PARTS III–IV OF
THE PRINCIPLES OF MATHEMATICS

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Here I continue earlier work devoted to the detailed study of Russell's manuscript copy of *The Principles of Mathematics*, in particular its relation to the text as published in May 1903. Previous studies have presented collations of Parts I, II, and V of *Principles*, together with assessments of the significance of the variations between manuscript and published text.[1] In the present study, I examine the manuscripts of Parts III (Quantity) and IV (Order).

As I and others have argued, the manuscripts render likely the conclusion that Russell began his work on *Principles* in the fall of 1900 by writing Parts III through VI of the book.[2] Parts I and II were written later. A draft of Part I, now published in the *Collected Papers*, Vol. 3, was written in May 1901, and an expanded and much altered version was written in May 1902, the month before Russell sent the book off to Cambridge University Press.[3] Part II was written originally in May or June of 1901, and not, in my opinion, extensively revised thereafter prior to its submission.

The collations of Parts I, II, and V reveal substantial alterations

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[1] The studies are Kenneth Blackwell, "Part I of *The Principles of Mathematics*", *Russell* n.s. 4 (1984): 271–88; Michael Byrd, "Part II of *The Principles of Mathematics*", *Russell*, n.s. 7 (1987): 60–70; Michael Byrd, "Part V of *The Principles of Mathematics*", *Russell* n.s. 14 (1994): 47–86.

[2] Alejandro Garciadiego, *Bertrand Russell and the Origins of the Set-theoretic 'Paradoxes'* (Basel: Birkhäuser, 1992), pp. 88–92; Byrd, "Part V", pp. 48–56.

[3] See *Papers* 3: 181–208.

made by Russell subsequent to their initial composition. In the case of
Parts I and II, these represent primarily Russell's ongoing reflections
on the paradoxes about classes and relations and also his growing
acquaintance, during the summer and fall of 1902, with Frege's logical
and philosophical ideas. In the case of Part v, on the other hand,
many alterations reflect the fact that the manuscript was not substan-
tially altered after its initial composition in November 1900 and was
probably replaced by a typescript, now lost, on which some of the
alterations may have been made. (The manuscripts of Parts I, II and
VII have, instead, several characteristics that identify their service in
the University Press's printshop.[4]) Given the rapid development of
Russell's thought during the years 1901–02, the fall 1900 manuscript is
terminologically and doctrinally at variance with the newer Parts writ-
ten in 1901 and 1902.

The existing manuscripts of Parts III and IV, are, in this respect,
more similar to Part v than Parts I and II, as is that of Part VI. The
manuscripts for these parts were written primarily in the fall of 1900,
and, I think, not substantially revised thereafter. Unlike Part v, the
revisions in Parts III and IV are considerably less extensive and sub-
stantial. The alterations in Part III, for example, amount to about 650
words; in the manuscript of Part v, the alterations number about
3,500 words in a text that is two and one-half times as long. This
difference is not surprising; Russell's views on quantity and order were
more stable through this period than his views on number and class,
because they were less affected by the concerns about foundational
issues in logic that arise in Parts I and v.

The manuscripts of Parts III and IV confirm some conclusions that
I reached based on the manuscript of Part v. In particular, the explicit
logicist definitions of cardinal and ordinal number in terms of equival-
ence classes are not part of Russell's early post-Peano work. No such
definitions appear in the fall 1900 manuscripts of Parts III and IV.
The relatively few mentions of them in the published texts are later
insertions. (See, for example, the newly added line 4 on page 158 and
the added footnote on page 167.) Instead, as Rodríguez-Consuegra has

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[4] See I. Grattan-Guinness, "How Did Russell Write *The Principles of Mathemat-
ics?*", this issue, pp. 101–27.

argued, Russell's first reflections in the fall of 1900 give a central place
to what Russell calls the *axiom* of abstraction: "whenever a relation, of
which there are instances, has the two properties of being symmetrical
and transitive, then the relation in question is not primitive, but is
analyzable into sameness of relation to some other term ..." (*PoM*, p.
166).[5] This principle is used by Russell in Part III to adjudicate
between what he calls the absolute and relative theories of quantities.
It is used in Part IV to clarify certain issues about relations and their
properties (§§210 and 211) and was apparently used to offer a defini-
tion by abstraction of ordinal numbers in the now missing manuscript
version of Chapter 29. I will examine below the use to which the
principle is put in Part III.

I. THE MANUSCRIPT TEXT

The initial leaves of the manuscript of both Parts III and IV are dated
November 1900.[6] The upper left-hand corner on the leaves from Part
III bear the notation "Q" for Quantity. Those from Part IV bear the
notation "O" for Order. There are no section numbers, and chapters
are ordered internally, the first chapter of each part being labelled
"Chapter 1".

The list of variants is given at the end of the essay. It is constructed
on the model of previous collations in this series. The list is read as
follows. At the left is a number such as 157: 27. This means page 157,
line 27 from the top. To the right is the reading from the first impres-
sion of the published text of *The Principles of Mathematics*. This is
followed by a square brace and then the corresponding reading from
the fall 1900 manuscript. Editorial brackets enclose my comments.

The leaves in Part III are numbered consecutively from 1 to 80;
there are five leaves with non-standard numbers: 33a, 45a, 64a, 64b,
67a.[7] A number of leaves have double numbers.[8] They are taken

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[5] F. A. Rodríguez-Consuegra, *The Mathematical Philosophy of Bertrand Russell:
Origins and Development* (Basel: Birkhäuser, 1991), pp. 189–205.

[6] Since the manuscript of Part v is also dated November 1900, this suggests that
Russell produced about 400 manuscript pages of work during that month.

[7] 33a is an inserted paragraph on the comparability of quantities of a given kind.
This insertion is self-contained, and placed in the text of Chapter 20 (The Range of

from the 1899–1900 version of Part III of *Principles*. This includes all of Chapter 11 (The Range of Quantity), which becomes Chapter 20 of *Principles*, and a number of leaves from Chapter IV (Continuity, Zero, and Infinity).

The leaves from Part IV are numbered consecutively from 1 to 72, and then from 108 to 122.[9] Thus there is a substantial gap in the manuscript; the gap begins at 232: 12 of the text after the word "difficulty", and the manuscript resumes at 249: 35, with the word "but". It thus includes the last page and a half of Chapter 27, all of Chapters 28 (On the Difference between Open and Closed Series) and 29 (Progressions and Ordinal Numbers), and all but two pages of Chapter 30 (Dedekind's Theory of Number). The gap is not a natural unit, such as a chapter or a discussion of some particular set of topics. It begins and ends in the middle of sentences. It is an unfortunate loss, since especially Chapters 29 and 30 concern topics on which Russell was making rapid progress in the fall of 1900, as we can see from the contents of section 4 of the October 1900 draft of "The Logic of Relations".[10] That the lost material is not simply what is found in the published text is shown by a lengthy section of the manuscript (folios 108–10), which was omitted from the published text.

