HOW DID RUSSELL WRITE *THE PRINCIPLES OF MATHEMATICS* (1903)?

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> Parts III, IV, V, and VI of the book as published were written that autumn [of 1900]. I wrote also Parts I, II, and VII at that time, but had to rewrite them later, so that the book was not finished in its final form until May 1902. Every day throughout October, November and December, I wrote my ten pages, and finished the MS on the last day of the century, in time to write a boastful letter to Helen Thomas about the 200,000 words that I had just completed.
> (*Auto.* I: 145)

> During September 1900 I invented my Logic of Relations; early in October I wrote the article that appeared in RdM VII 2–3 [Peano's journal *Revue des mathématiques*]; during the rest of the year I wrote Parts III–VI of my *Principles* (Part VII is largely earlier, Parts I and II wholly later, May 1902).... (Russell, letter to Philip Jourdain, April 1910[1])

1. THE RECEIVED STORY: FROM PEANO TO *The Principles*

Russell's book *The Principles of Mathematics*, published in the spring of 1903, was one of his most important works; for in it he set down not only his logicist thesis, that pure mathematics was obtainable solely from a version of logic that he had formulated, but also many features of the so-called "analytic philosophy" which he and his friend G. E. Moore had launched around 1899 as a reaction to the domi-

[1] I. Grattan-Guinness, *Dear Russell—Dear Jourdain* (London: Duckworth, 1977), p. 133. Russell confirmed the dating of Parts III–VI to Jourdain in 1917 (p. 144).

nant neo-Hegelianism in Britain. Its great influence—it seems never to have been out of print after its reprint in 1937—has naturally excited much interest as to its origins; and Russell duly obliged in several reminiscences, along the lines of the first quotation above.[2] To give a little more detail:

(1) He had been trying for some years to find a comprehensive philosophy of mathematics, but had not found a satisfying version.
(2) The work of Giuseppe Peano, which he discovered at the International Congress of Philosophy at Paris in August 1900, provided him with the basis that he sought.
(3) Immediately learning Peano's system, he extended it by adjoining a logic of relations, and during the autumn of 1900 he wrote all seven Parts of *The Principles* with this extended logic as the basis for logicism.
(4) However, during 1901 he discovered his paradox of set theory; some time around then he also read Frege in detail for the first time. When published in 1903 the book contained material on both these aspects, mainly in the first two Parts (which had been rewritten, along with Part VII) and in two new appendices.

To my transcription of the second quotation above I added a footnote that "The manuscript of *The Principles* is in the Russell Archives and repays a detailed study." This paper is a long-delayed fulfilment of that sentiment; but the intervening time has not been lost, for four scholars have conducted precisely such analysis on portions of the manuscript: J. Alberto Coffa[3] and Alejandro Garciadiego,[4] largely centred on the paradox; and Kenneth Blackwell and Michael Byrd on the most significant Parts of the book, respectively the first,[5] and second[6] and fifth.[7]

[2] The main sources in Russell for the summary below are the *PoM* reprint (1937), p. v; "My Mental Development", in Schilpp, pp. 12–13; *MPD*, pp. 72–3; and *Auto.* 1: 145 (quoted at the head of this article).

[3] J. Alberto Coffa, "The Humble Origins of Russell's Paradox", *Russell*, nos. 33–4 (1979): 31–8.

[4] A. Garciadiego, *Bertrand Russell and the Origins of the Set-theoretic 'Paradoxes'* (Basel: Birkhäuser, 1992), esp. Chap. 4.

[5] K. Blackwell, "Part I of *The Principles of Mathematics*", *Russell*, n.s. 4 (1984): 271–88.

[6] M. Byrd, "Part II of *The Principles of Mathematics*", *Russell*, n.s. 7 (1987): 60–70.

[7] M. Byrd, "Part V of *The Principles of Mathematics*", *Russell*, n.s. 14 (1994): 47–86.

The revised story offered below depends much on these fine studies, in which many details not mentioned here can be found.

As this paper is primarily textual and documentary in character, little attention is given to mathematical and philosophical content of *The Principles*, and the references to documents and secondary literature are similarly restricted; the principal wander occurs in §11, on the neglected Part VII on dynamics. "(RA)" indicates an unpublished text to be found in the Russell Archives. On technical terms, I shall follow Russell in writing of "classes" when referring to Cantorian sets; he did not change this term when, as part of his adoption of Peano's system, he replaced the traditional part–whole theory of classes by Cantorian set theory (as we now normally call it, and which I shall take to include also his theory of infinitely large numbers).

2. SURPRISES IN THE MANUSCRIPT AND ELSEWHERE

Russell received back the manuscript of *The Principles* from Cambridge University Press after publication, and kept it in his files; it is now in the Russell Archives. Although there are various extra pages of uncertain date, it does corroborate his quoted recollection fairly well; Parts I–VII contain around 970 pages and the wordage is around 230,000, which is fairly close to his claim made in the letter to Helen Thomas (which is published in *SLBR* 1: 207–8).

However, the manuscript also confounds Russell's memory in several ways. For example, the dates put on it in various places suggest that the Parts were written in the order III–IV–V–VI–II–I–VII. More radical is the second quotation above, made to Jourdain only seven years after the event, that only Parts III–VI were written in 1900. This reading is corroborated by a text *written at the time*: Russell's wife Alys noted in her diary for the end of 1900 that "Bertie wrote an article on the Logic of Relations, also ⅔ of a book on the Principles of Mathematics".[8] Internal

While my paper was in press, Byrd completed a study of Parts III and IV, which confirms my interpretation in several respects. See his "Parts III and IV of *The Principles of Mathematics*" in this issue, pp. 145–68.

[8] This diary, in which Russell also wrote occasionally, belongs to Camellia Investments (London); a photocopy is held at the Russell Archives (Rec. Acq. 434). The pages are not numbered: this passage appears near the end.

evidence also supports a revised story. Parts I and II are referred to only in general ways in the later ones; in particular, a mention in Part V that "irrationals could not be treated in Part II" (p. 278) refers to "I or II" in the manuscript, and in a similar remark four pages later "I" was altered to "II" for publication.[9]

Of course Russell did not try deliberately to mislead his readers; but his autobiographical writings are known to contain several inaccuracies, some quite wild,[10] and his recollection of *The Principles* seems to belong to this category. He drafted his autobiography in 1931 (revising it on occasion later before its publication in the late 1960s), and he may have looked through *the whole* manuscript at that time to lead his memory to create his sincere but mistaken history. He may have just guessed 200,000 words to Thomas without making an accurate count; the profusion of very short words and symbols makes it quite difficult, with the number per page varying between 160 and 300 (Parts III–VI seems to contain around 140,000). Maybe he counted in other writings of the period, such as the draft of his paper on the logic of relations and a popular essay on Peano's group of logicians (*Papers* 3: 590–612, 350–79). With the evidence as it stands, the testament of Alys written down at that time, reminiscences to Jourdain made nearer to the time than the autobiography,[11] and various internal features of the manuscript argue that a different tale needs to be told.

