* terms are not themselves single terms of *b**.

308: 27–309: 7 single terms of b*. ¶The definitions of multiplication ... than Cantor has carried it.] single terms of *b**.

308: 44 Chapter XLIII] Chapter XII

308: 44 <*The right-hand fn. is new.*>

309: 36 the number of finite numbers.] the number of numbers.

309: 45–6 <*fn. added*>

310: 7–8 a finite number, or the smallest transfinite ordinal, is still denumerable*.] a finite number is still denumerable*.

310: 21–2 are asserted by Cantor to be well-ordered,] are well-ordered,

310: 23 all (if there be a last) has an immediate successor] all has an immediate successor,

310: 27–9 no last finite number. But Cantor's grounds for his assertion ... must remain an open question.] no last finite number.

310: 39–40 Cantor's assertion that there is no greatest transfinite cardinal is open to question. See Chap. XLIII, *infra*.] Cantor's assertion that there is no greatest transfinite cardinal is not correct. See Chap. XII, *infra*.

310: 41 p. 404. a_1 is the number next after a_0.] p. 404.

311: 4 finite number or α_0.] finite number, or by multiplying by any finite number, or by raising to any finite power.

311: 6–12 cardinals. It is known that ... both of these classes*.] cardinals.

311: 13 the finite and transfinite cardinals together] the transfinite cardinals

311: 30–1 a single series, if it were known that of any two cardinals one must be greater. But] a single series. But

311: 39 Chapter XXXVI] Chapter V

311: 39 classes of finite integers] classes of integers

311: 41–3 Hence the number of all infinite classes ... the number of the continuum.] Hence the number of all classes of finite integers is $2\alpha_0 + \alpha_0$ and since $2\alpha_0 > \alpha_0$, the number of infinite classes is $2\alpha_0$, and this is therefore the number of the continuum.

311: 44 classes of finite integers] classes of integers

311: 45 all the finite integers;] all the integers;

CHAPTER XXXVIII. TRANSFINITE ORDINALS.