A striking fact about the extant parts of the manuscript of Part IV is the absence of the familiar phenomena of inserted or pirated leaves.

Quantity), a chapter taken from the 1899–1900 manuscript. 67a is a footnote added to a leaf taken from the 1899–1900 version of *PoM*; so it was probably written with the main text in the Fall of 1900. 45a is a leaf also taken from the 1899–1900 manuscript. 64a and 64b are most of the first paragraph of Section 180. It is unclear to me whether these leaves were part of the original text or not. They are not simply newly inserted leaves, since folio 64b connects with 65 in the middle of a sentence. On the basis of the facts mentioned in footnote 20 below, they may be leaves inserted as Russell separated the material in Chapter 22 from Chapter 23. They appear in the text at 172: 24–36.

[8] The folios in question are numbers 29–37 (excluding 33a), 45a, 49, 50, 53, 67–9, and 74–5. They are respectively folio numbers 11–19, 27, 30, 31, 22(?), 34–6, and 38–9 from Part III of the 1899–1900 version of *Principles*.

[9] There are two folios numbered 36 and two numbered 37. Russell simply misnumbered them.

[10] *Papers* 3: 597–601. Interestingly, while Russell discusses and criticizes Dedekind's method of founding the real numbers, even in the 1899–1900 version, no mention is made of Dedekind's work on arithmetic in that version, and there is no explicit reference to Dedekind in the fall draft of "The Logic of Relations".

The version of Part IV written by Russell in 1899–1900 is completely intact; it is about half the length (68 leaves) of the manuscript written in the fall of 1900. The Fall 1900 version introduces substantial new chapters; on the distinction between open and closed series and on Dedekind's views. What this suggests is that Russell's thought in this area was considerably clarified and extended by his growing acquaintance with Peano and his school. Certain central philosophical claims, such as the importance and irreducibility of asymmetrical relations, are already firmly in place in the earlier manuscript.[11] But much material is new; this is especially true of the work on closed series and the relation of separation of couples, which draws heavily on the work of Vailati.[12]

In my discussion of Part V, I noted that the manuscript used terminology and expressed views different, in certain respects, from what we find in the spring of 1902.[13] As mentioned earlier, the manuscript contains repeated references to the "axiom" of abstraction, in line with the fact that the principle is taken as an unproved primitive proposition in the October 1900 draft of "The Logic of Relations". This is always changed in the published text to the "principle" of abstraction, since the spring draft of "The Logic of Relations" offers a proof of the principle (Theorem 6.2).[14] In the manuscript, Russell uses "concept" as the overarching general category, so that points, instants, and bits of matter are regarded as concepts. (See the collation at 212: 15–21.) In the published text, Russell uses "term" for the overarching category and "concept" for the smaller of terms that may occur in a proposition "otherwise than as terms", in conformity with the classification set out in Chapter 4 of Part I of the published text.[15]

It is also clear that Russell's understanding of Cantor's conception of continuity was defective when he originally wrote Part III. On page 193 of the text, Russell introduces "continuity" to mean merely that a

[11] See *Papers* 3: 89–94.
[12] *Principles*, pp. 205–6, 214–15.
[13] Byrd, "Part V", pp. 50–6.
[14] *Papers* 3: 320.
[15] In fact, the published text contains a new footnote to that chapter. It also introduces the new nomenclature with the words: "Such terms we agreed to call concepts", which, of course, do not occur in the manuscript.

series is dense. In a footnote to this introduction, he explains correctly why this does not properly capture the continuity of the real numbers. But this footnote supplants a footnote in the manuscript in which Russell says that the objection to the definition is that "it does not give a fixed property of a collection, but depends sometimes upon the order in which the terms are taken."[16] Cantor's definition does not give a fixed property of a collection either, of course. Very soon thereafter, Russell's had dramatically clarified his understanding when he wrote Chapters 35 and 36 of Part v.

The manuscript text also sets out a conception of numbers that does not sit easily with the logicist definitions of the published text. In §200 where he is discussing whether there are unanalyzable three-place relations, Russell gives a brief account of his basic metaphysics of terms, concepts, and propositions, roughly as set out in Chapter 4 of the published text. However, the manuscript version of several sentences in this passage treat numbers as predicates whose role in numerical predications is like that of relations. That is, numbers do not occur as terms in such sentences. Rather, they are adjectives, as Russell calls them, which apply to multiplicities of objects.[17] Here are the relevant passages (ms. fos. 30–1, my emphasis).

There are, on the other hand, concepts which can occur otherwise than as terms: such are being, *numbers*, and relations.

This gives the opinion that relations are always between only two terms, for a relation may be defined as any concept, *other than a number*, which occurs in a proposition containing more than one term.

In the first place, when a number is asserted of a collection, if the collection has n terms, there are n terms and only one concept (namely n) which is not a term.

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[16] See the List of Variants for 197: 41–4. A similar footnote occurs at the comparable point in the 1899–1900 version of Part III. See *Papers* 3: 72.

[17] These ideas of Russell's survive in *Principles* in his talk of classes as many. For a stimulating discussion of this conception, see Peter Simons, "Numbers and Manifolds" and "Plural Reference and Set Theory", both in Barry Smith, ed., *Parts and Moments: Studies in Logic and Formal Ontology* (Munich: Philosophia, 1982), pp. 160–198, 199–256.

In the published text, the italicized words are omitted, and in the third passage, Russell speaks of the "concept of a number", not a number directly.

2. THE MISSING(?) PARTS I AND II

As with the manuscript of Part v, the manuscripts of Parts III and IV have few back references to Parts I and II. They are quite interesting in content and again strongly indicate that these Parts had not yet been written in the fall of 1900. In the manuscript version of the passage at 229: 26–8, Russell writes: "But the addition of integers, as we saw in Part I, is a complicated matter, and the relation R is prior to it." Strikingly, the back reference here is to Part I, not Part II, as one would expect. Part I of the 1899–1900 version of *Principles* contained an extended discussion of arithmetical addition in Chapter III. So here Russell appears to anticipate that Part I of the new manuscript would begin with a part on Number just as his previous attempt had done.

A similar case is found in the manuscript version of the passage at 178: 23, where Russell writes:

But the addition of two quantities of divisibility, i.e. two wholes, does yield a new single whole, provided the addition is not of the kind explained in Part I, but of that other kind which we discussed in Part II.