3. A REVISED STORY: FROM "PRINCIPLES" TO *The Principles*

The rest of this article is based upon these four assumptions:

(1) *Parts* I *and* II *did not exist at all in 1900*, at least not beyond sketch form (which has not survived, if it ever existed): they were written not earlier than the summer of 1901.

(2) *The book conceived in 1900 did not advocate logicism:* Russell came to that position only around January 1901.

(3) It seems to have come to him as a generalization of his view of geometries.

(4) His paradox of set theory, and also two papers which he published in Peano's journal in 1901 and 1902, are *integral* to the conception of *The Principles*, not adjoined to it *after* its drafting.

The scenario goes as follows:

(5) Between August 1899 and June 1900 Russell had written a book manuscript entitled "Principles of Mathematics" (hereafter "Principles", and published in *Papers* 3: 9–180). It played an important role in the preparation of *PoM*, even to the extent of providing some of the folios.[12]

(6) After encountering Peano and his followers in August 1900, Russell was sure that Peano's system was important for him (September), and so could provide Parts I and II of a revision of "Principles" with the new grounding that he had been seeking; however, a logic of relations had to be introduced (October). He followed the Peanists in maintaining some distinction between mathematics and logic, although he was not sure what or where it was, especially regarding set theory. So he rewrote Parts III–V (November) and VI (December) of "Principles" to try to develop and also clarify his theory. He probably sketched out the content of Parts I and II and maybe even some of their structure; but he did not write them then.

(7) In the new year Russell envisioned a solution to his demarcation problem: the distinction did not exist. Instead, *pure* mathematics, *not* mathematics, was contained in Peanesque logic. (The word "pure" was used in a special sense explained in §5.) However, he did not yet have a detailed conception of this logic, apart from the need for relations; still awaiting clarity were the constants and

[9] See Garciadiego (n. 4), p. 91; for similar evidence from Part V see Byrd (n. 7), pp. 52–7.

[10] See especially K. Blackwell, "Our Knowledge of *Our Knowledge*", *Russell*, no. 12 (1973): 11–13, à propos of the fiction told of *Our Knowledge of the External World* (1914) in *Auto.* 1: 210. For another important example of unreliability, see n. 26 below.

[11] Russell seems to have written at least part of an autobiography for Ottoline Morrell in 1912; unfortunately it is lost.

[12] The Russell Archives hold an unpublished analysis by John King of the textual relationships between "Principles" and *The Principles* and some anterior manuscripts.

indefinables, and the status of set theory. Parts I and II had to wait; the corresponding Parts in "Principles" could not suffice, as there he had started with the number concept and moved on to the part–whole theory of collections (not Cantorian set theory).

(8) In January 1901, and definitively in May, he rethought a discussion in Part V of Cantor's diagonal argument, and thereby found his paradox. Thus Parts I and II became still harder to plan!

(9) Around the same time he thought out rather more clearly the basic notions of his logic, and thereby refined logicism to some extent: Part I was sketched out in detail, with the title "The Variable". Part II on "Number" was also written, including the nominal definition of cardinal integers as classes of classes, basic for arithmetic and therefore for logicism.

(10) By the spring of 1902 Part I could be developed further, especially regarding "The Indefinables of Mathematics" (its new title). The prominence of the variable was tempered by deeper consideration of propositional functions. Despite the presence of the paradox, logicism could be expounded in more detail.

(11) The book was completed and readied for publication by May 1902; but then further changes were made and two appendices added. Many of the references to other literature, and still further changes, were made at the proof stage from June to the following February.

This proposed chronology, outlined in more detail in Table 1, is elaborated in §§4–11 below. Table 2 shows the structures of "Principles" and of *The Principles*; after a preface and an elaborate analytical table of contents, the main text of the latter was divided into seven Parts with 59 chapters and 474 numbered articles, 498 numbered pages in all.

The following apparent feature of the manuscript is not indicated on the Table, for if it happened, no dating is available. At some stage another version of Parts III–VI seems to have been prepared, possibly a typescript, on which he and the printer worked.[13] For, unlike the other three Parts, the chapters are numbered from 1 onwards in each Part

[13] Russell had been using the new technology of typing bureaux since about 1897. He wrote on 20 July 1898 to G. E. Moore: "I hope your Dissertation is growing with all speed, and that you will have it typed by my people", mentioning "The Columbia Literary Agency, 9 Mill Str. Conduit Str. W." (Cambridge U. Libr., Add.MS. 8830, 8R/33/7).

TABLE 1. Russell's Progress with *The Principles*, Aug. 1900–Feb. 1903

ProM = "Principles of Mathematics" (1899–1900); *PoM* = *The Principles of Mathematics* (1903). The *Papers* entry gives the first page(s) of the text(s).

Month(s)	*Papers* 3	Activity
August '00		Hears Peanists; likes their logic & use of *Mengenlehre*
September '00		Learns Peanese: invents logic of relations
October '00	590	Drafts paper on relations
Oct.–Dec. '00	351	Writes manuscript on Peanists
November '00		Writes Parts III–V of *PoM* in Peanist spirit, using ProM
December '00		Writes Part VI of *PoM* in Peanist spirit, using ProM
January '01?		Envisions logicism: "pure mathematics" in his logic
January '01	363	Writes popular essay on mathematics
Jan–May '01?	385	Approaches his and Burali-Forti's paradoxes
February '01	310/614	Completes paper on relations: sent to Peano
March–April '01	630	Drafts paper on well-ordered series
?–May '01		Refines logicism: clarifies logical indefinables & constants
Apr–May '01?		Finds his paradox of set theory
May '01	181	Drafts "Part I Variable" for *PoM*; includes his paradox
June '01?	423	Writes on cardinal numbers for Whitehead
June '01		Writes Part II of *PoM*, using ProM
August '01	284/661	Completes paper on series: sent to Peano
April–May '02	208	Writes Part I of *PoM*
May '02		Writes Part VII of *PoM* (much from ProM)
May '02		Readies manuscript of *PoM*
June '02–Feb. '03		Handles proofs: adds many footnotes, rewrites passages
July?–Nov. '02		Writes Appendix A on Frege's work
November '02		Completes Appendix B on the theory of types
December '02		Writes preface of *PoM*
February '03		Indexes *PoM*
May/June '03		*PoM* published in Britain / in USA

TABLE 2. Summary by Parts of ProM and *PoM*

The summaries of *PoM* do not always follow the order of chapters in the Part.