312: 4–5 transfinite cardinal, or at any rate for any one of a certain class,

there is] transfinite cardinal, there is
312: 6–7 same as or less than that of all cardinals.] same as that of all cardinals.
312: 10–12 classes of series, or better still, ... some relation.] classes of series, and are defined, for the most part, by a certain relation
312: 13–14 the ordinal number *n*] the number *n*
312: 14 "a serial relation of *n* terms;"] "a series of *n* terms";
312: 17 a class of serial relations.] a class of series.
312: 30 <*fn. added*>
313: 31–2 class of series, or rather of their generating relations.] class of series.
313: 40–1 of the class *progression*, or of the generating relations of series of this class.] of a progression.
313: 43–4 any ordinal α which is obtained] any number α obtained
314: 1, 26 ordinal] number
314: 27 the class of generating relations of progressions.] the class of progressions.
314: 36 the principle of abstraction] the axiom of abstraction <*Also at 315: 3.*>
314: 40–4 13. It is important ... indemonstrable and implausible.] 13
314: 45 *Mannigfaltigkeitslehre*, p. 34.] *Ib.* p. 34.
315: 4 the type or class of serial relations,] the type of series,
315: 9 the finite ordinals or cardinals—] the finite ordinals—
315: 10 starting from 0 or 1,] starting from 1,
315: 15–16 provided, of course, that the number assigned is] provided, that is to say, that the number assigned was
315: 21–2 applies to *all* ordinal or *all* cardinal numbers.] applies to *all* numbers.
315: 40–2 A collection of two ... one serial relation.] A collection of terms may contain two or more relations of the kind generating series.
316: 12–18 number of such interchanges. The general principle ... are of the above form $\breve{R}PR$. But] number of such interchanges. But
316: 18–19 not reducible to a permutation,] not reducible to interchanges of pairs,
316: 27 Chapter XXXVI;] Chapter V;
316: 31–2 the types of well-ordered series] the types of series
316: 39–40 make the whole ordinally different] make the whole, in general, ordinally different
317: 1–317: 8 <*This entire paragraph is new.*>
317: 11–12 is true, in general, only in the form] is true only in the form
317: 12: $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$] $(\alpha + \beta)\,\gamma = \alpha\gamma + \beta\gamma$
317: 33–4 is less than] < <*Also at 318: 18.*>
317: 39–42 p. 39; $\alpha + \beta$ will be the type ... of the type $\omega.2$.] p. 39.
318: 26 elements which belong] elements belong <*MS. error*>
319: 2 an infinity of roots;] an infinite of roots; <*MS. error*>
319: 17: represents] represent <*MS. error*>
319: 26–7 All possible functions of ω and finite ordinals only, to the exclusion of other types such as that of rationals, represent] All possible functions of ω represent
319: 27: hold. In every well-ordered series] hold. Every ordinal number represents some completed segment of the series of cardinals, and every completed segment of the series of cardinals is a well-ordered series. In a well-ordered series
319: 28–9 term, except the last if there be one; and provided] term; and provided
319: 31 A term which comes next after a progression has] The terms which come next after progressions have
319: 32–3 and the type of the segment formed of its predecessors is of ...] and are of
319: 34–5 and the types of the segments formed of their predecessors are said] and are said
319: 43–5: 12. The definition ... has a first term.] 12
320: 7 negative and positive] positive and negative <*Also at 320: 14.*>
320: 9 a serial type $^*\omega$,] a number $^*\omega$,
320: 11–12 generates] also generates
320: 23 ordinal type] ordinal
320: 24 type] number <*Also at 320: 26.*>
320: 30–1 applies also, for example,] applies also
320: 38–9 is contained.] can be contained.
321: 6–7 correlation that the type of the rationals] correlation—the general and simplest expression of which is, a class which forms two series with regard to two distinct relations—that the type of the rationals
321: 10–11 irrationals.] irrationals. Thus Cantor's ordinal definition of continuity gives no method of defining a single independent continuous series.
321: 13–14 are to be considered—as I suggested] are scarcely to be considered as numbers but rather—as I suggested
321: 14–15 serial relations,] series,
321: 15 now apparently adheres;] now adheres;
321: 18 restricts] confines
321: 20–1: analogies to more familiar kinds of numbers.] analogies to number.
321: 22–3: which begin with some cardinal.] which begin with some cardinal, as all such series must.
321: 25–323: 20 <*§§299–301 are entirely new in the text. In the MS., the text section numbered 298 ends at the bottom of folio 120, and §302 begins at the top of folio 121. The footnote on p. 321 and the first three on p. 323 are new as well.*>
323: 22 Chapter XXXVI] Chapter V <*Also at 323: 45, 324: 31.*>
323: 34–5 the finite ordinals.] the finite integers.
323: 37–324: 1 of the 1st genus and the vth species.] of the 1^{st} species and the v^{th} genus.
324: 8–9 the method of limits, or rather of segments:] the method of limits:
324: 29 ordinal] number

CHAPTER XXXIX. THE INFINITESIMAL CALCULUS.

325: 10–11 that, if metaphysical subtleties are left aside, the Calculus] that the Calculus
325: 30 Gerhard's ed., Vol. V, pp. 220 ff.] Gerhard's ed., Vol.
325: 31–2 p. 305. Cf. Cassirer ... pp. 206–7.] p. 305.
326: 4 fluxion] <*In the MS. the word "fluxion" is underlined, but it is in no way distinguished in the text; typically what is underlined in the MS. is italicized in the text, and that would seem appropriate here. See e.g. 333: 35.*>
326: 16 little known] unknown

326: 32 (Chap. XXXII)] (Chap. I)
326: 33 (Chap. XXXVI)] (Chap. V)
326: 38 the field of the function] the function
327: 17 any positive value] every positive value
327: 44 <*This added footnote explains the change made at 327: 17.*>
328: 8 greater than 0 and less than ϵ.] > 0 and $< \epsilon$.
328: 22–3 the derivative of a function, or differential coefficient.] the differential coefficient.
329: 33 $(n-1)$] $\overline{n-1}$
329: 35 , say $f(\zeta_s)$,] (say $f(\zeta_s)$)
330: 16–17 descends and has no last term cannot reach its limit;] descends cannot reach its limit;
330: 17 other infinite series] other series
330: 33 something to say] much to say

CHAPTER XL. THE INFINITESIMAL AND THE IMPROPER INFINITE.