As a back reference to the published text of *Principles*, it is hard to know what to make of this. But in the context of the 1899–1900 version, the intended reference is quite clear. There Chapter II of Part I is entitled "Arithmetical Addition", and it concerns what Russell calls the addition of terms. Chapter III of Part II (Whole and Part without the Use of Numbers. The Logical Calculus) discusses what Russell calls the addition of wholes to form a new whole.[18] It is clearly this latter kind of addition to which Russell is referring in the manuscript passage.

There are a couple of other back references in the manuscript. One of these occurs at 160: 17–18:

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[18] *Papers* 3: esp. 45–7.

The kind of equality which consists in having the same number of parts has already been discussed in Part 11. If this indeed be the meaning of quantitative equality, then quantity introduces no new idea. But it may, I think, be shown that greater and less have a wider field than whole and part, and an independent meaning.

The back reference here is to a discussion of equality in connection with whole and part. Part 11 of the 1899–1900 version of *Principles* is entitled "Whole and Part", but it contains no discussion of the kind of equality mentioned by Russell. The relevant chapters in *Principles* are Chapters 16 and 17 of Part 11, but they likewise consider no such notion of equality. There is, of course, the notion of similarity of classes, used to define cardinal numbers. Russell was already using this notion in the October draft of "The Logic of Relations"; so, it seems likely that we have here an anticipation on Russell's part of consideration of this relation in the second part.

The only other back reference to Parts 1 and 11 in the manuscript occurs at 178: 20: "In this case, although the magnitudes are even now incapable of addition of the sort required, the quantities can be added in the manner explained in Part 11." Once again the sort of addition in question here is the combination of two wholes to form a new whole. As noted above, this kind of addition is isolated and discussed in Part 11 of the 1899–1900 version of *Principles*, and this discussion recurs in the published text at several places, for example, at §131. Certainly, the back reference here is to material that Russell could readily have anticipated being in Part 11 on the basis of his own previous writing on the subject.

By way of contrast to the back references that occur in the fall 1900 manuscript, the published text contains a number of significant and specific additions that involve Parts 1 and 11. On page 167, there is a new footnote that shows how to apply Russell's proof of the principle of abstraction to the definition of magnitude: a magnitude is a class of equal quantities. There are two back references in the published text to §55 at 186: 44 and 211: 37. Since Russell allows that relations may be magnitudes, he claims that the quantities which have these relational magnitudes are particular instances of the relation, "a particularized relation". However, Russell interprets the results of §55 as showing

that there are no such entities as particularized relations.[19] At 211: 37, he writes: "The particularized relation is a logically puzzling entity, which in Part 1 (§55) we found it necessary to deny." This is, of course, not in the manuscript. Here Russell finds that his developing logical views undermine his conception of relational quantities.

There is a striking footnote on page 192 of the published text. It refers to the discussion in Part 11 of two different criteria for being a finite class; one using mathematical induction and the other Dedekind's definition in terms of similarity to a proper subclass. The footnote in the manuscript says simply, "See Part v." These examples confirm the view that Russell wrote Parts 111–v1 without Parts 1 and 11 in hand, and that when he came to submit the manuscript in June 1902, he did not adjust the manuscripts to fit with his evolving views. Moreover, it is clear that the logicist conception of mathematics that forms the unifying theme of the published text is absent from the fall 1900 manuscript of Parts 111 and 1v.[20]

3. THE DISAPPEARANCE OF QUANTITY

In the collection of manuscripts on the philosophy of mathematics, which Russell produced in the years 1898–1900, there is always a part

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[19] Russell seems wrong in this respect. The most §55 shows is that not all relations are particularized relations. See N. Griffin and G. Zak, "Russell on Specific and Universal Relations: *The Principles of Mathematics*, §55", *History and Philosophy of Logic*, 3 (1982): 55–67.

[20] In his recent biography of Russell, Ray Monk assumes that the fall manuscript of *Principles* already contained the basic logicist definitions. For example, at page 132, he writes, "By the end of the year he had produced a work of astonishing breadth and equally astonishing confidence, which sought to show that the whole of mathematics could be based on a mere handful of logical notions and axioms." No substantial evidence, based on the extant manuscripts, is offered in support of this very questionable assumption. He does bring forward a letter to Helen Thomas, written on 31 December 1900, the relevant part of which reads, "In October I invented a new subject, which turned out to be all of mathematics, for the first time treated in its essence." The subject invented in October was the logic of relations, of course. But this rather general remark, with its suggestive use of the verb "turned out", does not justify the assumption that the fall manuscript already incorporated and developed the idea that the logic of relations was all of mathematics, especially not in the face of the clear countervailing evidence of the manuscripts themselves. See Ray Monk, *Bertrand Russell: the Spirit of Solitude, 1872–1921* (New York: Free P., 1996).

on Quantity. It invariably is the third or fourth part, usually third, and follows an initial part on number and one on some topic which Russell associates with logic, such as "Whole and Part". The overall structure of this Part can already be seen in the synoptic table of contents that Russell prepared for the manuscript of *The Fundamental Ideas and Axioms of Mathematics* in 1899. It has four chapters that correspond in topic and order to Chapters 19, 20, 21, and 23 of *Principles* (*Papers* 2: 268–9). The same is true of the 1899–1900 version of *Principles*; it has four chapters on the same topics in the same order.[21]

There is also a considerable degree of doctrinal and argumentative stability in the first three of these four chapters. Russell argues that magnitudes are primitive, and that they are needed in addition to the quantities which have the magnitudes. Both relations and qualities can have magnitudes. Distances and divisibilities are in some sense naturally measurable by numbers. Other magnitudes are at best only indirectly measurable by numbers.

The situation is very different with respect to the fourth of these chapters, corresponding to Chapter 23 of the published text. Chapter IV of Part III of the 1899–1900 version is entitled "Continuity, Zero, and Infinity". In it, Russell sets out the opposing triads of propositions listed as (1), (2), and (3) and (*a*), (*b*), and (*c*) on page 190 of the published text.

(1)   That no two magnitudes of the kind are consecutive.
(2)   That there is no least magnitude.

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[21] Chapter 22 (Zero) has no predecessor in the manuscripts. In these, "zero" is linked with continuity and infinity in the fourth chapter of the part. Zero is thought of as the supposed least quantity of a kind, parallel to infinity, the greatest quantity of the kind. In Chapter 22, this is rejected as a true definition of a zero quantity on the grounds that it fails to bring out the intrinsic connection of a zero quantity to some kind of negation.