ProM; Chaps.	*PoM*; Chaps., Pp.	Summary of Main Contents
I: "Number"; 6	I: "The Indefinables of Mathematics"; 10, 105	"Definition of Pure Mathematics"; "Symbolic Logic", "Implication & Formal Implication"; "Proper Names, Adjectives & Verbs", "Denoting"; "Classes", "Propositional Functions", "The Variable", "Relations"; "The Contradiction"
II: "Whole & Part"; 5	II: "Number"; 8, 43	Cardinals, definition & operations; "Finite & Infinite"; Peano axioms; numbers as classes; "Whole & Part", "Infinite Wholes"; "Ratios & Fractions"
III: "Quantity"; 4	III: "Quantity"; 5, 40	"The Meaning of Magnitude"; "The Range of Quantity", numbers & measurement; "Zero"; "Infinite, the Infinitesimal, & Continuity"
IV: "Order"; 6	IV: "Order"; 8, 58	Series, open & closed; "Meaning of Order", "Asymmetrical Relations", "Difference of Sense & of Sign"; "Progressions & Ordinal Numbers", "Dedekind's Theory of Number"; "Distance"
V: "Continuity & Infinity"; 9	V: "Infinity & Continuity"; 12, 110	"Correlation of Series'; real & irrational numbers, limits; continuity, Cantor's & ordinal; transfinite cardinals & ordinals; calculus; infinitesimals, philosophy, & of infinite & of continuum
VI: "Space & Time"; 4	VI: "Space"; 9, 91	"Complex Numbers"; geometries, projective, descriptive, metrical; definitions of spaces; continuity, Kant; philosophy of points
VII: "Matter & Motion"; 7	VII: "Matter & Motion"; 7, 34	"Matter"; "Motion", definition, absolute & relative, Newton's laws; "Causality", "Definition of Dynamical World", "Hertz's Dynamics"
	Appendix A; 23 pp.	Frege on logic & arithmetic
	Appendix B; 6 pp.	"The Doctrine of Types"

instead of the consecutive system that was printed (roman numbers 19–52), the texts are not divided into the numbered articles printed (149–436), and (most significantly) there are no printers' markings. These features accentuate the differences between these Parts and the other three.

4. THE FIRST EFFECTS OF PEANO, AUGUST–DECEMBER 1900

Russell stopped working on "Principles" apparently in June 1900. Perhaps he wanted to reflect upon its contents, or maybe he anticipated that something new and useful would turn up at the International Congress of Philosophy at Paris forthcoming in August. He cannot have expected that The Light would fall upon him with such glorious intensity.

One can determine Russell's magic time precisely: the morning session on Friday, 3 August 1900, which organizer Louis Couturat had given over to four lectures by Peano and his three main disciples. Peano and Alessandro Padoa were present to speak on definitions in mathematics and on deductive theories respectively; Couturat read abstracts of papers from Cesare Burali-Forti (definitions again) and Mario Pieri (the logic of geometry).[14] Russell was initially struck that morning by a dispute after Peano's paper between Peano and the German algebraic logician Ernst Schröder, in which Peano maintained the need for a symbol for "the" when defining classes.[15]

In later recollections Russell stated that he received and read Peano's works at the Congress (*MPD*, p. 65); but in his letter to Jourdain quoted above he stated that Peano had with him only the current issue (Vol. 7, no. 1) of his journal, currently called *Revue de mathématiques*, for sale. He had to wait until the end of August before the other material came in

[14] These four papers were published in sequence in *Bibliothèque du Congrès International de Philosophie*, Vol. 3 [ed. L. Couturat] (Paris: Colin, 1901; repr. Liechtenstein: Kraus, 1968), pp. 279–365. Russell himself read a paper on the absoluteness of space and time (pp. 241–77; repr. in *Papers* 3: 570–88).

[15] This exchange probably occurred between 10.00 and 10.30 on the Friday morning. Russell emphasized its importance in 1913 in an unsigned note to Norbert Wiener: see my "Wiener on the Logics of Russell and Schröder: an Account of His Doctoral Thesis, and of His Subsequent Discussion of It with Russell", *Annals of Science*, 32 (1975): 103–32 (p. 110). It was noted in one of the reports of the session: E. O. Lovett, "Mathematics at the International Congress of Philosophy, Paris, 1900", *Bulletin of the American Mathematical Society*, 7 (1900–01): 157–83 (pp. 169–70).

the post: the first two editions of Peano's edited compilation *Formulaire des mathématiques,* the first six volumes of the *Revue,* and Peano's short book *I principi di geometria logicamente esposti* (1889).

Upon reading these sources Russell was bowled by the Peanists' approach; mathematical range combined with logical power, the use of propositional functions and quantification, and especially the overthrow of subject–predicate logic with the distinction between membership and inclusion drawn from Cantorian set theory (in which Russell's own interest had gradually been growing). But he was surprised to find no logic of relations, of whose importance for philosophy in general he had been convinced from his neo-Hegelian days. So, thinking out many of the required details in September, he wrote out a draft manuscript of a paper for Peano's *Revue* the next month, in which he affirmed his belief in the central importance of relations for logic and mathematics (*Papers* 3: 590–612). It contained material on groups with applications to distance and angles, topics which unfortunately hardly appeared again in his logicist writings.

Russell also annotated "Principles" in various places with expressions of enthusiasm for Peano's system; but he must have soon realized that more radical surgery on it was needed. So he spent the rest of 1900 rewriting four relatively clearly conceived Parts of his "Principles" in the new Peanist logic together with relations, and a greater role for Cantorian set theory. As Table 1 suggests, the coverage broadly followed the corresponding Parts of its predecessor—quantity and magnitude, order and ordinal numbers, infinity and continuity, and space and geometries—but with a far more detailed analysis of virtually all components, with relations deployed very frequently. In addition, Cantorian set theory was much more prominent than before, especially the point-set topology; previously Russell's interest had lain (somewhat negatively) in Cantor's way of defining numbers and in his theory of the actual infinite, and (rather positively) in Cantor's doctrines on the *many kinds* of order-type and their bearing upon relations. The most striking novelties include his definition of irrational numbers as certain classes of rationals (Chap. 34); and the correlation of relations (Chap. 32), which was eventually to flower into his largest mathematical contribution to *Principia Mathematica,* "relation-arithmetic". But some of the most striking passages in these Parts as printed were *not* written during this period; two adjacent examples are described in §5 and §7 below.

5. ENVISIONING LOGICISM OUT OF METAGEOMETRY, JANUARY 1901

Much of Russell's discussion of geometries in Part VI stressed their hypothetical character; that the theorems of a geometry followed as consequences of the pertaining definitions and axioms. In his first mathematical book, *An Essay on the Foundations of Geometry* (1897), he had used the term "metageometry", then quite common, to describe this approach; perhaps the massive change of underlying philosophy stopped him from using it here (or in "Principles"), but the same logical pattern was emphasized. In particular, Russell wrote this passage, which looks on the manuscript as having been written in 1900 and was printed unchanged on page 373:

> ... Geometry has become (what it was formerly mistakenly called) a branch of pure mathematics, that is to say, a subject in which the assertions are that such and such consequences follow from such and such premisses, not that entities such as the premisses describe actually exist.

Later reflection upon this line of thought may well have solved for Russell his demarcation problem between logic and mathematics; *generalizing this conception of metageometry,* he envisioned logicism *as the philosophy which defined* all *pure mathematics as hypothetical, and that the Peanist line between this pure mathematics and logic did not exist.* All mathematics, or at least those branches handled in this book, could be obtained from mathematical logic as an all-embracing implication, or perhaps inference, for this new category of "pure mathematics"; the propositional and predicate calculi (including relations) with quantification provided the means of reasoning, while the set theory furnished the "stuff": terms or individuals, and/or classes or relations of them.