333: 26 finite spaces] elliptic space <*Also at 333: 36.*>
333: 28–9 by von Staudt's] by Staudt's
333: 34 ordinally infinitesimal with] infinitesimal with <*Also at 333: 35 and 333: 37.*>
333: 44 See Part VI, Chap. XLV.] See Part VI, Chap. II.
333: 44 <*2nd fn. added*>
334: 24 but give] but all give
334: 35 Chapter XXXIV] Chapter III
335: 29–41 cannot be terminated. ¶In the case of the rational ... in some radically new sense.] cannot be terminated.
336: 17 $1/g$.] 1g.

CHAPTER XLI. PHILOSOPHICAL ARGUMENTS CONCERNING THE INFINITESIMAL.

338: 16 with modern mathematics] with mathematics
339: 43–4 highly important, though portions ... text.] highly important.
340: 5 no relevance] no relevance whatever
340: 24–5 unnecessary] irrelevant
341: 10 endless] infinite
345: 5–6 have a finite ratio to each other.] have a finite ratio to each other.* <*The following fn. is deleted in the text:*> *Neglecting such as are defined by functions having no derivative.

CHAPTER XLII. THE PHILOSOPHY OF THE CONTINUUM.

346: 10 we may conjecture] we see
346: 29 Cantor's definition.] Cantor's definition, together with such modification (if any) as may be found desirable hereafter.
347: 11 alleged] held
347: 12 maintained] held
347: 14–15 selected by definition,] created by definition,
347: 43 This consequence by no means follows,] This consequence by no means follows, as we shall see in Part VII,
348: 9–11 But it is instructive ... into arithmetical language*.] In their actual form, they will be considered in Part VII; for the present, I wish to translate them, as far as possible, into arithmetical language.
349: 14 quite] perfectly
349: 26 wholes which are defined extensionally, *i.e.* by enumerating their terms,] collective wholes which are defined by enumerating their terms,
349: 27 such as are defined

intensionally, *i.e.* as the class] distributive wholes, which are defined as the class
349: 29 (For a class of terms, when it forms a whole, is merely] (For a class of terms is merely
349: 30 extensional] collective
349: 41 subsists as a genuine entity] subsists as a legitimate concept
349: 45 For precise ... VI and X.] I doubt whether the above use of *collective* and *distributive* is exactly the traditional use: but at any rate it is precise and serviceable.
350: 9–10 lies in the theory of denoting and the intensional definition of a class.] lies in clearly distinguishing the class from the sum of its terms.
350: 21 simultaneity establishes a one-one correlation] there is a one-one correlation
351: 12–13 enough has been said on this subject in Part I.] enough has been said to show what is meant by a variable.
351: 24–5 taken as roughly correct*.] taken for the present.
351: 44 <*fn. added*>

CHAPTER XLIII. THE PHILOSOPHY OF THE INFINITE.

355: 21–2 it remains conceivable that we don't know what we don't know;] it remains conceivable that, unlike the late Master of Balliol, we don't know what we don't know;
355: 26 Plato's *Parmenides*—which is perhaps the best collection] Plato's Parmenides—which is far the best collection
356: 6–43: to which we have been led. Accepting as indefinable ... if not, not. In this way,] to which we have been led. There are among concepts some which are susceptible of a certain special relation, which may be called that of the class to the individual. This is the relation of number or prime to 2, of colour to a particular shade of yellow, of man to Socrates. The terms of the relation may be both simple, both complex, or either simple and the other complex. A term which can be on the one side of such a relation is called a *class*; a term which can be on the other is called an *individual.* Every class is an individual; for every class (including *class* itself) has to *class* the relation of the individual to the class. But not every individual is a class; for example, 2 or 1/2 or Socrates is not a class. Among individuals there are the fundamental relations of identity and diversity, one of which may be defined as the negation of the other. Every class has a certain property, called its cardinal number, which depends on all the individuals composing it. It is to be observed that the whole composed of the terms of a class is totally different from the class: for example; *numbers* differs from *number.* But given the class, the whole composed of the terms of the class is determinate, and so is the number of the class. Where finite numbers are in question, the class is unimportant as compared with the whole formed of its terms; but where infinite numbers are in question, the whole is only defined by the class, so that the class is essential. When a class is such that, if *a* belongs to the class, and *b* differs from *a*, then it follows that *b* does not belong to the class, the class is said to have the

