Here I continue earlier work devoted to the detailed study of Russell's manuscript copy of *The Principles of Mathematics*, in particular its relation to the text as published in May 1903. Previous studies have presented collations of Parts I, II, and V of *Principles*, together with assessments of the significance of the variations between manuscript and published text. In the present study, I examine the manuscripts of Parts III (Quantity) and IV (Order).

As I and others have argued, the manuscripts render likely the conclusion that Russell began his work on *Principles* in the fall of 1900 by writing Parts III through VI of the book. Parts I and II were written later. A draft of Part I, now published in the *Collected Papers*, Vol. 3, was written in May 1901, and an expanded and much altered version was written in May 1902, the month before Russell sent the book off to Cambridge University Press. Part II was written originally in May or June of 1901, and not, in my opinion, extensively revised thereafter prior to its submission.

The collations of Parts I, II, and V reveal substantial alterations...
made by Russell subsequent to their initial composition. In the case of Parts I and II, these represent primarily Russell's ongoing reflections on the paradoxes about classes and relations and also his growing acquaintance, during the summer and fall of 1902, with Frege's logical and philosophical ideas. In the case of Part V, on the other hand, many alterations reflect the fact that the manuscript was not substantially altered after its initial composition in November 1900 and was probably replaced by a typescript, now lost, on which some of the alterations may have been made. (The manuscripts of Parts I, II and VII have, instead, several characteristics that identify their service in the University Press's printshop.) Given the rapid development of Russell's thought during the years 1901-02, the fall 1900 manuscript is terminologically and doctrinally at variance with the newer Parts written in 1901 and 1902.

The existing manuscripts of Parts III and IV, are, in this respect, more similar to Part V than Parts I and II, as is that of Part VI. The manuscripts for these parts were written primarily in the fall of 1900, and, I think, not substantially revised thereafter. Unlike Part V, the revisions in Parts III and IV are considerably less extensive and substantial. The alterations in Part III, for example, amount to about 650 words; in the manuscript of Part V, the alterations number about 3,500 words in a text that is two and one-half times as long. This difference is not surprising; Russell's views on quantity and order were more stable through this period than his views on number and class, because they were less affected by the concerns about foundational issues in logic that arise in Parts I and V.

The manuscripts of Parts III and IV confirm some conclusions that I reached based on the manuscript of Part V. In particular, the explicit logicist definitions of cardinal and ordinal number in terms of equivalence classes are not part of Russell's early post-Peano work. No such definitions appear in the fall 1900 manuscripts of Parts I and II. The relatively few mentions of them in the published texts are later insertions. (See, for example, the newly added line 4 on page 158 and the added footnote on page 167.) Instead, as Rodríguez-Consuegra has argued, Russell's first reflections in the fall of 1900 give a central place to what Russell calls the *axiom* of abstraction: "whenever a relation, of which there are instances, has the two properties of being symmetrical and transitive, then the relation in question is not primitive, but is analyzable into sameness of relation to some other term ..." (*PoM*, p. 166). This principle is used by Russell in Part III to adjudicate between what he calls the absolute and relative theories of quantities. It is used in Part IV to clarify certain issues about relations and their properties (§§210 and 211) and was apparently used to offer a definition by abstraction of ordinal numbers in the now missing manuscript version of Chapter 29. I will examine below the use to which the principle is put in Part III.

I. THE MANUSCRIPT TEXT

The initial leaves of the manuscript of both Parts III and IV are dated November 1900. The upper left-hand corner on the leaves from Part III bear the notation "Q" for Quantity. Those from Part IV bear the notation "O" for Order. There are no section numbers, and chapters are ordered internally, the first chapter of each part being labelled "Chapter 1".

The list of variants is given at the end of the essay. It is constructed on the model of previous collations in this series. The list is read as follows. At the left is a number such as 157: 27. This means page 157, line 27 from the top. To the right is the reading from the first impression of the published text of *The Principles of Mathematics*. This is followed by a square brace and then the corresponding reading from the fall 1900 manuscript. Editorial brackets enclose my comments.

The leaves in Part III are numbered consecutively from 1 to 80; there are five leaves with non-standard numbers: 33a, 45a, 64a, 64b, 67a. A number of leaves have double numbers. They are taken

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6 Since the manuscript of Part V is also dated November 1900, this suggests that Russell produced about 400 manuscript pages of work during that month.

7 33a is an inserted paragraph on the comparability of quantities of a given kind. This insertion is self-contained, and placed in the text of Chapter 20 (The Range of
from the 1899–1900 version of Part III of Principles. This includes all of Chapter 11 (The Range of Quantity), which becomes Chapter 20 of Principles, and a number of leaves from Chapter IV (Continuity, Zero, and Infinity).

The leaves from Part IV are numbered consecutively from 1 to 72, and then from 108 to 122. Thus there is a substantial gap in the manuscript; the gap begins at 232: 12 of the text after the word “difficulty”, and the manuscript resumes at 249: 35, with the word “but”. It thus includes the last page and a half of Chapter 27, all of Chapters 28 (On the Difference between Open and Closed Series) and 29 (Progressions and Ordinal Numbers), and all but two pages of Chapter 30 (Dedekind’s Theory of Number). The gap is not a natural unit, such as a chapter or a discussion of some particular set of topics. It begins and ends in the middle of sentences. It is an unfortunate loss, since especially Chapters 29 and 30 concern topics on which Russell was making rapid progress in the fall of 1900, as we can see from the contents of section 4 of the October 1900 draft of “The Logic of Relations”. That the lost material is not simply what is found in the published text is shown by a lengthy section of the manuscript (folios 108–10), which was omitted from the published text.