Later on in Part VI occurs a similar passage: "And when it is realized that all mathematical ideas, except those of Logic, can be defined, it is seen also that there are no primitive propositions in mathematics except those of Logic" (p. 430). Unfortunately, unlike the passage from page 373 just cited, this one belongs to a sector of the manuscript which is lost,[16] so we cannot tell if it was written thus in December 1900: it may

[16] No manuscript survives between folios 81a and 169 of Part VI. The corresponding

well have been largely rewritten, as earlier in it Russell referred to his eight logical constants, the number given in Part I on page 11. This article also shows other features of the developing logicism, such as an emphatic discussion of the need for nominal definitions as he had been learning from Peano and Burali-Forti:

> ... a definition is no part of mathematics at all, and does not make any statement concerning the entities dealt with by mathematics, but is simply and solely a statement of a symbolic abbreviation; it is a proposition concerning symbols, not concerning what is symbolized. I do not mean, of course, to affirm that the word *definition* has no other meaning, but only that this is its true mathematical meaning. (P. 429)

I am sure that this change occurred in or around January 1901; for that month he wrote a popular essay on "Recent Work on the Principles of Mathematics", which was published in July in the American journal *International Monthly*.[17] It has become well known, largely because he included it in his anthology volume *Mysticism and Logic* in 1918, under the revised title "Mathematics and the Metaphysicians". The essay contains a frequently quoted aphorism which seems to me to be the announcement of the birth of logicism: "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true" (*Papers* 3: 365). Its kernel is the hypothetical character given to mathematics; but its import *as* the birth announcement of logicism has understandably escaped readers.[18] Another hint came a little later in the essay with the identity thesis that "formal logic, which has thus at last shown itself to be identical to mathematics

published version starts around the middle of page 413 (in a much rewritten passage) and ends at page 453, line 15↓ with "(4) succession;". In addition (or subtraction), folios 74–106 of Part IV are missing, between page 232, line 12↑ at "no difficulty" and page 249, line 9↓ at "but no reason".

[17] B. Russell, "Recent Work on the Principles of Mathematics", *International Monthly*, 4 (1901): 83–101; repr. in *Papers* 3: 363–79.

[18] For an example at the time, see the ramble by the Peanist Giovanni Vailati around "La più recente definizione della matematica", *Leonardo*, 2 (June 1904); repr. in *Scritti*, ed. M. Calderoni, U. Ricci and G. Vacca (Leipzig: Barth; Florence: Seeber, 1911), pp. 528–34; and in *Scritti*, Vol. 1, ed. M. Quaranto (n.p.: Forni, 1987), pp. 7–12. He taught Russell's logicism at that time in the University of Turin (see his letter of 26 July 1904 to Vacca in his *Epistolario 1891–1909*, ed. G. Lanaro [Turin: Einaudi, 1971], p. 235).

..." (p. 367); for elaboration, "those who wish to know the nature of these things need only read the works of such men as Peano and Georg Cantor" (p. 369).

6. RELATIONS AND CARDINALS, FEBRUARY–JUNE 1901

Russell finished his paper on relations the next month, and sent his French translation to Peano in March. To explain the absence of a logic of a theory of relations from his programme, Peano's letter of thanks and acceptance contained the extensionalist statement that "Classes of couples correspond to relations."[19] But this must have confirmed the need of his work to Russell, who always advocated an externalist reading of relations. The paper appeared in two consecutive issues of Peano's *Revue*, in July and November.[20]

Russell felt encouraged to try a second paper, on well-ordered series and arithmetic. Drafted in March and April 1901, it was finished and translated in July and August respectively and again appeared in two parts, in May and August 1902.[21] He developed Cantorian ordinal and cardinal arithmetic within the framework of relations of kinds appropriate to generate the numbers as its fields. He dwelt further on the correlation of relations; possibly around this time (maybe later) and certainly under the influence of this paper, he added to *The Principles* two important articles (299–300, not in the manuscript), on the basic ideas of "relation-arithmetic" and Cantor's methods of generating ordinals. He also cast doubt on Cantor's belief that any set could be well-ordered (pp. 321–3).

The next essential step, perhaps already conceived, came in connection with Whitehead. His reaction to the Peanists after Paris had been

[19] H. C. Kennedy, "Nine Letters from Giuseppe Peano to Bertrand Russell", *Journal of the History of Philosophy*, n.s. 13 (1975): 205–20 (p. 214).

[20] B. Russell, "Sur la logique des relations avec des applications à la théorie des séries", *Revue de mathématiques*, 7 (1900–01): 115–36, 137–48; the division occurred between arts. 4 and 5. The paper is partly reprinted in *Papers*, 3: 613–27, with R. C. Marsh's translation on pp. 310–49 and Russell's draft on pp. 590–612.

[21] B. Russell, "Théorie générale des séries bien-ordonnées", *Revue de mathématiques*, 8 (1902): 12–16, 17–43; the division occurred between *2.25 and *2.26 of art. 1. The paper is partly reprinted in *Papers* 3: 661–73, with G. H. Moore's translation on pp. 384–421 and draft on pp. 630–60.

different from Russell's, for he had turned to Cantor's theory of finite and infinite cardinals, spiced up with the new Peano/Russell logic.[22] Within two years he produced three papers and an addendum, slightly over 100 pages in total length. They appeared in the *American Journal of Mathematics*: it was edited by Frank Morley, who had been a fellow student with him at Trinity College in the mid 1880s but had emigrated to the USA and was professor at Johns Hopkins University.

Whitehead's second paper dealt with cardinal arithmetic, and included two important notions for logicism: the class "Cls" of all classes (also typeset "cls"); and the class "Cls excl" of disjoint classes, which was to be essential for defining multiplication.[23] Russell played an important role in the writing, not later than June 1901. He cast into Peano's system Cantor's diagonal argument, which proved that for any given class α the class $P(\alpha)$ of its sub-classes has a strictly greater cardinality. He also contributed the third section of the paper, on "finite and infinite", which contained in symbolic form his nominal definitions of cardinal integers as classes of similar classes and much of the finite arithmetic.[24] This gave an essential underpinning to his logicism, since now arithmetic could be captured. All was going well; however, "after an intellectual honeymoon such as I have never experienced before or since" in writing *The Principles* in the autumn of 1900, "early in the following year intellectual sorrow descended upon me in full measure" (*MPD*, p. 73).

[22] Unlike Russell, Whitehead had stayed on in August 1900 for the Congress of Mathematicians; presumably he heard David Hilbert's famous lecture on unsolved mathematical problems, of which the first was Cantor's continuum hypothesis. So he had a double dose of Cantoriana.

[23] A. N. Whitehead, "On Cardinal Numbers", *American Journal of Mathematics*, 24 (1902): 367–94. Whitehead's suite of papers deserve a detailed study; they include pioneering attempts to apply various algebraic theories to Cantorian set theory.

[24] B. Russell in *ibidem*: 378–83 (art. 3); this passage is reprinted in *Papers* 3: 422–30. On the tangle of Russell's progress to nominal definitions from those by abstraction, see F. A. Rodríguez-Consuegra, *The Mathematical Philosophy of Bertrand Russell: Origins and Development* (Basel: Birkhäuser, 1992), esp. Chap. 5; and companion textual analyses in Byrd (n. 6): 65–7.

