PART VI OF *THE PRINCIPLES OF MATHEMATICS*

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This paper continues a series of studies aimed at describing and analyzing the relations between the manuscript of Russell's *The Principles of Mathematics* and the printed text.[1] Previous studies have presented collations of the first five Parts of *Principles*, together with assessments of the significance of the variations between the manuscript and the published text. Here I examine the somewhat fragmentary manuscript of Part VI ("Space").

The earlier studies in this series, and related work by other scholars, have made it clear that Russell began writing *Principles* in the fall of 1900 by writing Parts III–VI.[2] In May of 1901, he wrote an initial version of Part I, and then an extensively revised version of the same Part in May 1902. Part II was originally written in June 1901, and not extensively revised prior to being sent to the printer in May 1902. This order is the same as that set out by Russell in a letter to Jourdain written in April 1910, but is in disagreement, as regards Parts I and II,

[1] The studies in this series are K. Blackwell, "Part I of *The Principles of Mathematics*", *Russell*, n.s. 4 (1984): 271–88; Michael Byrd, "Part II of *The Principles of Mathematics*", *Russell*, n.s. 7 (1987): 60–70; Byrd, "Part V of *The Principles of Mathematics*", *Russell*, n.s. 14 (1994): 47–86; Byrd, "Parts III–IV of *The Principles of Mathematics*", *Russell*, n.s. 16 (1996): 145–68.

[2] Alejandro Garciadiego, *Bertrand Russell and the Origins of the Set-Theoretic 'Paradoxes'* (Basel: Birkhäuser, 1992), pp. 88–92; Ivor Grattan-Guinness, "How Did Russell Write *The Principles of Mathematics* (1903)?", *Russell*, n.s. 16 (1996): 101–27, esp. 104–10; Byrd, "Part V", pp. 48–56; Byrd, "Parts III–IV", pp. 147–53.

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with the *Autobiography*.[3]

Since the manuscripts for Parts III–VI were written in the fall of 1900 when Russell is first assimilating the work of Peano and his school, they offer insight into Russell's initial attempt to integrate the new logic with the "Platonic realism" that he and Moore had been developing since 1898. The manuscripts of Parts III–V, particularly Part V, show substantial terminological and doctrinal variation from the printed text. In particular, in my studies, I have emphasized that the explicit logicist definitions of cardinal and ordinal number in terms of equivalence classes do not occur at all in the extant manuscripts from the fall of 1900. The manuscript of Part VI is particularly illuminating in identifying the place that logicist ideas had in the early evolution of *Principles*. For, as Grattan-Guinness has noted, the manuscript of Part VI contains several passages that clearly state one version of the idea that all of "pure Mathematics" is just a branch of logic.[4] I will discuss these passages in section 3 below.

The variation from the published text is considerably smaller in the case of Part VI than in the case of Part V. In the case of Part VI, there are roughly 1,000 words of altered text in the surviving manuscript of about 100 pages in length; in the 200-page manuscript of Part V, there are around 3,500. Part VI is similar to Parts III and IV in this regard. However, in the case of Parts III and IV, there is substantial overlap in organization, subject-matter, and argumentation with the version of *Principles* written in 1899 and the first half of 1900.[5] This is not true of the manuscript of Part VI, the structure of which is very different from that of its ancestor. The initial chapter (Chap. 44) in Part VI shares subject-matter (the nature of dimensions) with the fourth chapter of the 1899–1900 manuscript. The final two chapters of the fall 1900 manuscript address the same question as the first chapter of the 1899–1900 manuscript; namely, the logical coherence of the view that space is composed of indivisible spatial terms, points. There is almost no further overlap. Much of the fall 1900 manuscript (Chaps. 45–8) is devoted to a careful study of Projective and Metrical Geometry and their relations. In section 2 of this paper, I examine what Russell aimed to show in this lengthy discussion.

[3] Ivor Grattan-Guinness, *Dear Russell—Dear Jourdain* (London: Duckworth, 1977), p. 133; *Auto.*, 1: 192–3.

[4] "How did Russell write *The Principles of Mathematics*?", pp. 111–13.

[5] This version may be found in *Papers* 3: 3–180.

I. THE MANUSCRIPT TEXT

The initial leaf of the manuscript is dated December 1900. The leaves from the manuscript have the annotation "S" in their upper left-hand corners. As in the case of Parts III–V, there are no section numbers and no printer's markings on the leaves. The chapters are ordered internally, beginning with "Chapter 1".

The list of variants is given at the end of this essay. It is constructed on the model of previous collations in this series. The list is read as follows. At the left is a number such as 372: 44. This means page 372, line 44 from the top. To the right is the reading from the published text of *The Principles of Mathematics*. This is followed by a square bracket, then the reading from the manuscript of Part VI. Single angle brackets enclose my editorial comments.

The leaves from Part VI are numbered consecutively 1 to 81, 169 to 180, and 182 to 187. Thus, there is a substantial gap in the manuscript. It begins on page 414, line 13, which is the beginning of §399, and ends at page 453, line 31. The first line of folio 169 of the manuscript begins "(5) causality;". The omitted section is not a natural unit, such as a chapter. It contains all of Chapters 48 ("Relation of Metrical to Projective and Descriptive Geometry"), 49 ("Definitions of Various Spaces"), and 50 ("The Continuity of Space"). This is unfortunate; the initial discussion in Chapter 49 is an especially clear statement of Russell's logicist conception of Geometry.

The manuscript of Part VI has two non-standardly numbered leaves, 54a and 81a.[6] Five leaves have double numbers, indicating that they were removed from earlier work of Russell's. The folio numbers

[6] Fol. 54a is an inserted section of the published text, §378. It indicates how primitive geometrical concepts (for example, *point*) can be treated as variables in the definition of a kind of space, and explicitly refers forward to the similar approach advocated in Chapter 49 of the text. Other evidence, discussed in sec. 3, makes it likely that this section was added sometime after the initial writing of the manuscript. Fol. 81a is an abbreviated version of §398 of the published text. Since the leaf occurs immediately prior to the large gap in the ms., it is hard to get a clear sense of the status of this leaf.

are 180, 182, 183, 184, and 185. Their second (or original) numbers are 65, 67, 68, 69, and 70. Thus they all come from the sections of *Principles* that discuss Kant's antinomies. In all likelihood, these pages come from the 1899–1900 manuscript of Part VI of *Principles*. First, according to John King, the paper matches in quality and size that of the 1899–1900 manuscript.[7] Second, the crossed-out material on some of the manuscript leaves seems to refer to doctrines characteristic of the 1899–1900 version of *Principles*.[8]

The manuscript of Part VI manifests its early compositional date through certain characteristic variations of terminology and viewpoint. Russell's logical terminology increases in clarity and precision as he writes *Principles*. In the fall manuscript, Russell does not clearly distinguish between a series (as a relation), the field of the series, and the domain of the series. In particular, the term "extension" is used to cover both the last two. These uses of "extension" are consistently replaced in the Parts IV, V, and VI of the published text by "serial relation", "field", and "domain" as appropriate.[9]

A second change in terminology involves a substantive change of view. Russell uses "term" broadly in *Principles* to cover everything which has being. Famously, this includes abstract objects and possibly non-existent concrete objects (e.g., the Homeric gods, four-dimensional spaces). In the fall 1900 manuscript of *Principles*, Russell adheres to the view of terms advocated by Moore in "The Nature of Judgment". On this view, all terms are concepts, or Platonic ideas. Thus, these manuscripts use "concept" and "term" interchangeably. In May 1901, when Russell wrote the initial version of Part I, he had come to hold

[7] John King, "A Report on the Manuscripts of 'Analysis of Mathematical Reasoning', 'The Fundamental Ideas and Axioms of Mathematics', and 'The Principles of Mathematics'" (unpublished ms.: copy in BRA), p. 13.

[8] On fol. 180, the deleted material includes the following: "We shall find them <*the first two antinomies*> reducible to the antinomy of infinite number, which was discussed, and in a manner solved, in Book V." Chapter V of Part V of the 1899–1900 ms. is devoted to just such a discussion arriving at the "solution" that there is a collection of finite numbers, but that this collection itself has no definite number of terms. See *Papers* 3: 119–25.

[9] As examples in Part VI, see the list of variants below at 394: 40 and 395: 32; for earlier Parts, see Byrd, "Parts III–IV" (List of Variants), 220: 29; Byrd, "Part V" (List of Variants), 288: 30.

that some terms were not concepts; his examples are Socrates, points, instants, bits of matter, and particular states of mind (*Papers* 3: 189–90). So in the published text, Russell clearly distinguishes "term" and "concept". This pattern of replacement also occurs in Part VI at 397: 28. The manuscript uses "term", where, from the later point of view, the narrower word "concept" is appropriate.[10] "Term" is replaced by "concept" in the published text.