A striking fact about the extant parts of the manuscript of Part IV is the absence of the familiar phenomena of inserted or pirated leaves.

The version of Part IV written by Russell in 1899–1900 is completely intact; it is about half the length (68 leaves) of the manuscript written in the fall of 1900. The Fall 1900 version introduces substantial new chapters; on the distinction between open and closed series and on Dedekind’s views. What this suggests is that Russell’s thought in this area was considerably clarified and extended by his growing acquaintance with Peano and his school. Certain central philosophical claims, such as the importance and irreducibility of asymmetrical relations, are already firmly in place in the earlier manuscript. But much material is new; this is especially true of the work on closed series and the relation of separation of couples, which draws heavily on the work of Vailati.

In my discussion of Part V, I noted that the manuscript used terminology and expressed views different, in certain respects, from what we find in the spring of 1902. As mentioned earlier, the manuscript contains repeated references to the “axiom” of abstraction, in line with the fact that the principle is taken as an unproved primitive proposition in the October 1900 draft of “The Logic of Relations”. This is always changed in the published text to the “principle” of abstraction, since the spring draft of “The Logic of Relations” offers a proof of the principle (Theorem 6.2). In the manuscript, Russell uses “concept” as the overarching general category, so that points, instants, and bits of matter are regarded as concepts. (See the collation at 212: 15–21.) In the published text, Russell uses “term” for the overarching category and “concept” for the smaller of terms that may occur in a proposition “otherwise than as terms”, in conformity with the classification set out in Chapter 4 of Part I of the published text.

It is also clear that Russell’s understanding of Cantor’s conception of continuity was defective when he originally wrote Part III. On page 193 of the text, Russell introduces “continuity” to mean merely that a
series is dense. In a footnote to this introduction, he explains correctly why this does not properly capture the continuity of the real numbers. But this footnote supplants a footnote in the manuscript in which Russell says that the objection to the definition is that "it does not give a fixed property of a collection, but depends sometimes upon the order in which the terms are taken." Cantor's definition does not give a fixed property of a collection either, of course. Very soon thereafter, Russell's had dramatically clarified his understanding when he wrote Chapters 35 and 36 of Part v.

The manuscript text also sets out a conception of numbers that does not sit easily with the logicist definitions of the published text. In §200 where he is discussing whether there are unanalyzable three-place relations, Russell gives a brief account of his basic metaphysics of terms, concepts, and propositions, roughly as set out in Chapter 4 of the published text. However, the manuscript version of several sentences in this passage treat numbers as predicates whose role in numerical predication is like that of relations. That is, numbers do not occur as terms in such sentences. Rather, they are adjectives, as Russell calls them, which apply to multiplicities of objects. Here are the relevant passages (ms. fos. 30-1, my emphasis).

There are, on the other hand, concepts which can occur otherwise than as terms: such are being, numbers, and relations.

This gives the opinion that relations are always between only two terms, for a relation may be defined as any concept, other than a number, which occurs in a proposition containing more than one term.

In the first place, when a number is asserted of a collection, if the collection has \( n \) terms, there are \( n \) terms and only one concept (namely \( n \)) which is not a term.

In the published text, the italicized words are omitted, and in the third passage, Russell speaks of the "concept of a number", not a number directly.

2. THE MISSING(?) PARTS I AND II

As with the manuscript of Part v, the manuscripts of Parts III and IV have few back references to Parts I and II. They are quite interesting in content and again strongly indicate that these Parts had not yet been written in the fall of 1900. In the manuscript version of the passage at 229: 26–8, Russell writes: "But the addition of integers, as we saw in Part I, is a complicated matter, and the relation \( R \) is prior to it." Strikingly, the back reference here is to Part I, not Part II, as one would expect. Part I of the 1899–1900 version of Principles contained an extended discussion of arithmetical addition in Chapter III. So here Russell appears to anticipate that Part I of the new manuscript would begin with a part on Number just as his previous attempt had done.

A similar case is found in the manuscript version of the passage at 178: 23, where Russell writes:

But the addition of two quantities of divisibility, i.e. two wholes, does yield a new single whole, provided the addition is not of the kind explained in Part I, but of that other kind which we discussed in Part II.

As a back reference to the published text of Principles, it is hard to know what to make of this. But in the context of the 1899–1900 version, the intended reference is quite clear. There Chapter 11 of Part I is entitled "Arithmetical Addition", and it concerns what Russell calls the addition of terms. Chapter 111 of Part II (Whole and Part without the Use of Numbers. The Logical Calculus) discusses what Russell calls the addition of wholes to form a new whole. It is clearly this latter kind of addition to which Russell is referring in the manuscript passage.

There are a couple of other back references in the manuscript. One of these occurs at 160: 17–18:

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16 See the List of Variants for 197: 41–4. A similar footnote occurs at the comparable point in the 1899–1900 version of Part III. See Papers 3: 72.


The kind of equality which consists in having the same number of parts has already been discussed in Part I. If this indeed be the meaning of quantitative equality, then quantity introduces no new idea. But it may, I think, be shown that greater and less have a wider field than whole and part, and an independent meaning.