7. FROM CANTOR'S "FALLACY" TO RUSSELL'S "CONTRADICTION", 1900–01

In his essay in the *International Monthly* Russell described as a "subtle fallacy" Cantor's belief that there is no greatest cardinal. He probably had in mind a passage of *The Principles* written the previous November on "The Philosophy of the Infinite" where, with his common enthusiasm for faulting Cantor before reading him carefully, he found two supposed errors:

(1) there *was* such a number, namely that of Cls; hence
(2) Cantor's diagonal argument could not be applied to it to create a class of still greater cardinality.

Applying it to Cls by setting up a mapping with its power-class in which each class of classes was related to itself and every other class to its own power-class, he thought that "Cantor's method has not given a new term, and has therefore failed to give the requisite proof that there are numbers greater than that of classes."[25]

But some time afterwards, maybe in May 1901,[26] Russell thought over this line of reasoning, and found a different malaise. Using a letter of 1913 from Russell which is now lost, Jourdain reported that "In January [1901] he had only found that there must be *something* wrong" concerning this argument.[27] He did not fault either the idea of no greatest cardinal or the diagonal argument; the trouble lay in the new class thrown up by the mapping—and the new news was very serious.

[25] Coffa (n. 3): 35. In a manuscript of 1897 on "multitude and number" C. S. Peirce analyzed inequalities arising from cardinal exponentiation, but he failed to handle them correctly and found no conclusive results (*Collected Papers*, Vol. 4, ed. C. Hartshorne and P. Weiss [Cambridge, Mass.: Harvard U. P., 1933], pp. 178–89). For discussion, see M. Murphey, *The Development of Peirce's Philosophy* (Cambridge, Mass.: Harvard U. P., 1961; repr. Philadelphia: Hackett, 1993), pp. 253–74.

[26] As an autobiographer Russell is again unreliable. He dated the discovery later as in Spring (*MPD*, pp. 75–6), May (*Auto.* 1: 147), and—surely wrongly—June ("Development", in Schilpp, p. 13, and *PfM*, p. 26). Even nearer the time he was no better, giving Jourdain the June date in the recollection of 1910 quoted at the head of this paper, but Spring in 1915 (see *Dear Russell—Dear Jourdain*, pp. 133, 144).

[27] P. E. B. Jourdain, "A Correction and Some Remarks", *The Monist*, 23 (1913): 145–8 (p. 146).

The revised diagnosis can be presented in various ways, but basically it goes as follows. Cantor's diagonal argument showed that the cardinality α was less than that of $P(\alpha)$ by trying to set up an isomorphism between the classes but finding a member x of $P(\alpha)$—that is, a class—to which there was no corresponding member of α. After noting that some classes belonged to themselves while the rest did not do so, Russell used his argument to show that the class of all classes which did not belong to themselves belonged to itself if and only if it did not do so, *and, by a repetition of the argument,* vice versa *also.* This is his paradox.

The passage in *The Principles* was withdrawn (but the folio kept), and in May 1901 the revised argument was expressed in terms of predicates in the chapter on "Classes and Relations" of an attempted "Book I The Variable" of *The Principles*:

> We saw that some predicates [for example, "unity"] can be predicated of themselves. Consider now those (and they are the vast majority) of which this is not the case.... But there is no predicate which attaches to all of them and to no other terms. For this predicate will either be predicable or not predicable of itself. If it is predicable of itself, it is one of those referents by relation to which it was defined, and therefore, in virtue of their definition, it is not predicable of itself. Conversely, if it is not predicable of itself, then again it is one of the said referents, of all of which (by hypothesis) it is predicable, and therefore again it is predicable of itself. This is a contradiction, which shows that all the referents considered have no common predicate, and therefore do not form a class.
>
> (*Papers* 3: 195)

Russell was to summarize this reasoning briefly in his second letter to Frege, in June 1902;[28] three years later he gave a more technical account to his mathematical companion G. H. Hardy.[29]

This result was a true paradox, a *double contradiction,* a problem central to set theory which was in turn central to Russell's logic. This was not another neo-Hegelian puzzle to be resolved by synthesis;[30] how-

[28] See G. Frege, *Wissenschaftlicher Briefwechsel,* ed. H. Hermes and others (Hamburg: Meiner, 1976), p. 216.

[29] See my "How Bertrand Russell Discovered His Paradox", *Historia Mathematica*, 5 (1978): 127–37.

[30] I demur somewhat from the interpretation interestingly argued by G. H. Moore and A. Garciadiego ("Burali-Forti's Paradox: a Reappraisal of Its Origins", *Historia Mathematica,* 8 [1981]: 319–50) that Russell appreciated the significance of his paradox

ever, his neo-Hegelian habit of seeking contradictions may have helped him to find it, and moreover early on in his Peanist phase. There is a striking contrast here with Frege, who had been in the same area of work for over twenty years but had not found it.

8. PARADOXES VARIOUS AND INTERTWINED

Russell was now in paradox territory, not only because of this result but also in connection with the numbers associated with Cls. His reasoning had drawn upon the diagonal argument, which can generate a paradox of its own using the power-class. Using Cantor's overbar notation to mark the cardinal number of a class, it takes forms such as

$$\overline{\overline{\mathrm{Cls}}} < 2^{\overline{\overline{\mathrm{Cls}}}} \text{ and } \overline{\overline{\mathrm{Cls}}} \geq 2^{\overline{\overline{\mathrm{Cls}}}}; \quad (1)$$

the first property follows from the power-class argument while the second relies upon the definitions of Cls. Russell was diverted from finding this paradox by his switch of thinking, and it has rarely been mentioned in the discussion of paradoxes.[31] Instead, the usual paradox of the greatest cardinal is a different one based just on the sequence of cardinals; in the above notation, it could read, say,

$$\overline{\overline{\mathrm{Cls}}} < \overline{\overline{\mathrm{Cls}}} \text{ and } \overline{\overline{\mathrm{Cls}}} = \overline{\overline{\mathrm{Cls}}}. \quad (2)$$

In his writings Russell never mentioned (2) at all; further, (1) appeared in only two places, where he named it after Cantor.[32] (By contrast, modern books which discuss the paradoxes of set theory usually give (2) and

only after hearing Frege's reaction in June 1902 in response to his first letter ([n. 28], pp. 211–12). Apart from anything else, Russell had sent off *The Principles* to the Press by then, with a chapter on the paradox in it; if he had doubts, then Whitehead (or Hardy) would surely have dispelled them. Ernst Zermelo had found the paradox somehow in 1899 (B. Rang and W. Thomas, "Zermelo's Discovery of the 'Russell Paradox' ", *Historia Mathematica,* 8 [1981]: 15–22); but he seems to have told nobody outside the Göttingen circle, so that it was new to both Russell and then Frege.

[31] See my "Are There Paradoxes of the Set of All Sets?", *International Journal of Mathematical Education in Science and Technology,* 12 (1981): 9–18.