number 1. Numbers are distinguished among the properties of classes by the fact that, when two classes have a one-one correlation, they both have the same number. The logical sum of a class *a* which has the number 1, and a class *b* which has the number 1, provided the one term of *a* differs from the one term of *b*, is a class having the number 2. And generally, the logical sum of any class whose number is *n*, and a class whose number is 1, and whose one term does not belong to the previous class, is defined to be n+1. In this way,

356: 43 0] 1 <*Also at 363: 30.*>

356: 44–5 leads to a new number n+1.] leads to a number n+1.

357: 10 that 0 and 1, or 1 and 2, are different numbers] that 1 and 2 are different numbers

357: 23 A very simple proof] The simplest proof

357: 33–4 similarity is reflexive for classes,] similarity is analyzable, and is reflexive for classes,

357: 36–9 infinite. A better proof ... the number of numbers. Again,] infinite. Again,

357: 40 propositions or concepts] terms or concepts

357: 41 term or concept] term or concept (I use these words as synonyms)

358: 44 <*fn. added*>

359: 9 of *f*(*x*); also let all the values of *f*(*x*) belong to *u*.] of *f*(*x*).

359: 37 not necessarily a segment,] not a segment,

360: 21 definitions] meanings

360: 22 only one of these, at least practically,] only one of these

360: 38–9 the class-relation. If now] the class-relation. *Any* individual of this class is a variable, which we call *x*; for any value of *x*, *x* is an *a*. If now

360: 39–40 such that for all values of *x*, "*x* is an *a*" implies] such that "*x* is an *a*" implies

360: 44 <*fn. added*>

361: 26 the number of finite numbers is *n*+1.] the number of finite numbers is *n*.

361: 41 What is necessary] All that is necessary

362: 16–363: 17 ... The usual objections to infinite numbers, ... the cases in which contradictions arise.] ¶There is a certain difficulty in regard to the number of numbers, or the number of individuals, or of classes. Numbers, individuals, and classes, each form a perfectly definite class, and it will remembered that we found a general proof, from the reflexiveness of similarity, that every class must have a number. Now the number of individuals must be the absolute maximum of numbers, since every other class is a proper part of this one. Hence, it would seem, the numbers have a maximum. But Cantor has given two proofs* [*Mannichfaltigkeitslehre, p. 44; Jahresbericht der deutschen Mathematiker-Vereinigung, 1 (1892), p. 77.] that there is no greatest number. If these proofs be valid, there would seem to be still a contradiction. But perhaps we shall find that his proofs only apply to numbers of classes not containing all individuals, in which case we should conclude merely that the maximum number is one of those that have no immediate predecessor. It is essential, however, to examine this point with care, before we can pronounce infinity to be free from contradictions. ¶In the first place, the number of classes is the same as the number of individuals. This results from the two facts, (1), that every class-concept is an individual, (2), that every individual defines a class of one term only, namely the class of terms not diverse from the said individual. Hence there is a one-one correlation between all classes and some individuals, and between all individuals and some classes. Hence the number of classes can be neither greater nor less than the number of individuals. In the second place, the number of numbers, which always formerly led to contradictions, can now be freed from all contradictions. Among finite numbers, the number of numbers up to *n* is *n*; hence if *n* were the number of numbers, *n* would be the greatest number. This property just manages to extend itself to α_0, and then ceases to hold. The number of numbers up to and including α_1 is still α_0, and remains so until we have run through as many cardinals as there are ordinals in the second class and so on throughout the hierarchy. Thus the number of numbers is less than the greatest number, and no contradiction arises, as it did formerly, from the fact that the number of individuals is greater than the number of numbers.