The back reference here is to a discussion of equality in connection with whole and part. Part I of the 1899–1900 version of *Principles* is entitled “Whole and Part”, but it contains no discussion of the kind of equality mentioned by Russell. The relevant chapters in *Principles* are Chapters 16 and 17 of Part I, but they likewise consider no such notion of equality. There is, of course, the notion of similarity of classes, used to define cardinal numbers. Russell was already using this notion in the October draft of “The Logic of Relations”; so, it seems likely that we have here an anticipation on Russell’s part of consideration of this relation in the second part.

The only other back reference to Parts I and II in the manuscript occurs at 178: 20: “In this case, although the magnitudes are even now incapable of addition of the sort required, the quantities can be added in the manner explained in Part I.” Once again the sort of addition in question here is the combination of two wholes to form a new whole. As noted above, this kind of addition is isolated and discussed in Part I of the 1899–1900 version of *Principles*, and this discussion recurs in the published text at several places, for example, at §131. Certainly, the back reference here is to material that Russell could readily have anticipated being in Part I on the basis of his own previous writing on the subject.

By way of contrast to the back references that occur in the fall 1900 manuscript, the published text contains a number of significant and specific additions that involve Parts I and II. On page 167, there is a new footnote that shows how to apply Russell’s proof of the principle of abstraction to the definition of magnitude: a magnitude is a class of equal quantities. There are two back references in the published text to §55 at 186: 44 and 211: 37. Since Russell allows that relations may be magnitudes, he claims that the quantities which have these relational magnitudes are particular instances of the relation, “a particularized relation”. However, Russell interprets the results of §55 as showing that there are no such entities as particularized relations. At 211: 37, he writes: “The particularized relation is a logically puzzling entity, which in Part I (§55) we found it necessary to deny.” This is, of course, not in the manuscript. Here Russell finds that his developing logical views undermine his conception of relational quantities.

There is a striking footnote on page 192 of the published text. It refers to the discussion in Part I of two different criteria for being a finite class; one using mathematical induction and the other Dedekind’s definition in terms of similarity to a proper subclass. The footnote in the manuscript says simply, “See Part V.” These examples confirm the view that Russell wrote Parts III–V without Parts I and II in hand, and that when he came to submit the manuscript in June 1902, he did not adjust the manuscripts to fit with his evolving views. Moreover, it is clear that the logicist conception of mathematics that forms the unifying theme of the published text is absent from the fall 1900 manuscript of Parts III and IV.

### 3. THE DISAPPEARANCE OF QUANTITY

In the collection of manuscripts on the philosophy of mathematics, which Russell produced in the years 1898–1900, there is always a part

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19 Russell seems wrong in this respect. The most §55 shows is that not all relations are particularized relations. See N. Griffin and G. Zak, “Russell on Specific and Universal Relations: *The Principles of Mathematics,* §55”, *History and Philosophy of Logic*, 3 (1982): 55–67.

20 In his recent biography of Russell, Ray Monk assumes that the fall manuscript of *Principles* already contained the basic logicist definitions. For example, at page 132, he writes, “By the end of the year he had produced a work of astonishing breadth and equally astonishing confidence, which sought to show that the whole of mathematics could be based on a mere handful of logical notions and axioms.” No substantial evidence, based on the extant manuscripts, is offered in support of this very questionable assumption. He does bring forward a letter to Helen Thomas, written on 31 December 1900, the relevant part of which reads, “In October I invented a new subject, which turned out to be all of mathematics, for the first time treated in its essence.” The subject invented in October was the logic of relations, of course. But this rather general remark, with its suggestive use of the verb “turned out”, does not justify the assumption that the fall manuscript already incorporated and developed the idea that the logic of relations was all of mathematics, especially not in the face of the clear countervailing evidence of the manuscripts themselves. See Ray Monk, *Bertrand Russell: the Spirit of Solitude*, 1872–1921 (New York: Free P., 1996).
on Quantity. It invariably is the third or fourth part, usually third, and follows an initial part on number and one on some topic which Russell associates with logic, such as "Whole and Part". The overall structure of this Part can already be seen in the synoptic table of contents that Russell prepared for the manuscript of The Fundamental Ideas and Axioms of Mathematics in 1899. It has four chapters that correspond in topic and order to Chapters 19, 20, 21, and 23 of Principles (Papers 2: 268–9). The same is true of the 1899–1900 version of Principles; it has four chapters on the same topics in the same order.21

There is also a considerable degree of doctrinal and argumentative stability in the first three of these four chapters. Russell argues that magnitudes are primitive, and that they are needed in addition to the quantities which have the magnitudes. Both relations and qualities can have magnitudes. Distances and divisibilities are in some sense naturally measurable by numbers. Other magnitudes are at best only indirectly measurable by numbers.

The situation is very different with respect to the fourth of these chapters, corresponding to Chapter 23 of the published text. Chapter IV of Part III of the 1899–1900 version is entitled "Continuity, Zero, and Infinity". In it, Russell sets out the opposing triads of propositions listed as (1), (2), and (3) and (a), (b), and (c) on page 190 of the published text.

(1) That no two magnitudes of the kind are consecutive.

(2) That there is no least magnitude.

(3) That there is no greatest magnitude.

(a) There are consecutive magnitudes....