[32] B. Russell, "On Some Difficulties in the Theory of Transfinite Numbers and Order Types", *Proceedings of the London Mathematical Society,* (2), 4 (1906–07): 29–53 (p. 31); and *IMP,* pp. 135–6. There is also an allusion to it in *MPD,* p. 77.

do not mention (1).) The most relevant passage in *The Principles* occurs in a passage of Part v on the diagonal argument (p. 362), fairly heavily reworked at some stage after writing it in November 1900 and occurring shortly before the passage on Cantor's "fallacy" which he was to replace.

Instead, Russell stressed much more strongly the corresponding paradox of the greatest ordinal number, which takes forms such as (2) with one overbar instead of two and inequality read in ordinal terms. Those two paradoxes are closely linked; for if the Cantorian cardinal $\aleph_\beta$ generates a paradox, then ordinal β must be pretty large also. Presumably he gave this paradox greater publicity than those of the cardinal numbers *because of its intimate connection with order and thereby with relations*, two staples of his philosophy.

Russell learned of trouble with ordinals in January 1901 when Couturat wrote to him about a relevant paper of 1897 by Burali-Forti, and he borrowed the offprint (*Papers* 3: 385). Burali-Forti had *not* claimed any paradox; instead he had defined a different kind of order and had shown that it did not satisfy trichotomy (<, = or >).[33] However, Russell's reaction was to apply Burali-Forti's line of reasoning to Cantor's well-order-type, obtain the results analogous to (2), and bestow upon them also the award of paradox. He named it after Burali-Forti, first in a note added at the end of his second paper in Peano's *Revue*, then in a third article (301) added to Part v of *The Principles* after the two described in §6, and on some later occasions.[34]

Whatever the historical situation about these three strange results, Russell did see them *as paradoxes*. The two that he recognized were very much his creations, including the names.

9. REFINING LOGICISM, MAY–JUNE 1901

These detailed forays into relations and Cantorian territory must have

[33] The history here is quite complicated and indeed messy; for excellent surveys, see the analyses and documents in Garciadiego (n. 4), pp. 21–32, and in Moore and Garciadiego (n. 30).

[34] The main references are Russell, "Théorie générale des séries bien-ordonnées", at *Papers* 3: 421; *PoM*, p. 323; *PM* 1: 60–65, and art. 1 of the paper heralding that work ("Mathematical Logic as Based upon the Theory of Types", *American Journal of Mathematics*, 30 (1908): 222–62). There are allusions to it in *MPD*, p. 77, and in *Auto.* 1: 147. Curiously, it is ignored in *IMP*.

helped Russell to understand which undefined notions and logical constants (whether undefined or not) logicism needed. In May 1901 he sketched "Part 1 Variable" of *The Principles* with a short text summarizing eight chapters. The first one, on the "Definition of Pure Mathematics", begins:

> Pure mathematics is the class of all propositions of the form "*a* implies *b*", where *a* and *b* are propositions each containing at least one variable, and containing no constants except constants or such as can be defined in terms of logical constants. And logical constants are classes or relations whose extension either includes everything or at least has as many terms as if it included everything. (*Papers* 3: 185)

He did not coin the word "logicism",[35] but its vision was clearly stated:

> ... the connection of mathematics with logic ... is exceedingly close. The fact that all mathematical constants are logical constants, and that all the premisses of mathematics are concerned with these, gives, I believe, the precise statement of what philosophers have meant in asserting that mathematics is *à priori*.
> (P. 187)

He outlined the main features of the theories of classes and of relations

[35] Russell only used the word "logicism" once, when writing of Frege "who first succeeded in 'logicizing' mathematics" (*IMP*, p. 7: note his scare-quotes). During the 1910s some German authors used "*Logizismus*", and even "*logizistische*" with a sense of denial and in the context of phenomenological logic (see references including to himself from 1913, in T. Zieher, *Lehrbuch der Logik* ... [Bonn: Marcus und Webers, 1920], pp. 172–3: my thanks to G. Sanchez Valencia for this reference, and to V. Peckhaus for this and others via "Russell-l"). Perhaps independently, "*Logizismus*" in the modern sense was proposed by Rudolf Carnap around 1927 in letters, hesitantly suggested in his *Abriss der Logistik* (Vienna: Springer, 1929), pp. 2–3, and publicized confidently in his "Die logizistische Grundlegung der Mathematik", *Erkenntnis*, 2 (1932): 91–105. Early converts include W. Dubislav in *Die Philosophie der Mathematik in der Gegenwart* (Berlin: Junker und Dünnhaupt, 1932), pp. 38–43; and W. Burkamp, *Logik* (Berlin: Mittler, 1932), p. 152. Probably Carnap was troubled by the ambiguity of the regular word "Logistik", which referred indifferently to the works of Peano's school, to Russell and Whitehead's different philosophy, and to other formal systems using these logical techniques. It owed its origins, as "Logistique", to Augustin Cournot in the 1840s: it was revived at the International Congress of Philosophy at Geneva in 1904, where Louis Couturat, André Lalande and the obscure Gregor Itelson came to it independently (see Couturat, "1[me] Congrès de Philosophie ...", *Revue de métaphysique et de morale*, 12 [1904]: 1037–77 [p. 1042]).

in a separate chapter, followed by a discussion of the variable in which any temporal connotation was condemned. The manuscript is incomplete (*Papers* 3: 185–208), but remaining is part of a survey of "Peano's symbolic logic".

With a reasonable-looking Part I now sketched, Russell could write Part II of *The Principles* on "Number", in June 1901. He laid out cardinal arithmetic within this logic: his definition of cardinals as classes of similar classes, the pertaining arithmetical operations and their arithmetic, and the definition of the infinite class of finite cardinals without reference to numbers themselves but by generation from transitive and symmetrical relations. Thanks to his own insights and Whitehead's exegesis, finite and infinite could be nicely distinguished, and mathematical induction did not have to be taken as primitive. He also showed that the Peano postulates for cardinal arithmetic came out as theorems, thus making clear by this example the deeper level of foundation which he could attain (*PoM*, pp. 127–8). Another major achievement was to clarify the tri-distinction between the empty class, the cardinal and ordinal zeros (pp. 128 and 244 respectively), and the notion of nothing (p. 73); his importance here, and that of his anticipator Frege, is too little recognized.

Comparisons with the corresponding Part of "Principles" show how Russell's priorities had changed with his conversion. That one had been entitled "Whole and Part" (Table 2); in *The Principles* the topic received one chapter, of six pages (137–42). Logicism was coalescing; but the paradox, surely important, lacked Solution. Part I still needed much cogitation.

10. THE DEFINITIVE LOGICISM, APRIL–MAY 1902

For several months after June 1901 Russell seems not to have much modified his book. If a typescript of Parts III–VI was prepared, as was mooted in §2, then perhaps it was done during this period. In August he completed his second paper for Peano; and during the winter he gave a lecture course in mathematical logic at Trinity College, the first in Britain (*Papers* 3: 380, 677–8), with Whitehead and student Jourdain among the audience. Some collections of notes seem to belong to this period: on continuity, likeness between relations, implication and the notion of class (pp. 431–51, 553–7, 566–9). Another suite attempts to solve his paradox (pp. 560–5), which was doubtless a main preoccupation of this period; another one will have been to determine the primitive logical-(/set-theoretical) notions among the wide repertoire available and surveyed in sketch form the previous May.