Another doctrinal change concerns the status of the notion of quantity and allied conceptions, such as magnitude of divisibility. As I noted in my study of Part III, Russell's conception of the place of the concept of quantity changes somewhat from the manuscript to the published text. While the fall 1900 manuscript subordinates quantity to order, contending that what is mathematically significant about quantity is that quantities exhibit order, the published text draws the further conclusion that quantity as such is simply not a concept of pure mathematics.[11]

This pattern is also found in Russell's treatment of Metrical Geometry in Part VI. He introduces distance as the fundamental concept of Metrical Geometry in §392 of Chapter 47. He then sets out axioms for this concept. In §393, Russell considers the replacement or identification of the concept *distance* with the concept *stretch*, where a stretch of points is the class of points intermediate between two points (*PoM*, p. 181). In assessing the prospects of this approach, the published text says two things: (1) the approach involves a new indefinable—the magnitude of divisibility of a stretch; (2) this new indefinable is not a purely logical concept and hence is not part of pure Mathematics. These claims do not occur in the manuscript. There he says that the approach using stretches involves several new axioms and no new indefinables (see the List of Variants, 408: 26–7).

The extant manuscript of Part VI confirms conclusions drawn in

[10] This pattern of change also occurs in closely related passages dealing with Russell's criticism of Lotze's views on relations in Chapter 51. (This is a section where the manuscript is absent.) At page 448: 38–9, Russell clearly pairs "concept" and "term" in a way that suggests their non-coextensiveness. In the papers from which this section derives, Papers 5 and 6 of *Papers* 3, Russell uses "concept" alone at the corresponding place. See *Papers* 3: 252, 275.

[11] Byrd, "Parts III–IV", pp. 156–7. The crucial fact is that 158: 38–45 is new in the published text.

earlier papers of the series about the order of composition of *Principles*. There are only two clear back references to Parts I and II in the extant sections of the fall 1900 manuscript of Part VI. One of these corresponds to the footnote at 457: 45. In the published text, this footnote refers back to the Russell's detailed axiomatization of propositional logic in Chapter 2, §18. Of course, this back reference does not occur in the manuscript. Rather the manuscript refers to Peano's 1900 paper "Formules de logique mathématique", the work of Peano cited in the first paragraph of the original version of "The Logic of Relations" (see *Papers* 3: 590).

The second back reference to Parts I and II occurs in the initial section of Part VI at page 371: 3–5. This material occurs unaltered in the manuscript. Russell begins Part VI by summarizing the results of earlier Parts:

> In the first two Parts, it was shown how, from the indispensable apparatus of general logical notions, the theory of finite integers and of rational numbers without sign could be developed.

This is a more precise and accurate description of what occurs in Parts I and II than occurs anywhere in the manuscripts of Parts III–V. As I have noted, the back references in these Parts make no mention of any general thesis relating logic and mathematics. In fact, many of them seem more appropriate as back references to Parts I and II of the 1899–1900 version of *Principles*, which held that basic notions, such as number, were indefinable.[12]

In this passage, for the first time in the fall manuscripts, Russell sets forth claims connecting arithmetic with logic, holding that arithmetic can be "developed" solely from logical notions. In the face of the strong evidence already presented that Russell had not written Parts I and II in the fall of 1900, it is natural to see this as a prediction of what Parts I and II would contain. Furthermore, Russell already had in hand substantial grounds on which to base this claim. Russell's first draft of "The Logic of Relations" dates from October 1900. Sections 1 and 3 of this draft show how to define such concepts as "cardinal number", "successor", "zero", and "finite number" in the logic of

[12] See Byrd, "Part V", pp. 52–3; Byrd, "Parts III–IV", pp. 151–2.

relations, assuming the so-called axiom of abstraction. This seems an adequate basis for Russell's prediction, even without Parts I and II in hand.

In contrast to the references to Parts I and II, the back references to Parts IV and V are detailed and specific. This is particularly true of Part IV. These concern the nature of relations that generate order, particularly in closed series. For example, at 386: 13, Russell is discussing how to generate order on the straight line from the quadrilateral construction in projective geometry. Having defined a four-place relation *Q* from the notion of harmonic conjugate, he says, "We have here a relation of four points from which, as we saw, in Part IV, Chapter 1, an order will result if certain axioms are fulfilled." He then presents a set of axioms for *Q*, and defines from *Q* a second four-place relation *T*. Of *T*, he says at 387: 6, "It is a relation which has the formal properties of separation of couples, as enumerated in Part IV, Chapter 1."[13] In both cases, Russell clearly has in mind the axioms for separation of couples, derived from Vailati, that he presents in §194 of *Principles*. The material on separation of couples is new to the fall 1900 version of *Principles*.[14]

2. PROJECTIVE AND METRICAL GEOMETRY

More than half of Part VI of *Principles*, Chapters 45 to 48, is devoted to presenting, developing, and comparing various formulations of what Russell calls Projective, Descriptive, and Metrical Geometry. At the end of Part VI, Russell summarizes what he takes himself to have accomplished in that Part. Strangely, this summary omits any specific reference to these chapters. He writes:

> We found that the abstract logical method, based upon the logic of relations which had served hitherto, was still adequate, and enabled us to deduce from the definitions all the propositions of the corresponding Geometries. (*PoM*, p. 461)

[13] The published text replaces "Chapter 1" by "Chapter XXIV".

[14] Back references to Part IV in the manuscript are found at passages corresponding to 376: 14, 381: 10, 386: 13, 387: 6–7, 387: 14, and 396: 20. There is a back reference to Part V at 461: 15.

While Russell arguably shows that the "abstract logical method" is adequate for the development of Projective, Descriptive, and Metrical Geometry, it is also clear that a much shorter discussion would have sufficed to establish this. Russell's own discussion in Chapter 49 makes the point. There, in the course of eight pages, Russell shows the adequacy of logic for the definition of three-dimensional projective space, three-dimensional Euclidean space, and two-dimensional Clifford space. So, it is natural to ask what Russell hoped to accomplish by the extended discussion of Chapters 45 to 48, beyond simply establishing the adequacy of logic to the definition of such spaces.

In his earliest philosophical book, *An Essay on the Foundations of Geometry*,[15] Russell argued that Projective Geometry is independent of Metrical Geometry and that the axioms of Projective Geometry are à priori in something like the Kantian sense. The axioms of Metrical Geometry are claimed to lack this strong kind of aprioricity; some are said to be flatly empirical. So, at an early stage of his philosophical career, Russell held that there was a substantive philosophical difference between Projective and Metrical Geometry. By 1898, the Kantian approach that Russell advocated in *Foundations* seemed to Russell unsatisfactorily psychologistic.

Yet Russell seems to have retained the belief that there is a fundamental logical difference between Projective and Metrical Geometry. When introducing the contrast between Projective, Descriptive, and Metrical Geometry in *Principles*, Russell writes:

> In the discussions of this Part, I shall not divide Geometries, as a rule, into Euclidean, hyperbolic, elliptic, and so on, though I shall of course recognize this division and mention it whenever it is relevant. But this is not so fundamental a division as another, which applies, generally speaking, within each of the above kinds of Geometry and corresponds to a greater logical difference. (*PoM*, p. 381)

I shall argue that, in these chapters of *Principles*, Russell is returning to a contrast that he had long held to be of substantial importance in Geometry, and he is attempting to see how his earlier ideas about the

[15] Cambridge: at the U.P., 1897; reprinted New York: Dover, 1956; and cited here as *EFG*.

logical relations of Projective and Metrical Geometry stand up in the setting of his newly developing general philosophy of mathematics. The results are largely negative, although it seems, from the manuscripts, that Russell came to see this conclusion clearly only after the initial composition of Part VI. The results are negative in the sense that Russell's newly developing philosophy of mathematics naturally leads him to the conclusion that the differences are less fundamental and substantial than he had originally thought.

I will begin by looking at Russell's claims about the logical relations between Projective and Metrical Geometry in *Foundations*. Chapter I of *Foundations* is a short history of non-Euclidean geometry (which Russell calls Metageometry). Russell divides the history into three periods. The first is work on alternatives to the axiom of parallels. The second period centres on the work of Riemann in which the fundamental conceptions are the idea of a manifold and the measure of curvature of that manifold. Russell sees this work as a generalized kind of Metrical Geometry:

> ... it is a pity that Riemann, in accordance with the metrical bias of the time, regarded space as primarily a magnitude, or assemblage of magnitudes, in which the main problem consists in assigning quantities to the different elements or points, without regard to the qualitative nature of the quantities assigned. (*EFG*, p. 15)

Russell's general characterization of Metrical Geometry in *Foundations* is this: "Metrical Geometry, to begin with, may be defined as the science which deals with the comparison and relations of spatial magnitudes" (p. 149). He contends that, despite its generality, work in this tradition makes certain basic assumptions. These assumptions Russell regards as basic axioms of Metrical Geometry. Russell identifies them as follows:

> ... Riemann, in spite of his desire to prove that all the axioms can be dispensed with, has nevertheless, in his mathematical work, retained three fundamental axioms, namely, Free Mobility, the finite integral number of dimensions, and the axiom that two points have a unique relation, namely distance. (*EFG*, p. 22)

It is these assumptions that Russell holds to be the relatively à priori axioms of Metrical Geometry. They are required if spatial measurement to be possible (*EFG*, pp. 147–9). When Russell comes to characterize Metrical Geometry in *Principles*, it is the third of these axioms that is taken to be central. Metrical Geometry introduces a concept *distance*; a distance is relation between a pair of points, having certain properties in virtue of which it is numerically measurable. He then introduces a set of axioms governing distance (*PoM*, pp. 407–8).