(b) There is a magnitude smaller than any other of the same kind.

(c) There is a magnitude greater than any other of the same kind.

In the 1899–1900 version, Russell says that (a), (b), and (c) can be strictly proved on the basis of what he calls the philosopher's axiom of finitude, an axiom which he clearly accepts. He states the axiom in the following passage:

The whole argument turns upon the principle by which infinite number is shown to be self-contradictory, namely: Many terms must be some definite number of terms.22

For the fall 1900 manuscript, Russell retains many of the leaves from the 1899–1900 chapter, though they are heavily overwritten. He contends that the arguments in support of (a), (b), and (c) ultimately rest on the assumption that the principle of mathematical induction governs all numbers:

This is the principle which the philosopher must be held to lay down as obviously applicable to all numbers, though he will have to admit that the more precisely his principle is stated, the less obvious it becomes. (PoM, p. 192)

Russell then likens the principle of mathematical induction to the axiom of parallels; while useful in its proper place, to suppose it always true, "is to yield to the tyranny of mere prejudice" (PoM, pp. 192–3).

21 Chapter 22 (Zero) has no predecessor in the manuscripts. In these, "zero" is linked with continuity and infinity in the fourth chapter of the part. Zero is thought of as the supposed least quantity of a kind, parallel to infinity, the greatest quantity of the kind. In Chapter 22, this is rejected as a true definition of a zero quantity on the grounds that it fails to bring out the intrinsic connection of a zero quantity to some kind of negation.

The first leaf of the manuscript copy of Chapter 23 was originally entitled "Zero, Infinity, and Continuity", but "Zero" has been overwritten and "the Infinitesimal" inserted. Moreover, the number of this leaf appears to have originally been "55", the number of the first leaf of Chapter 22. Furthermore, the manuscript summary of Part 111 (fos. 78) appears to treat Chapters 22 and 23 as a single chapter. (See the List of Variants, 195; 32.) This suggests that the new material now appearing as Chapter 22 was originally subsumed under Chapter 23. It is clear that a primary impetus for these new reflections is Meinong's Uber die Bedeutung des Weber'schen Gesetzes.

22 Papers 3: 72 reads "A given collection of many terms has some finite number of terms", but it is most important here to look at the Textual Notes to this passage on 768: 72: 6; 72: 7, and the manuscript itself. The relevant manuscript leaf was reused in the fall 1900 manuscript, and the particular principle at issue here has been overwritten by Russell. Its original form appears to me to have been: "Many terms must be some definite number of terms." Russell blots out the first two letters of the word "definite" to obtain the word "finite", which is what appears in the published text. This reading of the older manuscript certainly would have made appeal to the principle less blatantly circular. In the fall of 1900, of course, Russell was interested precisely in making the circularity obvious.
Here Russell is attacking his own earlier argument, presenting it by using the very leaves on which it was written.

A crucial fact about Part III is that its importance diminishes as Russell’s thought evolves. In his 1897 paper “The Relations of Number and Quantity”, it is clear that Russell regards quantity as a central concept of mathematics and that he considers the question of how number and quantity are related as among the central questions of mathematical philosophy (Papers 2: 70). This is reaffirmed in the opening sentence of Part III of the 1899–1900 manuscript. But Russell’s discussion there suggests something quite different. Russell says that the work of Weierstrass and others has shown that all of pure mathematics should be regarded as dealing exclusively with numbers. In addition, he claims that algebra has been extended so as to cover non-numerical areas. This has led to “a greater separation of number and quantity” than has traditionally been maintained. Thus, Russell will attempt to give a theory of quantity which is independent of number (Papers 3: 54). Russell also emphasizes that the antinomies concerning continuity and infinity have nothing specially to do with quantity. They are problems of a “strictly arithmetical nature” (Papers 3: 73–4).

The fall 1900 manuscript version further subordinates the importance of quantity. He argues that the work of Weierstrass, Dedekind, and Cantor and the development of non-numerical, non-quantitative mathematics, such as projective geometry and the logical calculus, show that quantity and number are “completely independent” (PoM, p. 158). Moreover, what is mathematically important about quantity is nothing peculiar to quantity. What is mathematically important about quantity is that quantities exhibit order. Theorems about quantity are in general simply special cases of theorems about order.

However, once the logicist ideas central to the published text of Principles are in place, it becomes clear that quantity is simply not part of pure mathematics, by Russell’s standards. Russell declares this in the passage at 158: 38–45. This passage does not occur in the manuscript, but was added by Russell, probably at the proofreading stage. There he says that the part on Quantity is only a concession to tradition. Quantity has traditionally been supposed to be a part of mathematics. Without this supposition, the Part could have been omitted. But in fact quantity is not a concept of pure mathematics since it cannot be defined in purely logical terms.23

4. THE AXIOM/PRINCIPLE OF ABSTRACTION

As noted earlier, Rodríguez-Consuegra has emphasized the importance of the principle of abstraction in the evolution of Russell’s thought during this period. It certainly plays a crucial role during the fall months of 1900. In Part III, Chapter 18, the axiom, as it is called by Russell at that time, is used to justify the assumption of magnitudes over and above the quantities that have them. In Part IV, Chapter 26, it is used as a way to clarify the notion of a reflexive relation. In an omitted portion of the manuscript of Part V, Russell used the axiom to justify a definition by abstraction of the concept of cardinal number.24 This idea is developed formally in §3 of the fall draft of “The Logic of Relations”. There is omitted text from Part IV, Chapter 30, that suggests that Russell also considered a definition by abstraction of ordinal numbers. (See the List of Variants for page 249.)