The first leaf of the manuscript copy of Chapter 23 was originally entitled "Zero, Infinity, and Continuity", but "Zero" has been overwritten and "the Infinitesimal" inserted. Moreover, the number of this leaf appears to have originally been "55", the number of the first leaf of Chapter 22. Furthermore, the manuscript summary of Part III (fos. 78) appears to treat Chapters 22 and 23 as a single chapter. (See the List of Variants, 195: 32.) This suggests that the new material now appearing as Chapter 22 was originally subsumed under Chapter 23. It is clear that a primary impetus for these new reflections is Meinong's *Uber die Bedeutung des Weber'schen Gesetzes*.

(3)   That there is no greatest magnitude.
...
(*a*)   There are consecutive magnitudes....
(*b*)   There is a magnitude smaller than any other of the same kind.
(*c*)   There is a magnitude greater than any other of the same kind.

In the 1899–1900 version, Russell says that (*a*), (*b*), and (*c*) can be strictly proved on the basis of what he calls the philosopher's axiom of finitude, an axiom which he clearly accepts. He states the axiom in the following passage:

The whole argument turns upon the principle by which infinite number is shown to be self-contradictory, namely: *Many terms must be some definite number of terms.*[22]

For the fall 1900 manuscript, Russell retains many of the leaves from the 1899–1900 chapter, though they are heavily overwritten. He contends that the arguments in support of (*a*), (*b*), and (*c*) ultimately rest on the assumption that the principle of mathematical induction governs *all* numbers:

This is the principle which the philosopher must be held to lay down as obviously applicable to all numbers, though he will have to admit that the more precisely his principle is stated, the less obvious it becomes.
(*PoM*, p. 192)

Russell then likens the principle of mathematical induction to the axiom of parallels; while useful in its proper place, to suppose it always true, "is to yield to the tyranny of mere prejudice" (*PoM*, pp. 192–3).

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[22] *Papers* 3: 72 reads "A given collection of many terms has some finite number of terms", but it is most important here to look at the Textual Notes to this passage on 768; 72: 6; 72: 7, and the manuscript itself. The relevant manuscript leaf was reused in the fall 1900 manuscript, and the particular principle at issue here has been overwritten by Russell. Its original form appears to me to have been: "Many terms must be some definite number of terms." Russell blots out the first two letters of the word "definite" to obtain the word "finite", which is what appears in the published text. This reading of the older manuscript certainly would have made appeal to the principle less blatantly circular. In the fall of 1900, of course, Russell was interested precisely in making the circularity obvious.

Here Russell is attacking his own earlier argument, presenting it by using the very leaves on which it was written.

A crucial fact about Part III is that its importance diminishes as Russell's thought evolves. In his 1897 paper "The Relations of Number and Quantity", it is clear that Russell regards quantity as a central concept of mathematics and that he considers the question of how number and quantity are related as among the central questions of mathematical philosophy (*Papers* 2: 70). This is reaffirmed in the opening sentence of Part III of the 1899–1900 manuscript. But Russell's discussion there suggests something quite different. Russell says that the work of Weierstrass and others has shown that all of pure mathematics should be regarded as dealing exclusively with numbers. In addition, he claims that algebra has been extended so as to cover non-numerical areas. This has led to "a greater separation of number and quantity" than has traditionally been maintained. Thus, Russell will attempt to give a theory of quantity which is independent of number (*Papers* 3: 54). Russell also emphasizes that the antinomies concerning continuity and infinity have nothing specially to do with quantity. They are problems of a "strictly arithmetical nature" (*Papers* 3: 73–4).

The fall 1900 manuscript version further subordinates the importance of quantity. He argues that the work of Weierstrass, Dedekind, and Cantor and the development of non-numerical, non-quantitative mathematics, such as projective geometry and the logical calculus, show that quantity and number are "completely independent" (*PoM*, p. 158). Moreover, what is mathematically important about quantity is nothing peculiar to quantity. What is mathematically important about quantity is that quantities exhibit order. Theorems about quantity are in general simply special cases of theorems about order.

However, once the logicist ideas central to the published text of *Principles* are in place, it becomes clear that quantity is simply not part of pure mathematics, by Russell's standards. Russell declares this in the passage at 158: 38–45. This passage does not occur in the manuscript, but was added by Russell, probably at the proofreading stage. There he says that the part on Quantity is *only* a concession to tradition. Quantity has traditionally been supposed to be a part of mathematics. Without this supposition, the Part could have been omitted. But in

fact quantity is not a concept of pure mathematics since it cannot be defined in purely logical terms.[23]

4. THE AXIOM/PRINCIPLE OF ABSTRACTION

As noted earlier, Rodríguez-Consuegra has emphasized the importance of the principle of abstraction in the evolution of Russell's thought during this period. It certainly plays a crucial role during the fall months of 1900. In Part III, Chapter 18, the axiom, as it is called by Russell at that time, is used to justify the assumption of magnitudes over and above the quantities that have them. In Part IV, Chapter 26, it is used as a way to clarify the notion of a reflexive relation. In an omitted portion of the manuscript of Part V, Russell used the axiom to justify a definition by abstraction of the concept of cardinal number.[24] This idea is developed formally in §3 of the fall draft of "The Logic of Relations". There is omitted text from Part IV, Chapter 30, that suggests that Russell also considered a definition by abstraction of ordinal numbers. (See the List of Variants for page 249.)

I want to briefly consider the use of the principle of abstraction in Part III. There Russell uses it to argue that since equality between quantities is a reflexive, transitive, symmetric relation, there must be a property that a class of equal quantities have in common. This is the magnitude of the quantities in the class. Magnitudes of a given kind are greater or less than each other, but are never equal.

Even in the fall of 1900, Russell objected to the "definitions" generated by the principle of abstraction.[25] In an omitted section of the manuscript of Part V, Russell offers the following objection:

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[23] Russell tries to justify the inclusion of Part III by saying that it aims to disprove the supposition that quantity occurs in mathematics. But it doesn't seem to me that this is even a peripheral aim of most of the discussions of Part III.

[24] See Byrd, "Part V", p. 78, in the List of Variants for p. 305.

[25] Russell's published objection to definitions by abstraction is that they do give a unique property shared by the members of the equivalence classes. See *Principles*, p. 114, and earlier *Papers* 3: 320. This objection does not occur in the manuscripts written in the fall of 1900.

And this point illustrates the weakness of definitions by abstraction. For the above method will only define such numbers as are the numbers of some class: if there be others they remain indefinable.[26]

The principle of abstraction only gives numbers where there are classes of similar classes that have that number. If there could be numbers that were not the number of some class, the principle of abstraction would give no grounds for affirming their existence.

This objection applies with considerable force to the use of the principle of abstraction in connection with quantities. The principle allows the inference of a magnitude shared by equal quantities. Quantities are, according to Russell, spatio-temporal instantiations of magnitudes (*PoM*, p. 167). But there is absolutely no reason to suppose that all magnitudes of a kind will have spatio-temporal instantiations. So in general there will be magnitudes of, for example, temperature, that are never spatio-temporally instantiated. So the class of quantities having that magnitude is empty. The assumption of these magnitudes is not justified by the principle of abstraction.