The third period of Metageometry is work in Projective Geometry. In *Foundations*, Russell holds that Projective Geometry differs from Metrical Geometry in the concepts that it treats as basic. Concepts such as congruence and distance are replaced by relations such as collinearity. He interprets this difference philosophically as a difference between quantitative and qualitative aspects of spatial figures. For instance, Russell writes:

> Distance is a quantitative relation, and as such presupposes identity of quality. But projective Geometry deals only with quality for which reason it is called descriptive—and cannot distinguish between two figures which are qualitatively alike. (*EFG*, p. 33)

Metrical Geometry is thus viewed by Russell as presupposing Projective Geometry:

> Metrical Geometry, therefore, though distinct from projective Geometry, is not independent of it, but presupposes it, and arises from its combination with the extraneous idea of *quantity*. (*EFG*, p. 143)

Russell argues at length in *Foundations* that this is an epistemologically important difference. First, Projective Geometry is fundamental:

> We have good ground for expecting, therefore, that the axioms of projective Geometry will be the simplest and most complete expression of the indispensable requisites of any geometrical reasoning.... (*EFG*, pp. 117–18)

Second, and this is Russell's Kantian claim, the axioms of Projective Geometry are à priori. Russell's argument depends on the notion of a form of externality, which he describes as "a 'principle of differentiation', by which the things presented are distinguished as various" (*EFG*, p. 136). In his excellent discussion of this part of *Foundations*, Griffin suggests that a form of externality might be usefully thought of as "something like a display rack on which diverse items may be simultaneously presented."[16] Russell then argues that (what he identifies) as the axioms of Projective Geometry can be deduced from the properties of a form of externality, and that a form of externality is required if knowledge is to be possible.[17]

Third, Russell holds that Metrical Geometry is not à priori in this sense. Projective Geometry is not sufficient for all of the purposes of Geometry; for example, spatial measurement. While certain axioms of Metrical Geometry are à priori, in the sense that they are required if spatial measurement is to be possible, they are not à priori in the stronger sense applicable in the case of Projective Geometry. Consequently, Metrical Geometry must be a genuine extension of Projective Geometry, one that is not already contained in Projective Geometry.

This last claim seems to fly in the face of the mathematical fact that something like *distance* can be defined within Projective Geometry. Cayley and Klein had shown that, holding two points a and b fixed, the logarithm of the anharmonic ratio for two points c and d, relative to a and b, has characteristic properties of a distance relation, such as additivity.[18] Russell discusses this matter at some length in both *Foundations* and *Principles*. In *Foundations*, he argues that this definition has merely "technical" importance. By this, he means that while the concept defined has certain formal similarities to the concept of distance, these formal similarities are insufficient to warrant calling it a definition of distance. Distance, Russell claims, is a relation between two points, not four points (*EFG*, p. 35). The concept defined in projective geometry is not distance, but a formally similar surrogate. Russell writes:

[16] Nicholas Griffin, *Russell's Idealist Apprenticeship* (Oxford: Clarendon P., 1991), pp. 144–5; I have found Griffin's account of Russell's central arguments in *Foundations* very useful in trying to understand Part VI of *Principles*. See especially Griffin's Chapters 4 and 8.

[17] There are flaws in Russell's attempted transcendental deduction of the axioms of Projective Geometry. Some of these are discussed in Griffin, §4.4.

[18] The relevant anharmonic ratio is the fraction whose numerator is the fraction $c-a/c-d$, and whose denominator is $d-a/d-b$. See *PoM*, p. 422.

> Quantities, as used in projective Geometry, do not stand for spatial magnitudes, but are conventional symbols for purely qualitative spatial relations.... Distance in the ordinary sense is, in short, the quantitative relation, between two points on a line, by which their difference from other points can be defined. The projective definition, however, being unable to distinguish a collection of less than four points from any other on the same straight line, makes distance depend on two other points besides those whose relation it defines. No name remains, therefore for distance in the ordinary sense.... This confusion, in projective Geometry, shows the importance of a name, and should make us chary of allowing new meanings to obscure one of the fundamental properties of space. (*EFG*, p. 36)

By 1898, Russell had abandoned the Kantian claims in his treatment of Geometry. Russell apparently took to heart central points in Moore's critical review of *Foundations* in *Mind*.[19] For example, in his 1900 book on Leibniz, Russell holds:

> The distinction between the empirical and the *a priori* seems to depend on confounding sources of knowledge with grounds of truth. There is no doubt a great difference between *knowledge* gained by perception and *knowledge* gained by reasoning; but that does not show a corresponding difference as to what is known. (*PL*, p. 24)

However, his earlier claims about the logical relations of Projective and Metrical Geometry seem to be independent of Russell's changing views on the aprioricity of parts of Geometry. The first of these claims is that Projective Geometry is independent of Metrical Geometry in the sense that its fundamental concepts and axioms can be framed independently of the concepts of Metrical Geometry. Second, Metrical Geometry is a genuine extension of Projective Geometry, whose fundamental concepts, such as distance, cannot be legitimately be defined in Projective Geometry. It is these contentions that Russell addresses in Chapters 45, 47, and 48.[20]

I now turn to consideration of these chapters in *Principles*. In Chapter 45, Russell shows that Projective Geometry, properly and clearly axiomatized, is independent of metrical notions. Here the primary advance over Russell's earlier work is how clearly primitive concepts and axioms are identified. Poincaré, in reviewing *Foundations*, had complained about the unclarity of Russell's set of axioms for Projective Geometry, and Russell had independently come to the same judgment (see *Papers* 2: 403–9). In his reply to Poincaré, Russell attempts to present a more rigorous axiomatization, although his success is limited by the algebraic, equational framework within which he works.

In *Principles*, Russell relies on the work of Pieri.[21] *Point* is a primitive concept, and so is *straight line*. Axioms are set out which are sufficient for the proof of the uniqueness of the von Staudt quadrilateral construction. Russell shows how to define, using the harmonic range construction, a four-place relation among collinear points that has the properties of the relation of separation of couples that Russell had used in describing order in closed series. He also addresses the question of added axioms to ensure continuity. At the beginning of Chapter 48, Russell claims correctly that the logical independence of Projective Geometry has been established.[22] Thus, the discussion in *Principles* shows that Russell's judgment in *Foundations* continues to hold in the setting of a more precise and thorough axiomatization of Projective Geometry.

In Chapter 47, Russell turns to the presentation of Metrical Geometry. In Russell's earlier treatments of Metrical Geometry, *distance* is its fundamental concept. So, Russell begins his discussion by asking how distance is to be treated in his new mathematical philosophy. He begins by saying that "Metrical Geometry is *usually* said to be distinguished by the introduction of quantity" (*PoM*, p. 407, emphasis added). This, of course, is the view Russell espoused in *Foundations*, and he is carefully stepping back from it here. He then states a more austere conception of distance, one more in keeping with the "abstract logical method" that he now takes to characterize pure

[19] *Mind*, n.s. 8 (1899): 397–405.

[20] Russell's continued interest in the logical interrelations of Projective and Metrical Geometry is shown by unpublished notes on order and geometry that date from 1898 and 1899. See *Papers* 2: 339–89, esp. 375–80.

[21] Pieri, *I Principii della geometria di posizione* (Turin, 1898). Russell read this work in August 1900 (see *Papers* 1: 363).

[22] When comparing Projective and Metrical Geometry in Chapter 48, Russell groups Projective and Descriptive Geometry under the general head "non-quantitative geometry" (*PoM*, p. 419).

mathematics. He writes:

> It is *sufficient* for the characterization of metrical Geometry to observe that it introduces, between every pair of points, a relation having certain properties in virtue of which it is numerically measurable i.e. such that numbers can be given a one-one correspondence with the various relations of the class in question. (*PoM*, p. 407, emphasis added)

Russell then sets out some basic axioms governing the assignment of numbers to relations, including, for example, Archimedes' axiom. In §393, he then indicates that, from his new point of view, even the reference to relations is inessential: "It is not necessary that distances should be magnitudes, or even relations; all that is essential is that they should form a series with certain properties" (*PoM*, p. 408).