I want to briefly consider the use of the principle of abstraction in Part III. There Russell uses it to argue that since equality between quantities is a reflexive, transitive, symmetric relation, there must be a property that a class of equal quantities have in common. This is the magnitude of the quantities in the class. Magnitudes of a given kind are greater or less than each other, but are never equal.

Even in the fall of 1900, Russell objected to the “definitions” generated by the principle of abstraction.25 In an omitted section of the manuscript of Part V, Russell offers the following objection:

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23 Russell tries to justify the inclusion of Part III by saying that it aims to disprove the supposition that quantity occurs in mathematics. But it doesn’t seem to me that this is even a peripheral aim of most of the discussions of Part III.


25 Russell’s published objection to definitions by abstraction is that they do give a unique property shared by the members of the equivalence classes. See Principles, p. 114, and earlier Papers 3: 320. This objection does not occur in the manuscripts written in the fall of 1900.
And this point illustrates the weakness of definitions by abstraction. For the above method will only define such numbers as are the numbers of some class: if there be others they remain indefinable.\footnote{Byrd, "Part v", p. 78, the variant for 306: 9.}

The principle of abstraction only gives numbers where there are classes of similar classes that have that number. If there could be numbers that were not the number of some class, the principle of abstraction would give no grounds for affirming their existence.

This objection applies with considerable force to the use of the principle of abstraction in connection with quantities. The principle allows the inference of a magnitude shared by equal quantities. Quantities are, according to Russell, spatio-temporal instantiations of magnitudes (\textit{PoM}, p. 167). But there is absolutely no reason to suppose that all magnitudes of a kind will have spatio-temporally instantiated. So in general there will be magnitudes of, for example, temperature, that are never spatio-temporally instantiated. So the class of quantities having that magnitude is empty. The assumption of these magnitudes is not justified by the principle of abstraction.

The problem is compounded by Russell's views about zero magnitudes. He holds that there are many kinds of magnitudes which have a zero magnitude (e.g. distance, pleasure). The zero magnitudes of distinct kinds are distinct magnitudes; zero pleasure is not zero distance. Finally, there is no quantity of any kind that has the zero magnitude of that kind. There is no quantity which has the magnitude of zero distance (\textit{PoM}, p. 187).

Since no quantity has a zero magnitude, the class of quantities that have a given zero magnitude is always the same, the null class. It is the same class for zero magnitudes of all kinds. So, the principle of abstraction cannot be used to justify or differentiate these magnitudes. So we cannot define magnitudes by abstraction for precisely the reason that Russell gives in his discussion of cardinal numbers. The method will only define such magnitudes as magnitudes of some quantity: if there be others, they remain indefinable. All zero magnitudes, by this standard, remain indefinable.

By the spring of 1901, Russell had a proof of the principle of abstraction. This is Theorem 6.2 of the final version of "The Logic of Relations". In essence, the proof is effected by the familiar device of letting the property shared be "membership in the equivalence class". Thus the property shared by similar classes is membership in the class of all and only those classes similar to one of them. This is the device Russell uses to give his famous logicist definition of cardinal numbers.

One might contemplate applying this idea in the case of quantities. In a footnote not in the manuscript, Russell proposes this application. This occurs on page 167: "Thus a magnitude may, so far as formal arguments are concerned, be identified with a class of equal quantities." This would have the effect of identifying all zero magnitudes of all kinds, since the class of quantities having zero magnitude is the empty class. It would also identify all magnitudes which were not spatio-temporally instantiated. The proof of the principle does not remove the objections that Russell himself lodged against it.

Russell's qualifier, "so far as formal arguments are concerned", also provides no assistance. A formal argument about the magnitudes of a kind would purportedly be one dealing with the formal properties and relations of the kind. So, for example, the claim that a kind of magnitude is densely ordered would be a claim about a formal property of magnitudes of the kind. However, it could easily turn out that a collection of magnitudes, defined using classes of equal quantities, was not densely ordered even though the original class of magnitudes would naturally be held to be densely ordered. It might be that certain intensities of a colour are never spatio-temporally realized, rendering the order of equivalence classes non-dense. Thus the defined magnitudes may have formal properties which are very different from the properties of the kind of magnitude that they are supposed to represent.
This axiom may seem somewhat elaborate, and considerably lacking in self-evidence.

166: 16–17 It is, however, capable of proof, and is merely] It is, however, merely

166: 17 assumption,] principle.

166: 18 generally] commonly

166: 26 usually] commonly

166: 27–8 and predicate, as the only form of which propositions are capable, and the whole denial] and predicate, and the whole denial

166: 38–9 of the relation, namely the referent, the other] of the relation, the other

166: 43–4 it will be found ... and 6.2. ] it will be found in the Appendix.

167: 26, 27 <twice> concept[ term

167: 40–4 <fn. added>

168: 6, 10 both have both

168: 9, 10, 11 <The ms. uses “<” and “>” where the print has “is less than” and “is greater than.”>

168: 9 A, and vice versa.] A.

168: 18 Chapter xix. Chapter I.