By April 1902 Russell had a plan of Part I of *The Principles* in eleven chapters (pp. 209–212), which included "Denoting", "Assertions" and as a finale "The Contradiction". He followed the scheme closely in the writing.

The Part began with a "Definition of Pure Mathematics" which elaborated upon the vision a year earlier:

> 1. Pure Mathematics is the class of all propositions of the form "*p* implies *q*", where *p* and *q* are propositions each containing at least one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth.
> 2. The above definition of pure mathematics is, no doubt, somewhat unusual. Its various parts, nevertheless, appear to be capable of exact justification—a justification which it will be the object of the present work to provide.

Curiously, unlike the sketch, Russell's list did not include the notion of variable, which he soon emphasized as "one of the most difficult which Logic has to deal, and in the present work a satisfactory theory ... will hardly be found" (pp. 5–6). However,

> 9. Thus pure mathematics must contain no indefinables except logical constants, and consequently no premisses, or indemonstrable propositions, but such as are concerned exclusively with logical constants and with variables. It is precisely this that distinguishes pure from applied mathematics. In applied mathematics, results which have been shown by pure mathematics to follow from some hypothesis as to the variable are actually asserted of some constant satisfying the hypothesis in question....
> 10. The connection of mathematics with logic, according to the above account, is exceedingly close.

Russell clearly stated logicism in this chapter as an inclusion thesis; pure mathematics is *part* of this logic. But we saw in §5 that he had recently proposed an *identity* thesis between mathematics and his logic, which is surely indefensible; logic can be used in many contexts where mathematics is absent (for example, a *modus ponens* involving only particulars). Unfortunately, he did not have this distinction always under control, for the identity thesis appeared later—in *IMP*, page 194, and even in the 1937 reprint of *The Principles*, where early on in his new preface he opined that "the fundamental thesis of the following pages, that mathematics and logic are identical, is one which I have never seen any reason to modify" (*PoM*, p. v)! Perhaps he had recalled the unclear end of the new Part I, where "Mathematics [was] brought into very close relation to Logic, and made it practically [*sic*] identical with Symbolic Logic" (p. 106). The point is not at all trivial; apart from the question of whether or not mathematics is running, say, syllogistic logic or the law courts, there is the possibility that *only some* of the fundamental notions of logic are required for grounding (pure) mathematics. In *The Principles*, however, he assumed that all of them were needed.

Russell gave this Part the title "The Indefinables of Mathematics", and the rest of it went through the basic and subsidiary components required in his logic. The extra notions promised in article I were "propositional function, class, denoting, and *any* and *every term*". They were all adopted precisely *and only* as the epistemological starting points of logicism, not as self-evident entities, the position which is frequently misattributed to him; as he was to warn clearly in the preface, "the indefinables are obtained primarily as the necessary residue in a process of analysis", so that "it is often easier to know that there must be such entities than actually to perceive them" (p. xv). The Part has been extensively discussed; my purpose here is only to indicate some chief differences over the plan of 1901.[36]

One example is the wide range of notions concerning "Denoting" (Chap. v); for "characteristic of mathematics" are the six words "*all, every, any, a, some* and *the*" (p. 55). Russell could not handle any of them to his own satisfaction, but "the" fared the best: doubtless recalling a morning in Paris, he noted that it had been emphasized by Peano, but "here it needs to be discussed philosophically" (p. 62). However, while he brought out well its importance for theories of identity, he could not

[36] Blackwell (n. 5) gives the precise details of changes wrought to Part I on proof.

find a workable criterion for its legitimate occurrence. Handling "the" was to be his major advance, in his famous theory of definite descriptions of 1905; but one regrettable consequence is that commentators very often misidentify that theory with his *much broader* theory of denoting.

The six little words also played roles in Russell's complicated philosophy of classes. For a class *u*, " 'all *u*'s' is not analyzable into *all* and *u*, and that language, in this case as in some others, is a misleading guide. The same remark will apply to *every, any, some, and*, and *the*" (p. 73). He came to an extensional view of classes, which he contrasted with intensional class-concepts (pp. 73–7); but the paradox still held its ground. As a result, his conceptions of classes were to fluctuate, during the rest of his logicist career, between a full-blown intensionalism for a time around 1904 to the comprehensive extensionalism of the second edition of *Principia Mathematica* (prepared in 1922 and 1923).

Another striking, and related, change in the Part was the greatly increased place given at last to propositional functions (Chap. 7); in "Part I Variable" of the previous year he had said really nothing explicitly about them (at least in the surviving portions). One of their main roles was to specify classes, via class-concepts and the indefinable *such that* (p. 72). He also wondered about functions *f* predicated of themselves to produce "$f(f)$"; but his paradox to which he devoted the last chapter (10), made all these matters uncertain (p. 88).

II. AFTERTHOUGHTS ON DYNAMICS: PART VII, MAY 1902

This final Part, "Matter and Motion", was put together largely by importation from "Principles"; after a new opening folio, the next one is dated "1899", and folio 30 "June 1900".[37] Russell treated some aspects of dynamics, following studies from around 1898 (*Papers*, 2: 83–110); for some reason he ignored statics. As well as containing much of the oldest text in the book, it is the weakest Part as well as the shortest (34 pages): he seemed to be unaware of a rich field of work in the foundations of mechanics at that time, especially in Germany.[38] One suspects an

[37] See K. Blackwell, "Part VII of *The Principles of Mathematics*", *Russell*, forthcoming.

[38] On these still overlooked developments, see Vol. 4, sec. I of the *Encyklopädie der mathematischen Wissenschaften*, especially A. Voss, "Die Principien der rationellen Mechanik", (1901): 3–121, and P. Stäckel, "Elementare Dynamik der Punktsysteme und

understandable desire to get this big and tiresome book finished as soon as possible, especially with the continuing deterioration of his marriage to Alys over the previous eighteen months or so.[39]

Russell's basic strategy was to treat "rational Dynamics" as "a branch of pure mathematics, which introduces its subject-matter by definition, not by observation of the actual world", so that "non-Newtonian Dynamics, like non-Euclidean Geometry, must be as interesting to us as the orthodox system" of Newton (*PoM*, p. 467). He then used the continuity of space, as established in Part VI by Cantorian means, to establish realms within which motion could take place (Chap. 54).

In the next chapter Russell sought to establish causal chains as implications; but unfortunately he made the obviously mistaken assumption that if "a sufficient [finite] number of events at a sufficient number of moments" were known, then new moments "can be inferred" (p. 478). Maybe he drew upon analogies from logic, such as the members of a finite class (p. 59); but it was an elementary gaffe to assume that there could be a sufficiently large number of given events. As Hardy pointed out in reviewing the book in September 1903, if a particle be "projected from the ground, and take the second time to be that at which it reaches the ground again. How can we tell that it has not been at rest?"[40]

Apart from this, the enterprise undertaken in this Part sounds too good to be true, or more especially to be logicistic. How, or why, should logic care about dynamics? Are the propositions of this Part really expressed *only* in terms of *logical* constants and indefinables? As the American mathematician E. B. Wilson remarked in his review of the book in 1904, "why not thermodynamics, electro-dynamics, biodynamics, anything we please?"[41]—ballet and symphonies, one might add, or the French Constitution. It is worth noting that *Principia Mathematica* contains no treatment of dynamics (although unfortunately also no explanation of its absence); by then Russell had thought out better

starren Körper", (1905): 435–684.