However, in the same section, Russell also introduces a second approach to the basic concepts of metrical Geometry. Here his description relies on terminology introduced in the chapter on measurement (Chap. 21) of Part III ("Quantity"):

> In all series there are terms intermediate between any two whose distance is not the minimum. These terms are determinate when the two distant terms are specified. The intermediate terms may be called the *stretch* from a_0 to a_n. The whole composed of these terms is a quantity, and has a divisibility measured by the number of terms, provided the number is finite. (*PoM*, p. 181)

Russell makes several claims about this approach, which might easily seem to be a simple variant of the approach previously described. First, he says that on this approach, certain claims, which must be taken as axioms on the other approach, follow directly from the notion of stretch. This is so, provided the serial relation underlying the definition of *stretch* is a dense linear order without endpoints. Second, he claims that some of the axioms about distance, such as Archimedes' axiom, do not follow from the concept of stretch, and again he is correct about this. Third, he says that, as regards "actual space", this second approach represents the "most correct" approach. He does not elaborate on the claim at this point or, I think, elsewhere.

Finally, he assesses the logical resources required by this approach, but what he says in the manuscript differs from what he says in the

published text. In the manuscript, he claims that the approach involves no new indefinables, whereas according to the published text, the approach introduces a new indefinable, the magnitude of the divisibility of a stretch. The published text further notes that magnitude of divisibility is not a concept of pure mathematics, since it cannot be defined using just logical constants. Both manuscript and published text maintain that either approach is "logically permissable".

Russell is here being drawn in two directions. On the one hand, *distance* seems to be a clearly mathematical conception. As such, what matters about distance is that it satisfies certain formal conditions. The "abstract logical method" that Russell endorses in Part VI regards applications of mathematics as in general not a matter of mathematics, and certainly not to be included in the actual analysis of mathematical concepts. On the other hand, as in *Foundations*, he is clearly attracted to the idea that the concept of distance is closely tied to particular applications. Distance tells us the magnitude of divisibility of stretches.[23] It should not be confused with formally similar surrogates. Russell does not simply decide between these conceptions; rather, questions about the relations between metrical and projective Geometry continue to be discussed with both conceptions in mind.

In Chapter 48, Russell turns to a question that he considered at considerable length in *Foundations*: is Metrical Geometry genuinely independent of Projective Geometry? What Russell then considers in detail (§§408–11) is the projectively generated theory of distance already considered in our discussion of *Foundations*. In this discussion, Russell criticizes the projective theory of distance in much the same way that he does in *Foundations*.[24] He describes the theory as "purely technical", a description also used in *Foundations* (*PoM*, p. 428). He repeats his criticism that distance is a strictly two-place relation, not a four-place one:

[23] It is clearly worth asking why *quantity* and *magnitude of divisibility* are not logically definable. It seems to me that these concepts admit of austere interpretations that are logically definable. In fact, Russell himself suggests such an approach to *magnitude of divisibility* in §397, a section not in the manuscript.

[24] There is an added criticism in §408: the projective theory of distance does not verify certain axioms governing distance that Russell has come to take as central to the notion; for example, the axiom of Archimedes. This turns on more properly mathematical matters than the criticisms derived from *Foundations*.

It is important to realize that the reference to two fixed ideal points, introduced by the descriptive theory of distance, has no analogue in the nature of distance or stretch itself. This reference is, in fact, a convenient device, but nothing more. (*PoM*, p. 425)

There are further criticisms of a similar sort:

... if such a function is to be properly geometrical and to give a truly projective theory of distance, it will be necessary to find some geometrical entity to which our conjugate complex numbers apply. (*PoM*, p. 426)[25]

It is important that Russell does not simply let matters rest with these criticisms. He admits that if *distance* is treated by the "abstract logical method" in the way his developing view of mathematics advocates, then the projective theory of distance is perfectly acceptable:

Although the usual so-called projective theory of distance, both in descriptive and projective space, is purely technical, yet such spaces do necessarily possess metrical properties which can be defined and deduced without new indefinables or indemonstrables. (*PoM*, p. 428)

Metrical Geometry can be understood in two senses, corresponding to the austere and richer conceptions of *distance* set forth earlier in Russell's text. On the austere conception, *distance* is a mathematical concept, but metrical Geometry is not independent of projective Geometry. On the richer conception, metrical Geometry is an "independent subject". However, it is not a part of pure mathematics, since it uses *magnitude of divisibility* as an indefinable, and this concept is not a purely logical one. Thus, in neither case do we have a subject which is both part of pure mathematics and independent of projective Geometry.

3. RUSSELL'S LOGICISMS

As noted earlier, the opening section of Part VI is the first place in the extant fall 1900 manuscript where Russell makes a general claim connecting mathematics and logic. I want first to examine how Russell articulates this connection in the course of writing Part VI. I will argue that there are two layers of text present, the second of which is clearer, more sophisticated, probably later, and not substantially different from current versions of structuralism. Second, I want to explain how Russell connects the form of logicism set forth in Part VI with the more famous form of logicism set forth in Part II.

In the second section of Part VI, Russell says that in the nineteenth century, Geometry became a branch of pure mathematics. It did so when it ceased to assert its axioms and theorems, and instead asserted only that the axioms imply the theorems:

... the geometer would only assert that A implies P, leaving A and P themselves doubtful. And he would have other sets of axioms, A_1, A_2... implying P_1, P_2... respectively: the *implications* would belong to Geometry, but not A_1 or P_1 or any of the other actual axioms and propositions. (*PoM*, p. 373–4)

In this formulation, the claims of pure mathematics are implications, which have genuine propositions as their antecedents and consequents. And since the antecedents and consequents are genuine propositions, they contain concepts. Some of these concepts are analyzable, but some are primitive, or indefinable. Crucially, however, in pure mathematics, the implications are asserted on strictly deductive grounds.

This is precisely the conception of mathematics that Russell attributes to Peano and his followers in a paper that he wrote in the late fall of 1900. This paper was entitled "Recent Italian Work on the Foundations of Mathematics", and it appears for the first time in Volume 3 of Russell's *Collected Papers*. Russell corresponded with Stout, the editor of *Mind*, about writing such a paper, but then did not choose to publish it.[26] In the paper, Russell describes the aim of Peano's school as the identification of the necessary and sufficient premisses for various branches of mathematics, together with the rigorous logical deduction of theorems of these branches from these premisses. Russell identifies the role of logic and the various branches of mathematics in this project as follows:

[25] There is a similar but more obscure complaint lodged in the last two sentences of §410.

[26] On the dating of the paper, see *Papers* 3: 350–1. Russell also describes the work of the Peano school in exactly this way in Part II, Chapter 14, of *Principles*.

What distinguishes a special branch of mathematics is a certain collection of primitive or indefinable ideas, and a certain collection of primitive or indemonstrable propositions concerning these ideas. When once these have been assigned, symbolic logic appropriates the subject, and effects whatever deductions are legitimate. The exhibition of all mathematics in this form has been undertaken by Peano—with the help of a number of very able collaborators.... (*Papers* 3: 353)

Russell holds that the fact that geometrical proofs can be made formal and deductive in this way undermines a central tenet of Kant's views on geometry.[27]

In the first five chapters of Part VI, Russell almost always presents what he is doing in just this way. He identifies the primitive terms of projective, descriptive, and metrical Geometry. He debates the virtues of various choices of primitive terms, and then states axioms governing these primitive terms.

However, even in the manuscript, one finds a more abstract conception of the primitive terms and axioms. One of the undefined concepts in Russell's presentation of projective Geometry is *point*. What content does this term have in pure mathematics? Only what symbolic logic can extract from it. Given the formal nature of symbolic logic, this leads to a conception of the primitive terms that makes only their "formal type" relevant. Thus, Russell explains the use of *point* in projective Geometry, as follows:

It [projective space] is defined, like all mathematical entities, solely by the formal nature of the relations between its constituents, not by what those constituents are in themselves. Thus we shall see ... that the "points" of a projective space may each be an infinite class of straight lines in a non-projective space. So long as the "points" have a requisite type of mutual relations, the definition is satisfied. (*PoM*, p. 382)

The merely formal role of the indefinables is emphasized in what appears to be a later layer of the text.[28] In this regard, I want to look

[27] Russell does not deny that mathematics is synthetic or that it is à priori, though given Russell's treatment of these terms, it is very misleading to leave it at that. For detailed discussion of matters relating to Russell on Kant on mathematics, see Alberto Coffa, "Russell and Kant", *Synthese*, 46 (1981): 247–63; also Michael Friedman, *Kant and the Exact Sciences* (Cambridge, Mass.: Harvard U.P., 1992), Chap. 1.