168: 43 relations of greater and less are relations are

168: 45 <fn. added>

169: 17–18 arise. This ... Chapter xxii.] arise.

CHAPTER XX. THE RANGE OF QUANTITY

170: 1 questions] question

170: 1 are these] is this:

170: 3 quantities] magnitudes

170: 12 and to inquire] and inquire

170: 24 cherry] berry

170: 25 berry] cherry

171: 38 the other. Thus] the other, or that the time when it happened bore more resemblance to the present

than the more remote time did. Thus

171: 40–5 <fn. added>

172: 3 application. The importance] application. For example, the difference of two points in space, or of two instants in time, is, I should say, a qualitative difference, which we call distance; and this serial arrangement of points and instants is due to greater and lesser degrees of this difference. The enormous importance

172: 13 a more or less continuous] a continuous

172: 21 a specific relation] a specific transitive relation

172: 26–31 For example, ... £100? This question] For example, there is obviously more difference between the average temperature of England and that of India than between that of England and that of Italy. But need there then be equality or inequality between the difference for Spitzbergen and Italy and that for England and Italy? This question

172: 3 relation] relations

172: 3–4 class of terms, usually regarded as magnitudes, apparently] class of magnitudes, apparently

173: 9–11 Part v; and we ... series.] Part v.

173: 16–17 But may ... divisibility? If so,] But the quantity which has magnitude may be a sum of parts, and the magnitude may be a magnitude of divisibility. That is to say,

173: 18 will be] is

173: 20 On this supposition.] Thus

173: 37–8 is that] is merely that

174: 1 finite] infinite

174: 3 “appear” is italicized in print, but
not underlined in ms.>

174: 4-5 property of a whole ... finite.]

174: 9-12 distances. At a later stage, ... relations.]

174: 16 Chapter xix. Chapter i

174: 17 Very many] Most

174: 45 <fn. added>

CHAPTER XXI. NUMBERS AS EXPRESSING MAGNITUDE: MEASUREMENT

177: 5 quantities] qualities

177: 9-10 is one suggested by Kant's] is that of Kant in

177: 12 hence] and thus

177: 23 maintain] deny

177: 28-37 built. There is ... real numbers.] built.

177: 42-3 allude. See ... p. 161.] allude.

177: 44 <fn. added.>

178: 23-1 provided the addition ... their terms.] provided the addition is not the kind explained in Part i, but of that other kind which we discussed in Part ii.

178: 32-3 In actual space ... wholes.] It is necessary to have immediate judgments of equality as regards two infinite wholes.

178: 41 accomplished; we are always accomplished. To return to space and superposition: we are always

179: 2-6 But where ... space.] Where immediate comparison is psychologically impossible, measurement remains impracticable. For example, it follows from the above theory that a year is either more or less divisible, or exactly as divisible as, a foot; but owing to the total impossibility of immediate comparison, we cannot make any attempt to decide the alternative.

179: 7-15 That divisibility ... Metrical Geometry.] Thus in order to obtain a measure of comparative divisibility, where all our quantities are infinitely divisible, we require two steps. First we require the judgment of equality, which is required many times in most measurements. We thus obtain as many equal quantities as we choose, and their common magnitude is then taken as unit. We now require that the axioms that the whole is greater than the part, and that sums of equals are equal. By sufficient subdivision of the unit, any two wholes can be numerically compared with any required degree of accuracy; and theoretically, by the method of limits, real numbers can always be found to effect the comparison exactly. Thus although our units are not indivisible, their number gives the relative divisibility of any aggregate of units. In this case, as in all cases except that of finite wholes, the measuring number expresses a relation to an arbitrary unit, not an intrinsic property of the magnitude measured.

179: 18-19 difference (in the sense of dissimilarity) between] difference between

179: 26 two inches is] two is

179: 26 1002 inches.] 1002.

179: 37-9 judgment, and ... it. Thus] judgment. Thus

179: 39 divisibility] divisions

180: 32 being the relative product of] being called the product of

181: 11-12 what may be called the axiom] what DuBois Reymond has called the axiom

181: 13 measurement] measure

181: 30-1 an axiom, which may or may not hold in a given case, that] an axiom that

181: 39-44 <fn. added.>

182: 4-5 On the straight line, if, as is usually assumed, there is such a relation as distance, we have] On the straight line, we have

182: 8 Angles may also be regarded] Angles are also

182: 40-3 <fn. added>

CHAPTER XXII. ZERO

183: 37-8 any other, but not zero, unless any other, unless

183: 44 xix] 1

186: 44</n. added>

187: 16 emendation] correction

188: 5-6 disposed of, and such as] disposed of, are

188: 9-10 the ambiguity in the meaning] the six-fold ambiguity in the meaning

189: 3 smallest member;] smallest member, which is called the limit of a;

189: 4 be <int. occurrences] exist

189: 6-7 there is no greatest magnitude;] there is a magnitude between any two,

189: 10-11 it is not condensed in itself, but does have a term between any two, another] be not condensed in itself, another

189: 19-20 magnitudes of a kind having no maximum] such magnitudes

189: 27 deduces] educes

189: 31 The problem] Now it will be observed, in the first place, that the problem

189: 44 Part v, Chapter xxxvi.] Part v.

190: 5, 6 <n.> is greater than ] >

190: 45 magnitude in the cases we are discussing.] magnitude.

191: 23 finite] definite

191: 23-4 a first term, and ... the first.

