THE "VILLAIN" OF SET THEORY

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Shaughan Lavine. *Understanding the Infinite.* Cambridge, Mass.: Harvard U. P., 1994. Pp. ix, 372. US$39.95.

This is a fairly technical work in the history and philosophy of mathematics, with an emphasis on the mathematical notion of a "collection" and the epistemology of set theory. (I will use the vague term "collection" in contrast to "class" and "set", which have technical meanings, to avoid confusion with particular theories of what sort of mathematical multiplicities may be treated as wholes.) Understanding infinite collections is the primary epistemological problem of set theory for Lavine, since finite sets may be easily grasped by finite minds (the problem of their abstractness is, as Lavine says [p. 162], a distinct issue). His approach to understanding infinite collections is to view them as idealizations of indefinitely large finite sets, but that is a story which cannot be developed further here. Moreover, infinite collections played a seminal role in the development of set theory, for it was the need to understand their use in the mathematical definitions of limit and irrational number which led Cantor to begin studying collections "in their own right" (pp. 38, 41).

The first half of this book is historical, and it is connected to the second, philosophical half by Lavine's methodology. He attempts to base the philosophy of set theory upon an accurate understanding of how that science has developed. It is Lavine's philosophical goal to show that the mathematical notion of collection studied by set theory can be understood through the history of its development, and through the actual practice of mathematicians (p. 5).

As Lavine presents it, there have been three historically important concepts of mathematical collection: the combinatorial concept originated by Cantor, the logical notion derived by Russell from Peano, and the iterative idea developed by Zermelo, which was an historical outgrowth of Cantor's concept (p. 65).

Cantor's combinatorial concept is based on the idea that a mathematical collection, in order to be treated as a whole, must be capable of being counted (pp. 53–4), in a broad sense of "count" which means well-orderable (this is not the usual technical sense of "countable"). Thus, Cantor first treated the Well-Ordering Theorem ("all sets can be well-ordered") as a triviality (p. 54).

The Peano–Russell logical concept treats collections as the extensions of concepts (p. 63). For a multiplicity to be treated as a mathematical whole, we must have some propositional function which acts as a rule for picking out all of the members. Lavine contends that the set-theoretic paradoxes are only a problem for the logical concept, which includes the inconsistent Comprehension Principle, and has Russell's Paradox as a result (p. 66).

The iterative notion is that all sets are the results of reiterated applications of the "set of" operation on some (possibly empty) universe of non-sets (p. 144). On this view, no set can be a member of itself, which rules out the set of all sets not members of themselves, for the collection of all sets is not itself a set.

If a history and philosophy of set theory can have a villain, that role in Lavine's story is played by Bertrand Russell. According to Lavine, Russell misinterpreted Cantor's consistent combinatorial concept of collection in favour of his own inconsistent logical concept (p. 59), although he "was at least dimly aware that Cantor's conception of a set was different from his own" (p. 64). By means of his eponymous paradox, Russell succeeded in muddying the waters of the philosophy of mathematics to such an extent that the true history of set theory has been replaced by the "baneful influence" (p. 5) of a myth.

This "myth" is to the effect that naive set theory, including that of Cantor, was shown to be inconsistent by Russell, and that the subsequent axiomatic development by Zermelo was an *ad hoc* attempt to rescue some of the naive theory. The truth, according to Lavine, is that Cantor's original combinatorial concept of set is not subject to Russell's Paradox, and has the additional virtue that Zermelo's Axiom of Choice is evidently true of it (p. 78). Lavine rhetorically begins his introductory chapter by recounting that "myth" without warning (pp. 1–2), then concludes with the words: "The story I have just told is a common one, widely believed. Not one word of it is true."

Lavine's historical argument is intended to show that the combinatorial concept was Cantor's intended interpretation of set theory, that Russell confused Cantor's concept with his own logical notion, that mathematicians have generally worked within Cantor's framework, and that the later iterative conception is an outgrowth of the combinatorial concept. While the "myth" parodied by Lavine is too simple to be true, there is much about Lavine's account that also seems simplistic.

For instance, Cantor's initial response to the set-theoretic paradoxes is *ad hoc* (pp. 55–6). Cantor simply dismissed such apparent paradoxes as not applying to sets; he calls collections whose existence leads to contradictions "inconsistent multiplicities" (p. 99). Cantor does not reject "inconsistent multiplicities" on the principled grounds that they cannot be well-ordered, but simply because the assumption of their existence leads to inconsistency. For instance, one such non-set is the multiplicity of all ordinal numbers, yet that multiplicity is well-ordered by its "natural order" (p. 56).

This makes it clear that Cantor's own conception of set was less precise than Lavine supposes. The problem for the interpretation of set theory is to find some principled way to delimit the collections so that no paradoxes result. The iterative conception, Russell's type theory, and even the combinatorial notion described by Lavine are all principled attempts to do this. The fact that Russell, dimly aware of Cantor's conception as he was, could mistake it for his own, is evidence that the history of set theory is also the history of the concept of set. Lavine's book is valuable both as an historical account of that development, as well as a philosophical analysis of the different concepts which resulted.