[28] I cannot discern, from the terminology used, how much after fall 1900 these

at two passages in the text not present in the manuscript, and also at a leaf, folio 54a, which appears to be a later addition to the manuscript.[29] All three make essentially the same claim, a claim not made clearly elsewhere in the extant manuscript. The point is this: given the purely formal way in which the primitive terms are employed in pure mathematics, it is more perspicuous to think of them as *variables*. In the new material at the end of §357, Russell writes,

In Universal Algebra, our symbols of operation, such as + and ×, are variables, the hypothesis of any one Algebra being that these symbols obey certain prescribed rules. (*PoM*, p. 377)

The new footnote on page 384 refers the reader ahead to Chapter 49 ("Definitions of Various Spaces"), with the indication that primitive terms will there be regarded as variables in definitions.

The newly inserted leaf, 54a, constitutes §378, and is the most interesting and detailed of the three. Immediately prior to this section, Russell presents two approaches to descriptive Geometry, one due to Pasch and Peano, the other to Vailati. They differ in the concepts taken as primitive; at the end of the earlier section, Russell writes, "It is most important to observe, that in the above enumeration of fundamentals, there is only one indefinable, not two as in Peano's system" (*PoM*, p. 396). The beginning of the new section says that we can go one very large step farther than this, and "dispense altogether with indefinables." And, this is not merely a feature of descriptive Geometry; the method can be applied to any other mathematical theory, except logic itself. The proposal is to take the axioms, replace the non-logical constants in them by variables and to regard the result as the definition of a certain kind of structure: "The axioms then become parts of a definition, and we have neither indefinables nor axioms" (*PoM*, p. 397). On this view, the propositions of pure mathematics are generalized implications, whose quantifiers range over logi-

passages were added. Since one finds a comparable description in the June 1901 manuscript of Part II, it seems likely to me that it may well date from a rewriting done around that time.

[29] The new material is at 377: 3–18 and the new footnote at 384: 38–9. See the List of Variants.

cal entities, such as classes and relations. The antecedents may be regarded as defining a class of logically characterizable structures. This is the conception of the propositions of pure mathematics that Russell sets out in the very first section of *Principles*.

As so expressed, this view is obviously quite similar to contemporary versions of structuralism; for example, that advocated by Geoffrey Hellman in *Mathematics without Numbers*.[30] In the case of arithmetic, Hellman's proposal is that pure number-theoretic statements be construed as elliptical for statements as to what would be the case in any structure of the appropriate sort. Since Hellman wants to provide an alternative to "objects-Platonism", he needs to bring out explicitly the modal element present in the use of "would" in the previous sentence. Thus, if B is a pure number-theoretic statement and PA^2 is a conjunction of the axioms of second-order Peano arithmetic, B is to be regarded as elliptical for the generalized modal conditional:

$$\text{H. } \Box\forall X\forall f(PA^2 \supset B)^X(f/s)$$[31]

Here, "s", the function constant for the successor function, is to be replaced throughout by a functional variable "f". The open sentence "$PA^2(f/s)$" defines a class of structures, what Russell, in Part IV of *Principles*, calls progressions. Russell, of course, has no need for the initial modal operator for two reasons. First, he countenances logical objects, and second, his quantifiers already include merely possibly existent objects in their range.

Statement H represents what Hellman calls the hypothetical component of his modal structuralism. But for paraphrases like H to impart the proper truth values, it must be the case that it is possible for their antecedents to be satisfied. This is what Hellman calls the categorical component of his modal structuralism. In the case of arithmetic, it amounts to establishing:

$$\text{C. } \Diamond\exists X\exists f(PA^2)^X(f/s)$$

[30] Geoffrey Hellman, *Mathematics without Numbers* (Oxford: Oxford U.P., 1989).

[31] Here the superscript "X" indicates that the first-order quantifiers are to be restricted to "X".

Quite strikingly, Russell sees this as the second crucial component of his own logicist view. In the final section of the main text of *Principles*, Russell sums up what he thinks he has accomplished, as follows:

> ... it was shown that existing pure Mathematics (including Geometry and Rational Dynamics) can be derived wholly from the indefinables and indemonstrables of Part I. In this process, two points are specially important: the definitions and the existence-theorems.... The existence-theorems of mathematics—i.e. the proofs that the various classes defined are not null—are almost all obtained from Arithmetic. (*PoM*, p. 497)

Russell's existence-theorems are of the same form as sentence C, without the initial modal operator. In the context of *Principles*, Russell might well have thought that even the existence-theorems were ultimately of conditional form, since *Principles* proposes to define negation from universal quantification and conditionality. This would yield a definition of the existential quantifier, one that actually appears in a limited context in *Principles* (see *PoM*, p. 21).

Russell's view is therefore similar in kind to the sort of philosophical view often associated with Hilbert's conception of Geometry in *The Foundations of Geometry* (1899). Famously, there Hilbert says that a certain group of axioms *define* the concept "between". Frege argues that in fact the proper way to understand this sort of remark is that the axioms constitute a definition of a second-level concept, such as that of a "Euclidean space".[32] Russell had not read Hilbert's *Foundations* when he wrote the fall 1900 manuscript. No references to Hilbert occur in the extant manuscript. He read it in German in February 1901, but the three footnotes in Part VI comment on strictly mathematical matters.

I turn next to a discussion of how the conditional logicism (as Alberto Coffa has called it) advocated in Part VI is connected with the familiar explicit logicist definitions of Part II. As I noted earlier, the explicit definitions do not occur in the fall 1900 manuscript. They first appear in the June 1901 version of Part II, and are roughly contemporaneous with Russell's realization that what he had called the axiom of

[32] Frege, "On the Foundations of Geometry", in Frege, *On the Foundations of Geometry and Formal Theories of Arithmetic* (New Haven: Yale U.P., 1971), pp. 36–7.

abstraction in the fall 1900 manuscript could be proved, using equivalence classes under a relation.[33]

Various commentators have noted these two "strains" or forms of logicism in *Principles* and have sought to explain how they are related. I think that none of them have discerned quite rightly how Russell sees the forms as fitting together. I'll begin by discussing two of these and what I take to be their shortcomings.

In "The Thesis That Mathematics Is Logic", Putnam distinguishes the views, calling conditional logicism "if-thenism", and calling the explicit definitions "Logicism". He writes:

> Before he espoused Logicism, Russell advocated a view of mathematics, which he somewhat misleadingly expressed by the formula that mathematics consist of "if-then" assertions. What he meant was not, of course, that all well formed formulas in mathematics have a horseshoe as the main connective! but that mathematicians are in the business of showing that *if* there is any structure which satisfies such-and-such axioms (e.g. the axioms of group theory), *then* that structure satisfies such-and-such further statements (some theorem of group theory or other).[34]

As these textual studies show, Putnam is right that Russell espoused conditional logicism before he developed the explicit definitions that Putnam calls "Logicism". (It is unclear what basis Putnam had for this chronological claim.) But the first precedes the second by only about seven months. Putnam's second sentence is an accurate depiction of the structuralist character of Russell's conditional logicism.

Putnam holds that Russell rejected "if-thenism" and came to embrace "Logicism" because the former could not give an adequate account of the applications of mathematics. This difficulty affects only the understanding of *applied* mathematics, according to Putnam. Putnam notes that certainly, an applied statement, such as "The number of planets is nine", is not a universally generalized logical truth, since, among other things, it is not a logical truth. According to Putnam, Russell's response is to "abandon" conditional logicism in favour of the explicit definitions: "... since the truth-value of these applied state-

[33] *Papers* 3: xxvii, 320, 593.

[34] Putnam, "The Thesis That Mathematics Is Logic", in his *Philosophical Papers*, Vol. 1: *Mathematics, Matter, and Method* (Cambridge: Cambridge U.P., 1975), p. 20.

ments must be well-defined, a particular model—*the* standard model—must be fixed once and for all, as the one to be used in interpreting *Principia*" (*ibid.*, p. 31). I think that there is much that is correct in what Putnam has to say, but it is incorrect, as regards *Principles*, to see Russell as "abandoning" conditional logicism in favour of the explicit logicist definitions.[35] Both remain in play throughout the text, including Russell's discussions of arithmetic, the case Putnam is considering. In Chapter 29 of Part IV, Russell presents the theory of progessions, basically Dedekind's account of simply infinite series. Progressions are defined in purely logical terms, and the truths of Arithmetic hold of any progression. Russell also introduces what he calls the logical conception of numbers into this discussion; this involves the explicit definition of "cardinal number". And while Russell does, at that point, discuss the relations between these approaches, he does not claim that one supersedes the other, even in the case of Arithmetic. I will discuss what Russell *does* say at this point shortly.

Secondly, it is clear that Putnam's conception is not appropriate for other branches of mathematics to which Russell certainly intended his ideas to apply. Consider group theory. (The original version of "The Logic of Relations" had a section on group theory.) Suppose we have an application of group theory in some scientific context. It would seem misguided to respond to this by "abandoning" the conditional treatment of group theory, trying to find a logically definable model of group theory, and then promoting it as *the* standard model of group theory. Russell, of course, never does any such thing.