192: 5, 11, 15 0]

192: 6 Angles may also be regarded] built.

192: 8 Angles are also

193: 41-4 *The objection ... present discussion.] *The objection to this definition (as we shall see in Part v) is that it does not give a fixed property of a collection, but depends sometimes upon the order in which the terms are taken. The rational numbers, for instance, though in order of magnitude continuous, are discontinuous in what may be called the logical order. Another objection is, that a series which is continuous in the above sense may become discontinuous by the addition of new intermediate terms (see Appendix). These objections are removed by Cantor's definition, which will be considered in Part v.

194: 27 xix] 1

195: 12 xx] 11

195: 22 xxi] 111

195: 22 possible as regards existing, actual or possible, though possible though

195: 32 In Chapter xxii ... zero. The
PART IV. ORDER

CHAPTER XXIV. THE GENESIS OF SERIES

199: 3 The discussion of continuity
The discussion of continuity and infinity
199: 4 this these
199: 10 of finite numbers; of numbers
199: 14–15 quadrilateral construction and Pieri's work on Projective Geometry have shown quadrilateral construction has
199: 17 descriptive Geometry the theory of positional manifolds
200: 17 c and b, or between c and d] c and b, or between c and b, and a, and a and d, or between c and d
200: 21 No further special assumption
No special assumption
200: 27 in which series in series (Misprint in m.)
200: 31, 32 two terms] pairs of terms
200: 38 the principal ways] all the ways
200: 43–4 <fn. added>
201: 7 relation to a] relation a. (Misprint in m.)
201: 31–2 d between ... c] d between e and f, then c or e will be said to be also between b and f
202: 19 some finite number] some number
202: 31 steps from a without passing through e] steps from a,

210: 8 is equivalent to] means that
210: 18–19 rendered at least verbally circular] rendered circular.
210: 24 as “belonging to the domain of R,” as belonging to the extension of R.
210: 25 domain] extension
210: 26–30 The print uses “E” where the ms. has “K.”
211: 1 would, if possible, be] are, if possible, to be
211: 37–8 a logically puzzling entity, which in Part I ($55$) we found it necessary to deny; a logically puzzling entity, and
211: 45 <fn. added>
212: 15, 17, 19 terms] concepts
212: 20 being, adjectives generally, and relations, being, numbers, and relations.
212: 21 relations. Such terms we agreed to call concepts. Its relations. It
212: 33–4 any concept which] any concept, other than a number, which
212: 37 when the concept of a number] when a number
212: 43 <fn. added>
212: 46 Chap. LIV] Chap. I
213: 21 as in fact it is,] as it is
213: 25 might] may
213: 37 that sets] that some sets
213: 37 many] several
213: 45 denumerable. The logical ... 4, ...] denumerable.
214: 5 two terms] pairs
214: 7 Chapter XXIV] Chapter I
214: 24 two terms] pairs of terms
214: 42 abed and acde together imply abde] abed and acde imply abde
These axioms are equivalent in the presence of the second axiom. The circular diagram in the ms. places point c appropriately on the clockwise path from a to b to represent the ms. version of axioms v, v, and v, the accompanying figure, the figure.
215: 10, 23, 27 twelve] 12
215: 23 acde] acbe
215: 23 abede] aedbc
215: 41–216: 1 having possibly no] having no
216: 40 Chapter XXIV] Chapter I

CHAPTER XXVI. ASYMMETRICAL RELATIONS

218: 4 Critical Philosophy] critical philosophy
218: 8 Part VI, Chapter L] Part VI
218: 13 xby always excludes] xby implies
218: 13 <"symmetrical" is italicized in print, but not underlined in m.>
218: 14 together always imply] together imply
218: 15 <"transitive" is italicized in print, but not underlined in m.>
218: 17 xby always excludes] xby excludes
218: 19–20 xbe I shall call intransitive.] xbe, or the property that xby excludes there being any y such that ybe I shall call intransitive.
218: 26–7 and not transitive, if third ... intransitive, and intransitive.
219: 1 father] husband
219: 12 <important> is italicized in print, but not underlined in m.>
219: 3 Chap. XIX] Chap. I
219: 5 Chap. XIX] Chap. I
219: 7 Chap. XXIV] Chapter I
219: 11 Chap. LIV] Chap. I
219: 14 <fn. added>
219: 15 Chap. LI] Chap. I
219: 19 asserts] assert
219: 36 propositions, or relations.] propositions.
219: 31, 36 principle] axiom also <Also at 220: 8, 13, 44>
219: 42 this principle,] mine,  
219: 43 precision, and not demonstra­ 
ted, will be found] precision, will be found  
220: 5–6 this process, as set forth by  
Peano, requires] this process requires  
220: 13 property,] property.* <fn.  
 omitted in print.> *For a mathe­ 
matical statement of this axiom, see  
Appendix.  
220: 20 proposition] axiom  
220: 29 domain] extension  
220: 39 Chap. xix, §514, Part I i,  
Chap. t  
221: 10–11 But it follows from the prin­ 
ciple of abstraction that there is some  
relation] But if our axiom be  
allowed, there will be some relation  
221: 42–3 <fn. added>  
221: 44 Phil. Werke, Gerhardt's ed.,]  
Gerhardt's ed.,  
223: 33 extrinsic] not intrinsic  
223: 34 Hence] Thus  
223: 39 blos] blos <misprint in text>  
224: 6 <Throughout this § and through­ 
out §§215, 218, 239, and 221, in rep­ 
resenting the terms of a relation, the ms.  
has a capital “A” and a capital “B”,  
where the print has, respectively a  
small “a” and a small “b”>>  
224: 30 “We, in brief, are led] “We are  
led  
225: 20 if we are to explain] in order to  
explain  
225: 36 mean, if the monistic theory be  
correct, to assert] mean to assert  
226: 37 Chap. t i. ] Chap.  