362: 42–3 <*fn. added*>

362: 44 <*fn. added*>

363: 18 345. In the first of Cantor's proofs, the argument] I come now to Cantor's two proofs that there is no maximum cardinal number. In the first of above passages, the proof

363: 29 from 0 up to] from 1 up to

363: 30–2 that no class ... a greater number of terms.] that the ordinals obviously have no maximum.

363: 45 <*fn. added*>

363: 46 <*fn. added*>

364: 11–12 can be well-ordered;] is capable of order;

364: 16–18 it is impossible to prove that there must be a maximum ordinal, which in any case there are reasons for denying.] it is harder to prove that there must be a maximum ordinal.

364: 20 well-ordered series,] series, <*Also at 364: 29.*>

364: 22 (*Math. Annalen*, XLVI, § 2)] (*Math. Annalen* XLVI, §   )

364: 33 The second of the proofs] The proof in the second of the passages

364: 34 definite. The proof is interesting] definite. But here again, I think, we shall find that, when the impossibility of repetition of any individual is realized, the proof fails to be universal. The proof is interesting

364: 35 produced] reproduced

364: 40 Chap. XXXII,] Chap. 1,

364: 41–3 <*fn. added*>

364: 44 <*fn. added*>

364: 45 <*fn. added*>

364: 46–7 <*fn. added*>

365: 7 some fixed term. Thus the *x*'s may be rational numbers] some fixed term, and I imagine, from the choice of the letters *m* and *w*, that this view was in Cantor's mind. Thus the *x*'s may be numbers

366: 23–368: 18 belonging to the class. <*This includes the last paragraph of §347 and all of §§348 and 349.*> ¶Another form of the same argument ... leave the problem to the ingenuity of the reader. ¶To sum up] belonging to the class. Now if *u* be the class of classes, this is plainly self-contradictory, for classes contained in *u* will be only classes of classes, whereas terms belonging to *u* will be all classes without restriction, so that the classes contained in *u* are a proper part of the class *u* itself. Hence there must be somewhere in Cantor's argument a concealed assumption not justified when *u* is the class of classes. ¶The argument by which it is to be shown that the number of classes of classes exceeds the number of classes may be disproved in the following manner. We have u = Class, so that "x is a u" means "x is a class". When x is not a class of classes, let k_x be the class of classes whose only member is x. When x is a class of classes, let k_x be x itself. Then we define a class u', in accordance with the above procedure, as containing every x which is not a member of its k_x, and no x which is a member of its k_x. Thus when x is not a class of classes, x is not a u'; when x is *class*, or *class of classes*, or *class of classes of classes*, or *etc.*, x is not a u'; but when x is any other class of classes, x is a u'. Then Cantor infers that u' is not identical with k_x, for any value of x. But u' is a class of classes, and is therefore identical with $k_{u'}$. Hence Cantor's method has not given a new term, and has therefore failed to give the requisite proof that there are numbers greater than that of classes. In fact, the procedure is, in this case, impossible; for if we apply it to u' itself, we find that u' is a $k_{u'}$, and therefore not a u'; but from the definition, u' should be a u'. In fact, when our original class consists of all possible combinations of all possible terms, the method, which assumes new combinations to be possible, necessarily fails, since, in this case, u' itself is a u. Thus what Cantor has proved is, that any power other than that of all classes can be exceeded, but there is no contradiction in the fact that this power cannot be exceeded. The exact assumption, in Cantor, which *class* fails to satisfy is, that if u be the class whose power is to be exceeded, not all classes of u are themselves terms of u. ¶With this we have, I think, disproved most of the allegations against the infinite. I do not know of any others of equal importance, and the methods explained seem adequate to deal with any contradictions that may be suggested as belonging to the infinite. ¶To sum up the discussions

366: 45–6 <*fn. added*>

367: 42–4 <*fn. added*>

367: 45 <*fn. added*>

367: 46 <*fn. added*>

368: 23–4 define, in a purely ordinal manner, the kind of continuity which belongs to real numbers,] define the kind of continuity, which belongs to the real numbers, in a purely ordinal manner,

368: 38–9 proved concerning either, although certain special infinite classes do give rise to hitherto unsolved contradictions.] proved concerning either.