In Coffa's "Russell and Kant", he correctly notes that there is no contradiction between conditional logicism and what he calls "standard" logicism. He also notes the fact that Russell holds both doctrines together in *Principles*. Coffa develops the important point that conditional logicism is central to Russell's criticism of Kant, although he also raises concerns as to whether this idea is so broadly applicable as to trivialize logicism. On Coffa's view, the two doctrines play "complementary" roles:

[35] In fairness, when Putnam discusses the "abandonment" of conditional logicism, he probably has in mind the discussions in Chapters 1 and 2 of *Introduction to Mathematical Philosophy*. See the first sentence in the first full paragraph on p. 31 of "The Thesis". I am limiting my attention here to *Principles*.

roughly speaking, those mathematical theories for which there appeared to be no alternatives (i.e. arithmetic) were to be reduced to logic in the standard sense; those for which there were colegitimate alternatives (e.g. geometry) were to be reduced to logic only in the conditional sense. ("Russell and Kant", p. 252)

Such a view might be suggested by Russell's reflections in the 1937 Introduction to *Principles*; there he writes:

I was originally led to emphasize this form [the conditional form] by the consideration of Geometry. It was clear that Euclidean and non-Euclidean systems alike must be included in pure mathematics, and must not be regarded as mutually inconsistent; we must, therefore, assert only that the axioms imply the propositions. (*PoM*, p. vii)

But no such division of labour is to be found in the text of *Principles*. Russell simply does not say that it is because there is no sensible alternative to arithmetic, that he is therefore supplying explicit logicist definitions of its fundamental concepts. In Chapter 29, for example, the two approaches are discussed together in the context of Arithmetic. Russell does not set the conditional approach aside or indicate that it is secondary or less important in the case of Arithmetic. He in fact describes their relation differently.[36]

In my opinion, Russell offers a clear and reasonable answer about how the approaches are related in the case of Arithmetic. Here is what he writes in discussing how the theory of progressions (Arithmetic) is related to the "logical theory of numbers":

[36] I don't think Peter Hylton addresses the question I am considering in *Russell, Idealism and the Emergence of Analytic Philosophy* (Oxford: Clarendon P., 1990). On pp. 190–1, he contends that the initial characterization of "pure Mathematics" as a class of generalized implications misrepresents Russell's own considered view; some mathematical statements are existential, for example. Even if some mathematical statements were not conditional, we would still need an account of how these statements are connected with the explicit definitions. As noted earlier, at the time of *Principles*, Russell held that all logical connectives, including the existential quantifier, could be defined from the conditional and universal quantification. See my "Russell, Logicism, and the Choice of Logical Constants", *Notre Dame Journal of Formal Logic*, 30 (1989): 343–61. Similarly, in *Russell's Hidden Substitutional Theory* (Oxford: Clarendon P., 1998), Gregory Landini discusses conditional logicism primarily to dismiss it in favour of the project Landini calls "arithmetization" (see pp. 19–21).

It is numbers so defined [as equinumerous classes] that are used in daily life, and that are essential to any assertion of number. It is the fact that numbers have these logical properties that makes them important. But it is not these properties that ordinary mathematics employs, and numbers might be bereft of them without any injury to the truth of Arithmetic and Analysis. What is relevant to mathematics is solely the fact that finite numbers form a progression. (*PoM*, p. 241)[37]

So, in the case of the natural numbers, it is issues concerning application that the explicit definitions address. Putnam is right about the centrality of this issue. Numbers, Russell claims in Part II, apply essentially to classes. In an assertion of number, a class is asserted to belong with other classes on the basis of the equinumerosity relation. The explicit definition of "cardinal number" makes plain how this central kind of application works.

However, Putnam is wrong in thinking that this leads Russell to abandon conditional logicism. Rather, Russell sees the numbers, which are used in such assertions, as an *application* of Arithmetic, the general theory of progressions. It is the fact that the numbers used in daily life form a progression that makes Arithmetic important or interesting to us. Nor, I think, does Russell single the numbers of daily life out as somehow *the* standard model or interpretation of Arithmetic. Russell's account here does not suggest exclusivity. There might, as far as Russell's account goes, be other, rather different kinds of progressions, such as the ordinal numbers or the numerals, that might be of importance to us for some other reasons.

Russell's view has several clear virtues. It does not build some one particular application of Arithmetic into the very structure of Arithmetic. Here Russell's approach compares favourably, in my judgment, with Frege's.[38] In *Principles*, Russell clearly recognizes the multiplicity

[37] This text occurs in a portion of Part IV for which the manuscript is missing. We thus do not know precisely when Russell wrote the quoted material. The contrast with explicit nominal definitions was probably not present in the fall 1900 manuscript. I should also note that Russell expresses what I take to be the same view in Part II at pages 126–7, though I do not find what he says there to be as full and direct as the claim from Part IV.

[38] I am sympathetic with William W. Tait's criticism of Frege's approach in this regard. See his "Frege versus Cantor and Dedekind on the Concept of Number", in Tait, ed., *Early Analytic Philosophy: Frege, Russell, Wittgenstein* (Chicago: Open Court,

of structures that may satisfy a given theory. Note, for example, the discussion of models of projective Geometry. His discussion makes clear that different models may be useful for different purposes.

An important feature of the "logical" application of Arithmetic is that it is both central to an important use of Arithmetic and specifiable in purely logical terms. In this respect, the case of Arithmetic differs from the case of Geometry. In the case of Geometry, there are models that are purely logically specifiable; those, for example, that treat "points" as pairs of real numbers. There are also models that are central to our use of Geometry, ones where "point" is a predicate of certain spatial entities. But there appears, in this case, to be no model that has both properties.

Russell's discussion of Arithmetic and Analysis differ markedly in this respect. Russell doesn't identify a logically specifiable model of Analysis that illuminates a central use of Analysis. Rather, in the case of Analysis, Russell's concern is quite different. Here the problem to be addressed was the nature and intelligibility of the concept of continuity. Russell's concern, like Dedekind's, was to show that continuity, and the system of real numbers embodying it, could be explicated coherently and independently of spatiotemporal notions. So, for Russell, the role of the explicit definition of "real number" is simply to supply an "existence theorem" for Analysis that does not rely on spatiotemporal notions. No attempt is made to identify a set of logical definitions that will explicate central applications of Analysis.

Russell's logicism is not simply broader than Frege's, embracing logicism, at least in the conditional form, with respect to Geometry and Rational Dynamics, as well as Arithmetic and Analysis. It also, in my judgment, sees the relation between mathematics and its applications somewhat differently. As I have suggested, Russell takes central applications of Arithmetic as external to the pure theory of Arithmetic itself. Of course, these points are not unconnected. Frege, who sees a particular kind of interpretation as essential to Geometry, regards Geometry as non-logical. On the other hand, as these textual studies show, for Russell, Geometry is where logicism began.

1997), p. 229.

VARIANTS BETWEEN *The Principles of Mathematics*, PART VI, AND ITS MANUSCRIPT

PART VI. SPACE

CHAP. XLIV. DIMENSIONS AND COMPLEX NUMBERS.

371: 11 of finite numbers] of numbers

371: 28 of progressions or finite series respectively] of progressions, respectively

371: 30–1 continuity, except ... *all* classes.] continuity.

372: 44 nineteenth] present

374: 7 forms] form

374: 15 or upon the nature] or the nature

374: 16 on] upon

374: 36 belongs to the field of one] belongs to one

374: 37 class u_1 of serial relations.] class of series u_1.

374: 37 That is if] That is, if

374: 38 of the field ... u_1.] of some series which itself belongs to a class u_1.

375: 5 serial relations.] series. <*Also at 375: 13, 375: 16.*>

375: 6 the field of x_1,] the series x_1,

375: 6–7 serial relation,] series,

375: 8 be always a relation generating a simple series.] be always a simple series.

375: 9 belonging to the field of any serial relation,] belonging to any series

375: 11–12 belongs to the field of some serial relation,] belongs to some series

375: 14 belong to the field of only one serial relation] belong to only one series

375: 21 In the first place, we have just seen that alternative definitions] In the first place, alternative definitions

375: 38–9 if n be finite, or any infinite ordinal number, u_n can] if n be finite, u_n can

375: 40 discovered, for finite numbers or ω, by Cantor] discovered by Cantor

375: 41–5 <*fn. added*>

376: 5 any u_m will belong to some simple series of series u_{m+1}; and] $_{n-m}$ will be a class of series leading to a new class u_{n-m-1}; but

376: 14 Chapter XXIV] Chapter I

376: 32 examination] investigation

376: 36–7 Hamilton, De Morgan,] Hamilton, Boole, De Morgan

376: 38 add Boole and Grassman.] add Grassman.