CHAPTER XXVIII. DIFFERENCE  
OF SIGN AND DIFFERENCE OF SENSE  
27: 11 Raume (1768),] Raume",  
27: 21 space] it  

which the metrical straight line is a  
series.  
231: 31 non-coplanar] non-intersecting  
<Also at 232: 7.>  
231: 35 north] N  
231: 35 south] S  
231: 3 east] E  

232: 12–249: 37 <Folios 73–107 of the ms.  
of Part IV are missing. They include  
all of Chap. xxviii. On the differ­ 
ence between open and closed series, pp. 234–8, and  
chap. xxix. Progressions and ordi­ 
nal numbers, pp. 239–44.>  

CHAPTER XXX. DEDekIND'S THEORY  
of NUMBER  
249: 38 progressions] they  
249: 42–3 similarity between classes,  
which] similarity, which  
249: 43 principle] axiom  
249: 45 cardinals; for] cardinals. For  
249: 45–250: 29 suffice; we require ...  
set. For example, since] suffice; it is neces­ 
sary to have, not merely two classes,  
but two discrete series (which must  
be or contain progressions, if any  
ordinal is to be defined), and these  
must not merely be similar, but must  
be correlated so that their first terms  
are correlates, and if, in one series, a  
precedes b, then in the other, the  
correlate of a must precede the corre­ 
late of b. This is merely a new com­ 
plication added to the definition of  
cardinals in order to obtain that of  
ordinals. With this added complica­ 
tion, we may define the ordinals as  
the common property of terms in  
progressions which are correlated by  
a one-one relation having order  
unchanged. But to this definition by  
abstraction there are, as we saw in  
the preceding chapter, certain logical  
objec­tions. The definition which  
avoids the objections presupposes  
cardinals in a still more obvious way.  
It is as follows. Having first estab­ 
lished independently (which, as we  
know, is possible) the logical theory  
of cardinals, we find that in a pro­ 
gression, every term defines a certain  
class of terms, namely those which  
do not come after it, and that this  
class of terms has a cardinal number  
which uniquely determines and is  
uniquely determined by that term of  
the progression by which the class in  
question is defined. We have hence a  
one-one relation, which is not sym­ 
metrical, between the terms of a  
progression and the cardinal num­ 
bers—a relation, it should be  
ob­served, which no term can have to  
anything except a cardinal number,  
but which every cardinal number has  
to itself, and which is one-one only  
in regard to a given progression. The  
terms having this peculiar relation to  
the number n are called nth terms,  
and the ordinal number nth is, like  
Christian, Mahometan, etc., what  
may be called an adjective of relation  
—i.e. it expresses the fact that the  
terms to which it applies have a  
certain relation to a. A certain fur­ 
ther refinement is necessary, how­ 
ever, to make this definition quite  
correct.  
230: 29 For example, since] Since  
231: 1–2 series, and upon the relation by  
which they are ordered, so that]  
series, so that  
231: 5 Hence] Thus  
231: 5 four-cornered] three-cornered
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251: 5-6 (the first), a generating serial relation and the cardinal number \( n \)
(the first), and the cardinal number \( n \).

*<fn omitted in print:>* "The above account does not apply to transfinite ordinals, but we shall see in Part V how it may be adapted so as to apply.

251: 7-8 ordinals, ..., are more complex; ordinals are far more complex

251: 18 an absolute error; an error

251: 20 largely; quite

CHAPTER XXXI. DISTANCE

252: 21-3 stretches which fulfill the axiom of Archimedes and the axiom of linearity always are; stretches always are

252: 23 the idea, as; the idea of distance, as

252: 25 in most compact series; in compact series

252: 29 rationals or the real numbers are; rationals are

252: 31 Phil. Werke, Gerhardt's ed.

253: 20 <twice> respect; regard <The 2nd occurrence of "respect" in print is a misprint.>

253: 36 some distance; some one distance

254: 12 the \( n \)th power of \( n \) times

254: 13 second distance; second;

254: 14 has an \( n \)th root,; can be separated into \( n \) equal parts,

254: 15 whence; when

254: 24 as rationals or real numbers from; as rationals from

254: 40-5 The powers ... p. 46.] See Appendix.

255: 9-10 the opposite theory; the theory

255: 17 stretch; distance

255: 18 distance; stretch

255: 18-19 since ... distances,; since it is doubtful whether there are distances outside the theory of progressions,

255: 19 in almost all; in all

255: 21 for which, as a rule, no; for which no

255: 22 therefore generally better; therefore better

255: 26-7 elsewhere, except ... space; and if; elsewhere, and if

255: 29-32 without presupposing distance; the distances ... space, and in Part V; without distance; and in Part V

255: 32 how few; what

256: 8 how finite ordinals; how ordinals

256: 9 to be to a certain extent independent of; to be independent of