The problem is compounded by Russell's views about zero magnitudes. He holds that there are many kinds of magnitudes which have a zero magnitude (e.g. distance, pleasure). The zero magnitudes of distinct kinds are distinct magnitudes; zero pleasure is not zero distance. Finally, there is no quantity of any kind that has the zero magnitude of that kind. There is no quantity which has the magnitude of zero distance (*PoM*, p. 187).

Since no quantity has a zero magnitude, the class of quantities that have a given zero magnitude is always the same, the null class. It is the same class for zero magnitudes of all kinds. So, the principle of abstraction cannot be used to justify or differentiate these magnitudes. So we cannot define magnitudes by abstraction for precisely the reason that Russell gives in his discussion of cardinal numbers. The method will only define such magnitudes as magnitudes of some quantity: if there be others, they remain indefinable. All zero magnitudes, by this standard, remain indefinable.

By the spring of 1901, Russell had a proof of the principle of abstraction. This is Theorem 6.2 of the final version of "The Logic of

Relations". In essence, the proof is effected by the familiar device of letting the property shared be "membership in the equivalence class". Thus the property shared by similar classes is membership in the class of all and only those classes similar to one of them. This is the device Russell uses to give his famous logicist definition of cardinal numbers.

One might contemplate applying this idea in the case of quantities. In a footnote not in the manuscript, Russell proposes this application. This occurs on page 167: "Thus a magnitude may, so far as formal arguments are concerned, be identified with a class of equal quantities." This would have the effect of identifying all zero magnitudes of all kinds, since the class of quantities having zero magnitude is the empty class. It would also identify all magnitudes which were not spatio-temporally instantiated. The proof of the principle does not remove the objections that Russell himself lodged against it.

Russell's qualifier, "so far as formal arguments are concerned", also provides no assistance. A formal argument about the magnitudes of a kind would purportedly be one dealing with the formal properties and relations of the kind. So, for example, the claim that a kind of magnitude is densely ordered would be a claim about a formal property of magnitudes of the kind. However, it could easily turn out that a collection of magnitudes, defined using classes of equal quantities, was not densely ordered even though the original class of magnitudes would naturally be held to be densely ordered. It might be that certain intensities of a colour are never spatio-temporally realized, rendering the order of equivalence classes non-dense. Thus the defined magnitudes may have formal properties which are very different from the properties of the kind of magnitude that they are supposed to represent.

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[26] Byrd, "Part v", p. 78, the variant for 306: 9.

VARIANTS BETWEEN *The Principles of Mathematics,*

PARTS III–IV, & THEIR MS.

PART III. QUANTITY

CHAPTER XIX. THE MEANING OF
MAGNITUDE

157: 27 fractions] ratios

157: 30 with integers] with the integers

158: 4 integers, which, as we have seen
can themselves be defined in logical
terms.] integers.

158: 16 which are not] other than

158: 28–9 particular instances] any kind

158: 38–45 possible. The whole ...
quantity.] possible.

159: 30–1 All kinds of quantity have in
common] What all kinds of quantity
is

159: 32–3 *i.e.* a term which has this
relation at all has this relation to
itself;] *i.e.* a term which has this
relation to itself;

160: 12–13 will apply to quantities only
in virtue of] will only apply to quan-
tities owing to

160: 33 results on this hypothesis, sole-
ly] results solely

161: 19–20 the number of complex parts
that may be taken to make up the
whole] the number of complex parts

161: 24 number or type is] number is

161: 30 part] book

161: 36 the number] this number

161: 42 in certain spaces,] in space and
time

161: 43 *<fn. added>*

162: 38–9 *<Italicized text not underlined
in ms.>*

163: 12, 17, 18, 19, 20, 21, 23, 25,
26 *<The ms. uses "<" and ">" where
the print has "is less than" and "is
greater than".>*

163: 13–14 a *B*, which may be identical
with *A*, such that] a *B* such that
with *A*, such that

163: 21 *<Axiom* (h) *does not occur in the
ms.>*

163: 24 (*h*)] (*g*)

163: 25–6 *C*; from (*c*) ... than *C*. (In
the place] *C*. (In the place

163: 35 it may be laid down that] it may
be laid down as self-evident that

163: 36 the only unanalyzable symmetri-
cal and transitive relation] the only
unanalyzable relation

163: 38–40 Now to say that ... or that,
when] Now to say that a relation is
analyzable is to say that when

164: 1–2 Thus to assert ... mother.]
Thus to assert that *A* is *B*'s grandfa-
ther is to assert that there is some
third person *C* who is *A*'s son and
B's father.

165: 20 extension or field of] extension
of

165: 31–4 The quantities ... holds.] The
quantities which are instances of
magnitude are merely complex con-
cepts, of which magnitude is one
element, and spatio-temporal posi-
tion or (in the case of relations
which are quantities) the terms of
the relation form the other element.

165: 36–7 are distinct from] are only
elements in

165: 41–5 *<The ms uses "<" and ">"
where the print has "is less than" and
"is greater than".>*

166: 6, 7, 15 principle] axiom *<Also at
167: 1.>*

166: 10 and that this] and this

166: 13 son to father:] father and son:

166: 15–16 This principle ... elaborate.]

This axiom may seem somewhat
elaborate, and considerably lacking
in self-evidence.

166: 16–17 It is, however, capable of
proof, and is merely] It is, however,
merely

166: 17 assumption.] principle.

166: 18 generally] commonly

166: 26 usually] commonly

166: 27–8 and predicate, as the only
form of which propositions are
capable, and the whole denial] and
predicate, and the whole denial

166: 38–9 of the relation, namely the
referent, the other] of the relation,
the other

166: 43–4 it will be found ... and 6.2.]
it will be found in the Appendix.

167: 26, 27 *<twice>* concept] term

167: 40–4 *<fn. added>*

168: 6 both have] have both

168: 9, 10, 11 *< The ms. uses "<" and ">"
where the print has "is less than" and
"is greater than".>*

168: 9 *A*, and *vice versâ.*] *A.*

168: 18 Chapter XIX.] Chapter I.

168: 43 relations of greater and less are]
relations are

168: 45 *<fn. added>*

169: 17–18 arise. This ... Chapter XXII.]
arise.