376: 41 cannot, in my opinion,] cannot at present, in my opinion

377: 3–18 work. The possibility ... these properties.] work.<*end ¶*>

378: 31–2 definition, then the logical ... substitute.] definition, then the interpretation of the definition in terms of the theory of dimensions.

378: 45 Stolz, *ibid.*] Stolz, *Allgemeine Arithmetik*,

379: 7–36 360. The above ... various cases.] With regard to interpretation, it is plain, to begin with, that the class of all numbers a is a series of n dimensions. We may first form a simple series of classes of complex numbers by variations of α_1. Each member of this series of classes is a new simple series of classes by variation of α_2, and so on; until finally, the class of complex numbers in which α_1, α_2, ..., α_{n-1} are given, andonly α_n remains variable, is a simple series of complex numbers. Thus our complex numbers come within the definition of an *n*-dimensional series. In the second place, it is plain that our numbers are capable of the following interpretation: Let e_1, e_2, ... e_n be definite magnitudes of different kinds, but each of a numerically measurable kind. Then $\alpha_1 e_1$ will represent some magnitude of the same kind as e_1, and *a* will represent the assemblage of *n* different magnitudes, each of a distinct specified kind. Or, more generally, let e_1 be a specified term of some class having a one-one correspondence with the real numbers (or with the rationals or the integers, as the case may be), and let e_2, ... e_n be specified terms of other classes having the same property. Then making each of the specified terms correspond to 1, $\alpha_1 e_1$ will be the term of the same class as e_1, and corresponding to α_1, and so on; while *a* will be the assemblage of such terms. That is, *a* will represent (in case the α's may take all real values) a selection of one term each from *n* different continua; and every possible selection will be represented by one and only one value of *a*.

379: 26 <*"real" is a misprint; it should read "complex".*>

379: 37 In order that the complex numbers in the sense defined by Stolz should] In order, however, that complex numbers should

379: 38–9 considering assemblages] considering such assemblages

379: 40 in a metrical space] in a space

379: 40–1 to a circumstance which is essential to] to a further circumstance, essential to

379: 42 entities (points) be given] entities be given

379: 43–4 relation (distance),] relation,

380: 6 to a plurality of dimensions] to units—as many units, in fact, as our numbers have dimensions

380: 10–11 <*fn. added*>

CHAP. XLV. PROJECTIVE GEOMETRY.

381: <*chapter title*> Geometry] Space

381: 3–6 that results ... those axioms.] that inconsistent results followed from the denial of those axioms.

382: 5 All Geometries, as commonly developed agree] All Geometries agree

382: 40 requisite type of mutual relations,] requisite relations,

383: 31–2 same results. Since] same results. The two are strictly speaking inconsistent, since it must happen that either *R* is derived from *K*, or *K* from *R*. But as they lead to the same results, the choice is of no practical importance. Since

384: 25 and von Staudt's] and Staudt's

384: 26 with *c* but not on *ab*.] with *c*.

384: 33 <*fn. to the word "uniqueness" in text, to the word "range" in ms.*>

384: 38–9 <*fn. added*>

384: 43–5 <*fn. added*>

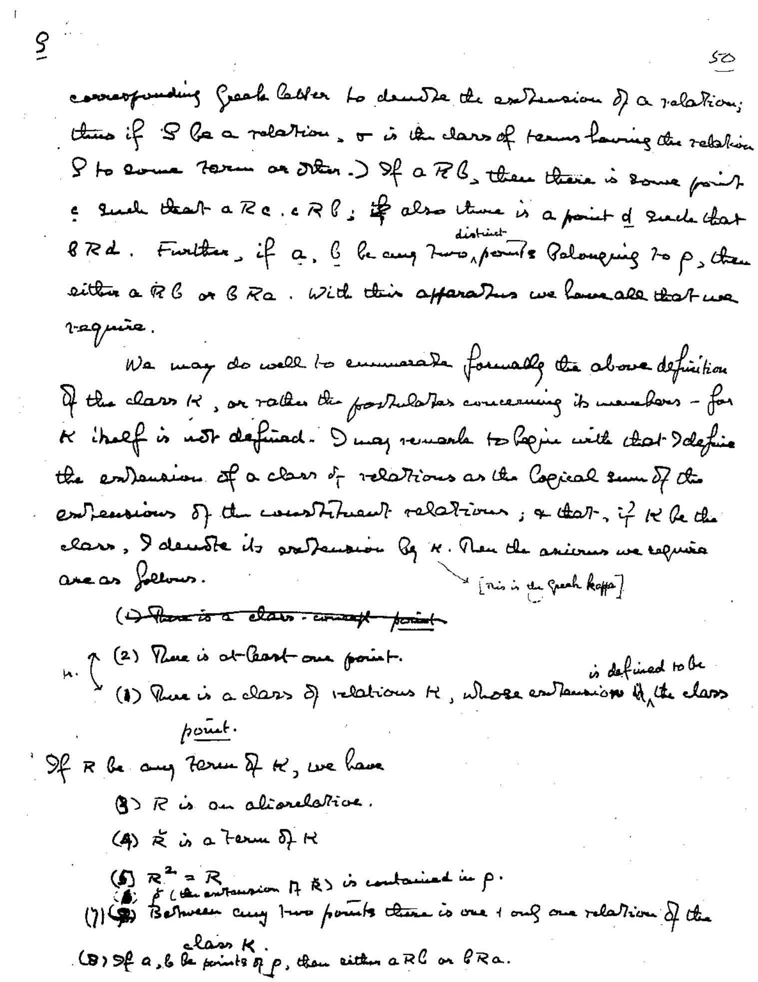
385: 19 <*Misprint in text: "an harmonic" should be "anharmonic".*>

385: 39–40 him. A simpler ... points.] him. <*end fn.*>

385: 44 <*fn. added*>

385: 45–6 7. Pieri's method ... No. 216.] 7. <*end fn.*>

386: 13 Chapter XXIV] Chapter I <*Also at 387: 7.*>


9 50

corresponding Greek letter to denote the extension of a relation; thus if ϱ be a relation, σ is the class of terms having the relation ϱ to some term or other.) If a R b, then there is some point c such that a R c . c R b; ~~if~~ also there is a point d such that b R d. Further, if a, b be any two distinct points belonging to ρ, then either a R b or b R a. With this apparatus we have all that we require.

We may do well to enumerate formally the above definition of the class K, or rather the postulates concerning its members – for K itself is not defined. I may remark to begin with that I define the extension of a class of relations as the logical sum of the extensions of the constituent relations; & that, if K be the class, I denote its extension by κ. Then the axioms we require are as follows. [This is the Greek kappa]

(1) ~~There is a class-concept point~~

(2) There is at least one point.

(1) There is a class of relations K, whose extension is defined to be the class point.

If R be any term of K, we have

(3) R is an aliorelative.

(4) $\breve{R}$ is a term of K

(5) $R^2 = R$

(6) ρ (the extension of R) is contained in ρ.

(7) Between any two points there is one & only one relation of the class K.

(8) If a, b be points of ρ, then either a R b or b R a.

Fol. 50 of the manuscript of *Principles*, p. 395 lines 22–44, showing Russell's (unheeded) direction to the printer re the Greek kappa.

387: 10 $aT_{bc}d$, $aT_{bd}c$, $aT_{cd}b$.] $aT_{bc}d$, $bT_{ca}d$, $aT_{bc}d$. *<The ms. is in error; interchanging "a" and "c" in "$aT_{bc}d$ rectifies the problem.>*

387: 36 that, when *h* and *k* are given, there can be] that there can be

387: 41–2 variable. This property] variable. In Schröder's notation, the property is expressed by the subsumption $T_{ac} \subset T_{ac} + T_{ac}$. This property

387: 43 the transitive relation Q_{ac}.] a transitive relation. $Q_{ac}^2 \subset Q_{ac}$ leads to the above equation immediately by contraposition.

389: 42 where proofs] where strict proofs

392: 25 Chapter XLIV] Chapter I

392: 26 it would seem (though this is only a conjecture) that] it would seem that

CHAP. XLVI. DESCRIPTIVE GEOMETRY.

393: 20 is in general not unaltered] is not necessarily unaltered

394: 7 Logic of Relations.] Algebra of Relatives.

394: 20 *c* is between] *c* between

394: 40 domain] extension *<Also at 395: 23, 41.>*

394: 44 † *Ib.* IV, p. 55 ff.] *Riv. di Math. IV, p. 55ff.

395: 24 *aRc* and] *aRc*.