[39] On the latter, see I. Grattan-Guinness, " 'I Never Felt Any Bitterness': Alys Russell's Interpretation of Her Separation from Bertie", *Russell*, n.s. 16 (1996): 37–44.

[40] [G. H. Hardy], Review of *PoM*, *The Times Literary Supplement*, 18 Sept. 1903, p. 263; repr. in his *Collected Papers*, 7: 851–4.

[41] E. B. Wilson, Review of *PoM* and *An Essay on the Foundations of Geometry*, *Bulletin of the American Mathematical Society*, 11 (1904–05): 74–93 (p. 88).

this aspect of logicism, and must have realized that Part VII belonged more to its neo-Hegelian background in the 1890s than to the new position of the 1900s. Maybe he also assumed unthinkingly that all seven Parts of "Principles" had to be rewritten into seven Parts of *The Principles.* His definitions of logicism, as quoted in §9 and §10, are unclear in that he did not lay down any restriction on the *kinds of magnitudes* over which variables could range; thus intruders such as terms from dynamics could be admitted.[42]

12. FINISHING (?) THE BOOK, JUNE 1902–FEBRUARY 1903

Russell wrote the date "May 23, 1902" on the last folio (72) of Part VII, stopped rewriting his book (or thought he had, anyway), sorted out the numberings of chapters and more or less of the articles. In June he signed a contract with Cambridge University Press, and shipped off the manuscript there. But it shows that the fiddling was far from over. While handling the proofs between June and the following February he added many of the footnotes, especially the majority of the references to pertinent literature which he now read at greater leisure. He also entirely rewrote a few articles and added two appendices, and maybe then prepared the lengthy analytical table of contents (the manuscript has also not survived).

One major source of change was Frege, whose work he read in detail for the first time in the summer of 1902. One early reaction was to add three remarks and five footnote references to his text, all but one to Parts I and II. He also altered article 128 and most of article 132 of Part II from doubts about treating a "number as a single logical subject" to a stress that the "*one* involved in *one term* or *a class*" should not be confused with the cardinal number one defined earlier, citing Frege's *Grundlagen* in support (*PoM*, pp. 132–3 and 135–6). The rewriting on proof of page 104 on classes and propositional functions was also partly inspired by Frege, but finally he omitted the most explicitly dependent passage.[43] By November he had also completed an appendix to his book on Frege's "logical and arithmetical doctrines", as he accurately

[42] With considerable apprehension, I reserve for another occasion the question of Russell's understanding of logicism in *Principia Mathematica.*

[43] Blackwell (n. 5), p. 288.

characterized them[44] (pp. 501–21, in slightly smaller font).

The other main change was a second appendix, also apparently finished by November (pp. 523–8, also in smaller font). In it Russell tried a theory of types to solve his paradox; but he noted that it was not successful, for he reformulated his paradox by associating a proposition P with the proposition *Q* "every member of a class *m* of propositions is true", and considering the class of propositions *P* which do not belong to the corresponding *Q*. Regrettably, he seems to have forgotten this paradox later, which can be constructed in his later simple type theories.[45]

In December Russell wrote the preface to the book. Then he surveyed it entirely in a new article (474) added to Part VII; it was received by the Press on 27 January 1903 (according to their date stamp on its first folio). After an analysis in Part I "of the nature of deduction, and of the logical concepts involved in it", among which "the most puzzling is the notion of *class* ... it was shown that existing pure mathematics (including Geometry and Rational Dynamics) can be derived wholly from the indefinables and indemonstrables of Part I. In this process, two points are specially important: the definitions and the existence theorems", the latter being "almost all obtained from Arithmetic". The known types of number and of order-type apparently provided the stuff of space and of geometries, which could be correlated with continuous series to "prove the existence of the class of dynamical worlds"; thus it followed that "the chain of definitions and existence-theorems is complete, and the purely logical nature of mathematics is established throughout." He thus finished his first presentation of logicism, in serene inclusion of dynamics but equally tranquil disregard not only of the mathematical physics which sits so akin to it but also of abstract algebras, probability and statistics, ..., ...

13. PUBLICATION AND RUMINATIONS, SUMMER 1903

During early February Russell prepared the index (*Papers* 12: 18), and at last the work was over. The book was published in May 1903, around his

[44] *Explicitly not* a philosophy of "mathematics", unlike the endlessly repeated modern misunderstanding.

[45] See P. de Rouilhan, *Russell et le cercle des paradoxes* (Paris: Presses Universitaires de France, 1996), pp. 178–94, 223–30.

31st birthday. The print-run was 1,000 copies at 12*s.* 6*d.* each, or \$3.50 when it went on sale in the USA in June. The intended audience comprised a sector of the philosophical and mathematical communities interested in each others' concerns, especially the audience for set theory which had been growing rapidly for around a decade. Indeed, the book played an important role in awakening the British to Cantor's set theory, and to mathematical as well as algebraic logic. It seemed to sell steadily; in June 1909 the Press told him that the last 50 copies were at the binders (RA1 410).

But Russell knew that the book was rather a shambles. Within days of issue he wrote to Frege on 24 May 1903 that in Parts I and II "there are several things which are not thoroughly handled, and many opinions which do not seem to me to be correct",[46] and two months later he told his friend the French historian Elie Halévy that "I am very dissatisfied with it" (*SLBR* I: 267). The previous 28 December, at preface-writing time, he even confessed to Gilbert Murray that "this volume disgusts me on the whole" (RA1 710).

The unsolved paradox was doubtless one main reason; but Russell must have recognized that the presentation was somewhat disordered and even contradictory across and even within some chapters. His apparent decision not to write Parts I and II until he had tested out the Peanists' approach in Parts III–VI was very sensible, since he had a good idea of what they would contain; books are often written out of order of presentation. But he did not bring the later Parts in line with positions and assumptions finally laid out in the openers, nor did he tidy up the overlaps (for example, on infinity and continuity between Parts II and V).

Russell's principles of mathematics, dependent so much upon order and relations, were in rather a mess in 1903. It must have been a big disappointment after the honeymoon of three years earlier.[47]

[46] Frege (n. 28), p. 242.

[47] Acknowledgements: Lectures based upon this material were delivered in November 1995 at the Fourth International Symposium on the History of Mathematics at Neuhofen (Austria), and to the Department of Philosophy in the University of Bologna (Italy), and in 1996 at the Institut d'Histoire et Philosophie des Sciences in Paris, and at the Departments of Philosophy of Indiana University (Bloomington) and of King's College, London. Answers to questions posed on these occasions have worked their way into this version, as have excellent promptings from Kenneth Blackwell and Michael Byrd. © I. Grattan-Guinness.