CHAPTER XX. THE RANGE OF QUAN-
TITY

170: 1 questions] question

170: 1 are these:] is this:

170: 3 quantities] magnitudes

170: 12 and to inquire] and inquire

170: 24 cherry] berry

170: 25 berry] cherry

171: 38 the other. Thus] the other, or
that the time when it happened bore
more resemblance to the present

than the more remote time did.
Thus

171: 40–5 *<fn. added>*

172: 3 application. The importance]
application. For example, the differ-
ence of two points in space, or of
two instants in time, is, I should say,
a qualitative difference, which we call
distance; and this serial arrangement
of points and instants is due to
greater and lesser degrees of this
difference. The enormous import-
ance

172: 13 a more or less continuous] a
continuous

172: 21 a specific relation] a specific
transitive relation

172: 26–31 For example, ... £100? This
question] For example, there is obvi-
ously more difference between the
average temperature of England and
that of India than between that of
England and that of Italy. But need
there then be equality or inequality
between the difference for Spitz-
bergen and Italy and that for Eng-
land and Italy? This question

173:2 relation] relations

173: 3–4 class of terms, usually regarded
as magnitudes, apparently] class of
magnitudes, apparently

173: 9–11 Part V; and we ... series.] Part
V.

173: 16–17 But may ... divisibility? If
so,] But the quantity which has
magnitude may be a sum of parts,
and the magnitude may be a magni-
tude of divisibility. That is to say,

173: 18 will be] is

173: 20 On this supposition,] Thus

173: 37–8 is that] is merely that

174: 1 finite] infinite

174:3 *<"appear" is italicized in print, but*

not underlined in ms.>

174: 4–5 property of a whole ... finite.]
property of a whole correlated with,
but distinct from, the number of
parts included in the whole.

174: 9–12 distances. At a later stage, ...
relations.] distances.

174: 16 Chapter XIX.] Chapter I

174: 17 Very many] Most

174: 45 <*fn. added*>

CHAPTER XXI. NUMBERS AS EXPRESS-
ING MAGNITUDE: MEASUREMENT

177: 5 quantities] qualities

177: 9–10 is one suggested by Kant's] is
that of Kant in

177: 12 hence] and thus

177: 23 maintain] deny

177: 28–37 built. ¶There is ... real
numbers.] built.

177: 42–3 allude. See ... p. 161.] allude.

177: 44 <*fn. added*>

178: 23–5 provided the addition ... their
terms.] provided the addition is not
the kind explained in Part I, but of
that other kind which we discussed
in Part II.

178: 32–3 In actual space ... wholes.] It
is necessary to have immediate judg-
ments of equality as regards two
infinite wholes.

178: 41 accomplished; we are always]
accomplished. To return to space
and superposition: we are always

179: 2–6 But where ... space.] Where
immediate comparison is psychologi-
cally impossible, measurement
remains impracticable. For example,
it follows from the above theory that
a year is either more or less divisible,
or exactly as divisible as, a foot; but
owing to the total impossibility of
immediate comparison, we cannot

make any attempt to decide the
alternative.

179: 7–15 That divisibility ... Metrical
Geometry.] ¶Thus in order to obtain
a measure of comparative divisibility,
where all our quantities are infinitely
divisible, we require two steps. First
we require the judgment of equality,
which is required many times in
most measurements. We thus obtain
as many equal quantities as we
choose, and their common magni-
tude is then taken as unit. We now
require that the axioms that the
whole is greater than the part, and
that sums of equals are equal. By
sufficient subdivision of the unit, any
two wholes can be numerically com-
pared with any required degree of
accuracy; and theoretically, by the
method of limits, real numbers can
always be found to effect the com-
parison exactly. Thus although our
units are not indivisible, their num-
ber gives the relative divisibility of
any aggregate of units. In this case,
as in all cases except that of finite
wholes, the measuring number
expresses a relation to an arbitrary
unit, not an intrinsic property of the
magnitude measured.

179: 18–19 difference (in the sense of
dissimilarity) between] difference
between

179: 26 two inches is] two is

179: 26 1002 inches.] 1002.

179: 37–9 judgment, and ... it. Thus]
judgment. Thus

179: 39 divisibility] divisibilities

180: 32 being the relative product of]
being called the product of

181: 11–12 what may be called the
axiom] what DuBois Reymond has

---

called the axiom

181: 13 measurement] measure

181: 30–1 an axiom, which may or may
not hold in a given case, that] an
axiom that

181: 39–44 <*fn. added*>

182: 4–5 On the straight line, if, as is
usually assumed, there is such a
relation as distance, we have] On the
straight line, we have

182: 8 Angles may also be regarded]
Angles are also

182: 40–3 <*fn. added*>

CHAPTER XXII. ZERO

185: 37–8 any other, but not zero,
unless] any other, unless

185: 44 XIX] I

186: 44 <*fn. added*>

187: 16 emendation] correction

187: 19–20 any concept that defines]
those concepts which define

187: 20, 21 <*twice*>, 22 kind] class

CHAPTER XXIII. INFINITY, THE
INFINITESIMAL, AND CONTINUITY

188: 5–6 disposed of, and such as
belong to the fundamentals of logic,
are] disposed of, are

188: 9–10 the ambiguity in the mean-
ing] the six-fold ambiguity in the
meaning

189: 3 smallest member;] smallest mem-
ber, which is called the limit of *u*;

189: 4 be <*1st occurrence*>] exist

189: 6–7 there is no greatest magni-
tude,] there is a magnitude between
any two,

189: 10–11 is not condensed in itself,
but does have a term between any
two, another] be not condensed in
itself, another

189: 19–20 magnitudes of a kind having

no maximum] such magnitudes

189: 27 deduces] educes

189: 32 The problem] Now it will be
observed, in the first place, that the
problem

189: 44 Part V, Chapter XXXVI.] Part V.

190: 5, 6 <*twice*> is greater that] >

190: 45 magnitude in the cases we are
discussing.] magnitude.

191: 23 finite] definite

192: 6, 11, 15 0] 1

192: 23–4 a first term, and ... the first.]
a first term.

192: 43–4 This has already ... in Part
V.] See Part V.

193: 39–40 the series ... numerator] the
series of fractions having one for
numerator

193: 41–4 *The objection ... present
discussion.] *The objection to this
definition (as we shall see in Part V)
is that it does not give a fixed prop-
erty of a collection, but depends
sometimes upon the order in which
the terms are taken. The rational
numbers, for instance, though in
order of magnitude continuous, are
discontinuous in what may be called
the logical order. Another objection
is, that a series which is continuous
in the above sense may become dis-
continuous by the addition of new
intermediate terms (see Appendix).
These objections are removed by
Cantor's definition, which will be
considered in Part V.