395: 31 field] extension *<Also at 395: 33, 395: 35; 396: 20 (twice), 396: 25, 398: 5.>*

395: 32 fields] extensions *<Also at 396: 24.>*

395: 33 *k*.] κ. *<Greek kappa; also 3 times at 397: 27–8; see also 397: 13.>*

397: 12–13 (*a*). If we combine … false. Nevertheless] (*a*). I know of no illustration of the falsity of (*b*) only; the nearest approach is in the antipodal form of elliptic space, but here the line is a closed series, so that other axioms fail. Nevertheless

397: 13 $k_1 = k_2$] *<a possible misprint for* $\kappa_1 = \kappa_2$*>*

397: 17 all except one] almost all

397: 18 of this one] of the remainder

397: 28 concept] term

397: 39 importance. The two] importance. But philosophically, the two definitions are inconsistent, that is, where one is correct, the other is not so, and *vice versa*. Any given relation is either simple or complex, and a space in which two points have a simple relation, by means of which the relation of the couple to their straight line is defined, is different from one in which this latter relation is simple, and is used to define the relation between the two points. The two

397: 44 chap. XXIV] chap. I

399: 43–4 dimensions, unless … axiom.] dimensions. Spaces of an even number of dimensions have no projective properties not possessed by spaces of one less dimension; thus four dimensions give nothing of any interest. Five, or seven, would give some new properties, but these are not philosophically important. *<end fn.>*

400: 32 the cardinals or the ordinals] the natural numbers *<Also at 400: 33.>*

401: 13 itself, and so,] itself. And so,

401: 27–8 ideal lines. For this … *Ebenenbüschel*). An] ideal lines, *i.e.* pencils of planes, axial pencils, *Ebenenbüschel*). An

402: 21 or through an ideal line] or an ideal line

402: 41–3 *<fn. added>*

403: 29–37 chapter. ¶It … notions.] chapter. *<new ¶ added>*

CHAP. XLVII. METRICAL GEOMETRY.

404: 32 Section 5.] Section, § 8.

405: 31 *DE*.] *EF*.

405: 42–4 *<fn. added>*

407: 16 this proposition] it

407: 17–18 three non-intersecting lines] three lines

407: 24 Chapter XXIV] Chapter I

407: 38 *i.e.* such that numbers] *i.e.* numbers

408: 17 stretches, while (2)] stretches. (2)

408: 27–34 (8), applied … this work.] (8), not a new indefinable. But on this point either course is logically permissable.

408: 38–9 a series with certain properties.] a series

408: 40 Chap. XXII] Chap. IV

408: 44 Chap. XXXI] Chap. VIII

409: 26 three collinear points] three points

410: 9 need a method of defining the straight line.] need a relation distinct from distance in order to define the straight line.

410: 10–411: 5 line. Pieri … important.] line. It is only on a given straight line that two and only two points have a given distance from a given point, and we must have a definition of the straight line which is independent of distance. Some readers may suppose that the straight line could be defined by $AB \pm BC = AC$, i.e. by the fact that, if *A*, *B*, *C* be three points, the distance *AC* is never equal to the sum or difference of the distances *AB*, *BC* unless *A*, *B*, *C* are collinear. If distances were given, to begin with, as numerically measured, such a definition of the straight line would do well enough; for example, we may define in this way a class of complex numbers which may be called a straight line by analogy. But in the present case, both addition and numerical measurement are derivative: we have to assign to points such properties as will make distances additive and measurable. Now addition is fundamentally of two kinds: that of classes, which is the logical kind, and that of individuals, which underlies Arithmetic. Hence we have to make one or other of these kinds applicable in the case of distance. We have seen, in Parts III and IV, how both will apply to stretches; if, then, distance is a relation by which series of points are generated, all will be well. But distance alone generates, not a series of points, but a series of spheres. Hence it is absolutely necessary that the straight line should be generated independently of distance, that there should be only two points on a given line at a given distance from a given point on the line, and that distance should obey the five axioms enumerated in the preceding paragraph. We then have stretches corresponding to distances, and we can render distances *on a line* asymmetrical by reference to the series of points on the line. Thus *AB*+*BC*, where *AB* and *BC* each have sign, will be *defined* as the distance *AC*. We can then proceed to the numerical measurement of all distances on one line.

(Observe that the sign of distances applies only to a single line.) On another line we can make a similar measurement. Without introducing signs, by means of the axiom that, if $BC = BC'$, AC is greater and AC' less (or vice versa) than AB or BC, we can define $AB+BC$ as the greater of AC, AC', and $AB-AC$ as the less. But our store of axioms (if distance is to be distinct from stretch) must be increased still further. ¶If A, B, C, D be collinear, and B be between A and C, DB (regard being had to sign) must be greater than DA and less than DC. Also, if A', B', C' be points on a different line, B' being between A' and C', and $A'B' = AB$, $B'C' = BC$, then we must have $A'C' = AC$. This is a new axiom, permitting, at last, a numerical measurement of all distances by the same standard. In this case, if A, B, C, D, E, F be any six points, the equation $AB+CD = EF$ must be interpreted as follows. Let D' be any point on the line AB, let B be between A and D', and let $BD' = CD$; then we shall have $AD' = EF$.

410: 41 *<fn. added>*

410: 42 *<fn. added>*

410: 43 *<fn. added>*

410: 44–5 *<fn. added>*

410: 45 *<fn. added>*

411: 14 We define] We now define

411: 23 is the divisibility of] is that of

411: 44–5 *<fn. added>*

411: 34–413: 4 *<§397 is absent from the ms.>*

413: 6 leads] would lead

413: 7 We now no longer] We should now no longer

413: 8 we have] we should have

413: 10–37 awkward. We may ... distance. ¶I shall not work out] awkward. I shall not work out

414: 5–6 constant. It is ... divisibility. The divisibility] constant. The divisibility

414: 14–453: 31 *<Folios 82–168 of the ms. of Part VI are missing. They include all of* CHAP. XLVIII. RELATION OF METRICAL TO PROJECTIVE AND DESCRIPTIVE GEOMETRY, *pp. 419–28;* CHAP. XLIX. DEFINITIONS OF VARIOUS SPACES, *pp. 429–36; and* CHAP. L. THE CONTINUITY OF SPACE, *pp. 437–44.>*

CHAP. LI. LOGICAL ARGUMENTS AGAINST POINTS.

453: 35 supposition. But] supposition; but

453: 43–4 pew-openers; they ... entities.] pew-openers; they do not *have* positions, since they *are* positions; and like all other entities, they eternally have the relations which they have.

454: 1 seems to be designed] seems designed

454: 5 psychological] subjective

454: 6–7 necessity of thought.] "necessity of thought".

454: 9 to the curious] to a somewhat curious

454: 10 What we cannot help believing, in this case, is something] What we cannot help believing (except, it should be added, after long practice in the Kantian philosophy) is, in this case, something

454: 15 the constitution of our minds remains] the constitution of our minds (however Kantians may protest the contrary) remains

454: 20–1 the premisses ... assumed.] the premisses themselves, and the logical rule or rules by which the deduction is effected, have to be simply assumed.

454: 21 Thus any ultimate premiss] Any premiss

454: 25–33 and concerning ... imaginations.] and unless the truth or falsehood in question is one of those that can be deduced from other truths or falsehoods, there is nothing further to be said. Necessity seems, in fact, to be mainly psychological. There are some propositions—notably those implied in a large number of other propositions which we believe—that seem almost impossible to doubt. Such we call necessary propositions*; *<fn. 454: 45>* and such are some of the propositions of Geometry. Thus there seems no valid reason for inferring, from our inability to imagine holes in space, that there can be no space except in our imaginations.* [*It should be observed that in a theory of space which eschews distance, and uses only stretch, holes in space are self-contradictory. It is only by making distance independent of stretch that holes become logically possible. See an article by the present author on "The Notion of Order", Mind, N.S. No. 37, p.]

455: 5 among points is rejected] among the points of one space is rejected

455: 5–6 as the assertion ... because both are] as we reject the denial that any proposition must be true or false, namely because both denials are

455: 9–10 The point, in fact, would] Like anything else that is capable of coming into existence, the point would

455: 15–28 ¶I conclude ... terms.] ¶We have now refuted the two current arguments against absolute space, namely that derived from the difficulties of continuity and infinity, and that derived from the identity of indiscernibles and the subject–predicate theory of propositions. It will not be difficult, from this standpoint, to refute Kant's antinomies. But owing to their importance for the development of philosophy, it will be well to devote a chapter explicitly to the examination of them.

CHAP. LII. KANT'S THEORY OF SPACE.

457: 44 † See ... 18.] *See e.g. Peano, Riv. di Mat., Vol. VII, No. 1. (An instance of such a rule is e.g.; "If A implies that B implies C, then A and B together imply C."

458: 9 Chapter XLIX.] Chapter VI.

458: 39–40 also in Part V (especially in Chapter XLII) that] seen also that

459: 16–40 *<Folio 181 is missing.>*

459: 45–460: 1 chapter, ... arguments;] chapter, and will be disproved, as regards time, in the following chapter;

460: 37 the stretch of ratios between] the stretch between

461: 7 of real numbers] of numbers

461: 20–1 because ... order.] because it enables us to extend the field of numbers so as to make all equations soluble, and because it gives us new methods of generating order.