194: 27 XIX] I

195: 12 XX] II

195: 22 XXI] III

195: 22 possible as regards existents,
actual or possible, though] possible
though

195: 32 In Chapter XXII ... zero. The

problem] Finally, in the present chapter, we discussed the problems arising out of zero, continuity, the infinite, and the infinitesimal. The problem

195: 44 In the present Chapter the problems] The problems

PART IV. ORDER

CHAPTER XXIV. THE GENESIS OF
   SERIES

199: 3 The discussion of continuity]
   The discussion of continuity and
   infinity

199: 4 this] these

199: 4 notion] notions

199: 10 of finite numbers] of numbers

199: 14–15 quadrilateral construction
   and Pieri's work on Projective
   Geometry have shown] quadrilateral
   construction has

199: 17 descriptive Geometry] the the-
   ory of positional manifolds

200: 17 *c* and *b*, or between *c* and *d*] *c*
   and *b*, or between *c* and *b*, *b* and *a*,
   and *a* and *d*, or between *c* and *d*

200: 21 No further special assumption]
   No special assumption

200: 23 to the former in a simple man-
   ner.)] to the former.)

200: 27 in which series] in series <*Mis-
   print in ms.*>

200: 31, 32 two terms] pairs of terms

200: 38 the principal ways] all the ways

200: 43–4 <*fn. added*>

201: 7 relation to *a*.] relation *a*. <*Mis-
   print in ms.*>

201: 31–2 *d* between ... *b* and *c*.] *d*
   between *e* and *f*, then *c* or *e* will be
   said to be also between *b* and *f*.

202: 19 some finite number] some number

202: 31 steps from *a* without passing

through *e*,] steps from *a*,

203: 35 (1895)] (1)

203: 35 No. 7] No. 8

203: 36 Vol. 1. We ... so.] Vol. 1.

203: 39 Some relations] A few relations

203: 18–24 <*In the ms., Russell uses "xπ"
   and "yπ" at the places where the print
   has, respectively, "πx" and "πy".*>

203: 32 (3).] (3), if our collection be
   suitably chosen.

203: 40–3 asymmetrical. Instead ...
   transitiveness.)] asymmetrical.

204: 24–30 <*In the ms., Russell uses "<"
   and ">", where the print has "is less
   than" and "is greater than".*>

204: 28 For "*xz* is less than *xw*" must
   involve "*wz* is less than *ww*,"] For *xz*
   < *xw* must involve *wz* < *ww*,

204: 36 are] seem

204: 37 order.] order, though I know of
   no instance of a series so generated

204: 44 <*fn. added*>

205: 6 With these assumptions, an
   unambiguous] With these assump-
   tions, it would seem, an unambigu-
   ous

205: 29 applied. The] applied. (This
   will be proved in detail in Chapter
   XI.) The <*There is no eleventh chapter
   of Part V; so this forward reference is
   very puzzling.*>

205: 43 <*fn. added*>

206: 17 in elliptic Geometry,] in
   projective and elliptic Geometry,

CHAPTER XXV. THE MEANING OF
   ORDER

207: 8, 24 Chapter XXIV] Chapter 1

207: 13–14 generated, unless ... it.]
   generated.

207: 14 To elicit this] To elicit clearly
   this

209: 3 implies] is

210: 8 is equivalent to] means that

210: 18–19 rendered at least verbally
   circular.] rendered circular.

210: 24 as "belonging to the domain of
   R."] as belonging to the extension of
   R.

210: 25 domain] extension

210: 26–30 <*The print uses "E" where
   the ms. has "K".*>

211: 1 would, if possible, be] are, if
   possible, to be

211: 37–8 a logically puzzling entity,
   which in Part 1 (§55) we found it
   necessary to deny;] a logically puzzl-
   ing entity, and

211: 45 <*fn. added*>

212: 15, 17, 18, 19 terms] concepts

212: 20 being, adjectives generally, and
   relations.] being, numbers, and rela-
   tions.

212: 21 relations. Such terms we agreed
   to call concepts. It] relations. It

212: 33–4 any concept which] any con-
   cept, other than a number, which

212: 37 when the concept of a number]
   when a number

212: 43 <*fn. added*>

212: 46 Chap. LIV] Chap. 11

213: 21 as in fact it is,] as it is

213: 25 might] may

213: 37 that sets] that some sets

213: 37 many] several

213: 45 denumerable. The logical ... 4,
   ......] denumerable.

214: 5 two terms] pairs

214: 7 Chapter XXIV] Chapter 1

214: 24 two terms] pairs of terms

214: 42 *abcd* and *acde* together imply
   abde] *abcd* and *acbe* imply *adbe*
   <*These axioms are equivalent in the
   presence of the second axiom. The
   circular diagram in the ms. places
   point c appropriately on the clockwise*

path from a *to* b *to represent the ms.
   version of axiom 5.*>

215: 2 the accompanying figure.] the
   figure.

215: 19, 23, 27 twelve] 12

215: 23 *acde*] *acbe*

215: 23 *abcde*] *aebcd*

215: 41–216: 1 having possibly no] hav-
   ing no

216: 40 Chapter XXIV] Chapter 1

216: 44 premises. See Part VI, Chap.
   XLV.] premises.

CHAPTER XXVI. ASYMMETRICAL
   RELATIONS

218: 4 Critical Philosophy] critical
   philosophy

218: 8 Part VI, Chapter LI),] Part VI),

218: 13 *xRy* always implies] *xRy* implies

218: 13 <*"symmetrical" is italicized in
   print, but not underlined in ms.*>

218: 14 together always imply] together
   imply

218: 15 <*"transitive" is italicized in print,
   but not underlined in ms.*>

218: 17 *xRy* always excludes] *xRy*
   excludes

218: 19–20 *xRz* I shall call intransitive.]
   xRz, or the property that *xRy*
   excludes there being any *z* such that
   yRz, I shall call intransitive.

218: 26–7 and not transitive, if third ...
   intransitive.] and intransitive.

219: 1 father] husband

219: 3 <*"important" is italicized in print,
   but not underlined in ms.*>

219: 9, 12 field] extension

219: 19 asserts] assert

219: 27 propositions, or relations.]
   propositions.

219: 31, 36 principle] axiom also <*Also
   at 220: 8, 13, 44.*>

219: 31 Chap. XIX,] Chap. 1

219: 41 p. 45, F. 1901, p. 193.] p. 45.

219: 42 this principle,] mine,

219: 43 precision, and not demon-
strated, will be found] precision, will
be found

220: 5–6 this process, as set forth by
Peano, requires] this process requires

220: 13 property.] property.* <fn.
omitted in print:> *For a mathemat-
ical statement of this axiom, see
Appendix.

220: 20 proposition] axiom

220: 29 domain] extension

220: 39 Chap. XIX, §154,] Part III,
Chap. I

221: 10–11 But it follows from the prin-
ciple of abstracion that there is some
relation] But if our axiom be
allowed, there will be some relation

221: 42–3 <fn. added>

222: 44 Phil. Werke, Gerhardt's ed.,]
Gerhardt's ed.,

223: 33 extrinsic] not intrinsic

223: 34 Hence] Thus

223: 39 blos] bloss <misprint in text>

224: 6 < Throughout this ¶ and through-
out §§215, 218, 219, and 221, in repre-
senting the terms of a relation, the ms.
has a capital "A" and a capital "B",
where the print has, respectively, a
small "a" and a small "b".>

224: 30 "We, in brief, are led] "We are
led

225: 20 if we are to explain] in order to
explain

225: 36 mean, if the monistic theory be
correct, to assert] mean to assert

226: 37 Chap. LI.] Chap.

CHAPTER XXVII. DIFFERENCE OF
SIGN AND DIFFERENCE OF SENSE

227: 11 Raume (1768),] Raume",

227: 21 space] it

227: 30 391.] 391. I shall, in Part VI,
base a somewhat longer argument for
absolute space upon the same fact.

228: 1 I mean, in the present discussion
at least, the difference] I mean the
difference

228: 7 either] the one

228: 8 the other or its negation;] the
other

228: 12 does not imply] excludes

228: 22, 32 < The ms. has "<" and ">",
where the print has "is less than" and
"is greater than".>

229: 23 Taking first finite numbers,]
Taking first numbers,

229: 26–8 But ... Part II.] But the
addition of integers, as we saw in
Part I, is a complicated matter, and
the relation R is prior to it.

229: 32 is easily shown to be the same]
is the same

229: 43 Progressions, §233.] Progres-
sions.>

230: 26, 27 Any pair] All pairs

230: 26–7 a new collection,] new collec-
tions,

230: 27–8 a new stretch.] new stretches.

230: 28–9 must be consecutive to the
beginning] must be the beginning

230: 33 negative and positive] positive
and negative

230: 41 asymmetrical relations.] asym-
metrical.

230: 41–4 In the former ... smaller.] In
the former case, the others may be
arranged in a series according as their
relations to A are greater or smaller.

230: 45 <fn. added>

231: 6 the relation of a to m] the rela-
tion AM

231: 28, 29 descriptive] projective

231: 28–9 a serial relation ... a series.] a
relation with sense, in virtue of

which the metrical straight line is a
series.

231: 31 non-coplanar] non-intersecting
<Also at 232: 7.>

231: 35 north] N

231: 35 south] S

232: 1 east] E

232: 12–249: 37 <Folios 73–107 of the ms.
of Part IV are missing. They include
all of CHAP. XXVIII. ON THE DIF-
FERENCE BETWEEN OPEN AND
CLOSED SERIES, pp. 234–8, and
CHAP. XXIX. PROGRESSIONS AND
ORDINAL NUMBERS, pp. 239–44.>

CHAPTER XXX. DEDEKIND'S THEORY
OF NUMBER

249: 38 progressions] they

249:42–3 similarity between classes,
which] similarity, which

249: 43 principle] axiom

249:45 cardinals; for] cardinals. For

249: 45–250: 29 suffice; we require ...
set. For example,] suffice; it is neces-
sary to have, not merely two classes,
but two discrete series (which must
be or contain progressions, if any
ordinal is to be defined), and these
must not merely be similar, but must
be correlated so that their first terms
are correlates, and if, in one series, a
precedes b, then in the other, the
correlate of a must precede the corre-
late of b. This is merely a new com-
plication added to the definition of
cardinals in order to obtain that of
ordinals. With this added complica-
tion, we may define the ordinals as
the common property of terms in
progressions which are correlated by
a one-one relation having order
unchanged. But to this definition by

abstraction there are, as we saw in
the preceding chapter, certain logical
objections. The definition which
avoids the objections presupposes
cardinals in a still more obvious way.
It is as follows. Having first estab-
lished independently (which, as we
know, is possible) the logical theory
of cardinals, we find that in a pro-
gression, every term defines a certain
class of terms, namely those which
do not come after it, and that this
class of terms has a cardinal number
which uniquely determines and is
uniquely determined by that term of
the progression by which the class in
question is defined. We have hence a
one-one relation, which is not sym-
metrical, between the terms of a
progression and the cardinal num-
bers—a relation, it should be
observed, which no term can have to
anything except a cardinal number,
but which every cardinal number has
to itself, and which is one-one only
in regard to a given progression. The
terms having this peculiar relation to
the number n are called nth terms;
and the ordinal number nth is, like
Christian, Mahometan, etc., what
may be called an adjective of relation
—i.e. it expresses the fact that the
terms to which it applies have a
certain relation to n. A certain fur-
ther refinement is necessary, how-
ever, to make this definition quite
correct.

250: 29 For example, since] Since

251: 1–2 series, and upon the relation by
which they are ordered, so that]
series, so that

251: 5 Hence] Thus

251: 5 four-cornered] three-cornered

251: 5–6 (the first), a generating serial relation and the cardinal number *n*.] (the first), and the cardinal number *n*.* <*fn. omitted inprint*:> *The above account does not apply to transfinite ordinals, but we shall see in Part v how it may be adapted so as to apply.

251: 7–8 ordinals, ..., are more complex] ordinals are far more complex

251: 18 an absolute error] an error

251: 20 largely] quite

CHAPTER XXXI. DISTANCE

252: 21–3 stretches which fulfill the axiom of Archimedes and the axiom of linearity always are] stretches always are

252: 23 the idea, as] the idea of distance, as

252: 25 in most compact series] in compact series

252: 29 rationals or the real numbers are] rationals are

252: 31 *Phil. Werke*, Gerhardt's ed.] Gerhardt's ed.

253: 20 <*twice*> respect] regard <*The 2nd occurrence of "respect" in print is a misprint.*>

253: 36 some distance] some one distance

254: 12 a finite integer] an integer

254: 12 the *n*th power of] *n* times

254: 13 second distance;] second;

254: 14 has an *n*th root,] can be separated into *n* equal parts,

254: 15 whence] when

254: 24 as rationals or real numbers from] as rationals from

254: 40–5 The powers ... p. 46.] See Appendix.

255: 9–10 the opposite theory] the theory

255: 17 stretch] distance

255: 18 distance] stretch

255: 18–19 since ... distances,] since it is doubtful whether there are distances outside the theory of progressions,

255: 19 in almost all] in all

255: 21 for which, as a rule, no] for which no

255: 22 therefore generally better] therefore better

255: 26–7 elsewhere, except ... space; and if] elsewhere, and if

255: 29–32 without presupposing distance; the distances ... space; and in Part v] without distance; and in Part v

255: 32 how few] what

256: 8 how finite ordinals] how ordinals

256: 9 to be to a certain extent independent of] to be independent of