FREGE, PEANO AND RUSSELL ON DESCRIPTIONS: A COMPARISON

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The main thesis of this paper is that some of the most important ideas and symbolic devices that made Russell's theory of descriptions possible were already present in writings by Frege and especially Peano that Russell knew well. The paper contains a detailed comparison between the relevant parts of Russell's theory—including manuscripts recently published—and some of Frege and Peano's insights, as well as a discussion of numerous possible objections that could be posed to the main claim. Even if Russell was not actually influenced by those insights, the parallelism is close enough to be worth analyzing, especially in the case of Peano, whose writings are not very well known.

I. INTRODUCTION

We can reduce the essentials of Russell's theory of descriptions—from the viewpoint of definability—to three claims: (1) the logical power of the definite article is reached only through the conditions of existence and uniqueness; (2) it can therefore be eliminated, from the whole expressions where it is *used*, in terms of these two conditions; (3) in cases where these conditions are not met, the

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sentence in which the descriptive phrase occurs has to be rejected as false. I will maintain here that (1) can be clearly found in Frege and Peano, that (2) was almost admitted by Frege and was admitted explicitly—including the symbolic expression—by Peano, and that (3) was partially stated by both, although for them sentences containing descriptions without involving existence or uniqueness (or both) also have to be rejected, but as meaningless rather than false.[2]

Almost nothing of this seems to have been noticed to date. Usually, the precedent of Frege is admitted only in a limited way, and Peano is presented as the mere inventor of a convenient notation for the descriptor. We shall see in the following how important was the conceptual role of Frege, and how Peano handled symbolic devices which are very similar in practice to Russell's famous definition.

2. FREGE'S EVOLUTION COMPARED WITH RUSSELL'S THEORY

Frege's view on descriptions evolved through three stages: *1884a*, *1892a* and *1893a*, but we are usually told only about the third one, the most sophisticated (Largeault *1970a*, pp. 179ff.; Tichý *1988a*, pp. 120ff.). Sometimes we are told about the second (Walker *1965a*, pp. 47ff.), and sometimes about both the second and third stages (Mosterín *1968a*, pp. 40ff.). However, the first one is also interesting because it preceded the appearance of the great logical and semantic resources (the full formalization of logic, the sense–reference distinction), and it offers a simpler set of ideas, which are not dependent upon a great formal system already constituted.

Frege *1884a* starts from three principles, one of which was fundamental for the future development of the theory of descriptions: the meaning of words is not something isolated, but depends upon the context (*1884a*, preface and §60). This permits a solution of Frege's main problem in this work (the definition of number in logical terms) by means of paraphrastic devices, which presuppose the "elimination" of the expression "the number which applies to the concept *F*" (§62) by means of a bijective function (to use modern terminology), allowing us to reach the concept of number (see my *1987a*). However, to be able to maintain the view that the number is not a property of things, i.e. a concept, but an object, Frege emphasizes the fact that only the definite article (or a pronoun) allows us to transform conceptual terms into proper names (§51).

Frege directly examined descriptions when he defined 0 as the number which applies to the concept "unequal to itself" (*1884a*, §74). At this point Frege claimed the existence of that concept—in the same way that the concept "square circle" exists—but avoided the supposition that something falls under it, which would happen if we use the concept to define some object. Thus, for instance, Frege denies any content to the expression "the greatest proper fraction", for the article presupposes reference to a definite object, although the concept which results by dispensing with the definite article "is wholly unobjectionable" (§74). But if we want to use this concept to define an object, it will be necessary first to prove that it meets the two conditions of *existence* and *uniqueness*, i.e. "1. that some object falls under this concept; 2. that only one object falls under it." As 1 is false for the concept in question, "it follows that the expression 'the largest proper fraction' is senseless [*sinnlos*]" (*ibid.*).[3]

As for Russell's theory, we already can find here claim (1), and some elements of (3). In particular, for Frege the expression in question (a description without involving existence) must be rejected as senseless. There are two differences with Russell: first, the thing rejected by Frege is, to be exact, the description in itself, not the whole sentence contain-

with some friends and colleagues for a long time. In particular, I am grateful to Juan J. Acero, Mario Gómez, Ivor Grattan-Guinness, Nicholas Griffin, Gregory Landini, Willard V. Quine, Michael Resnik and Gary A. Wedeking for their comments on earlier versions, which made several improvements possible. I am also grateful to the retired Russell Archivist for help and information provided. At that time, some of the manuscripts which I needed to study were still unpublished, but now all but one have all appeared in Russell's *Collected Papers*. Finally, my thanks are also due to two anonymous referees for their useful remarks.

[2] Dummett (*1981a*, p. 30) rejected the attempt to present Frege as a predecessor of Strawson by attributing to him the thesis that the phrases containing descriptions without reference are neither true nor false. But, as we shall see, Frege spoke about them as meaningless (though that took place before his famous semantic distinction between sense and reference). Curiously enough, Russell maintained Strawson's famous thesis, too, but only in a manuscript written 47 years before, "On the Meaning and Denotation of Phrases" (*1903a*, *Papers* 4: 286). To my knowledge, Skosnik *1972a* was the first to point out this fact.

[3] Curiously enough, the final part of the footnote to §74 from Frege *1884a* is *not translated* in the partial English version in Benacerraf and Putnam *1983a*.

ing it, although it has to be supposed that Frege would also admit that this whole sentence is meaningless (for its logical subject was so). Second, although Russell described such descriptions as meaningless, he did so with all descriptions as well, for in his theory no isolated description has a meaning. Claim (2) is not explicitly present in Frege here, although it seems to underlie the proof that Frege requires, for the two sentences offered seem to be equivalent to the one to be analyzed, and therefore they have to be *true* if the original sentence is to be accepted as involving an actual object.

In *1892a* the theory was developed through the sense–reference distinction: every grammatically correct expression has sense, although not always reference, as for instance happens with descriptions like "the least rapidly convergent series" (*1892a*, p. 58). By applying this to descriptions playing the role of grammatical subjects—although as subordinate clauses—Frege provided some steps in the anticipation of the essentials of Russell's theory as stated above.

Frege writes that in the sentence, "Whoever discovered the elliptic form of the planetary orbits died in misery", the grammatical subject ("whoever") "has no independent sense and only mediates the relation with the consequent clause 'died in misery'." But its reference, that of a proper name, is not a truth value (like that of a statement), but a particular man: Kepler. Furthermore, the fact that such expressions have reference, i.e. that they designate an object, does not depend on their grammatical form, which presupposes that designation, but "on the truth of the sentence: 'There was someone who discovered the elliptic form of the planetary orbits'." This sentence, as in Russell's theory, merely expresses the conditions of existence and uniqueness;[4] this is why it guarantees—if true—that the corresponding description has a reference (*1892a*, pp. 68–9). In addition, Frege stated the need for stipulating, at least in a logically reconstructed language, a conventional reference—the number 0—for such apparent proper names (p. 71).

It might be objected that "whoever" is not a definite article, as is shown by the fact that Frege called it the "grammatical subject", so that Frege's later "replacement" is not like Russell's. This is true, but I think

[4] It could be objected that the English "someone" need not imply uniqueness, but it seems to me that the point is clearer according to Frege's original German words: "es gab *einen*, der die elliptische Gestalt der Planeten bahnen entdeckte" (my emphasis).

some of the important elements of Russell's replacement are present anyway, as can be seen if we imagine the original example as reading "The person who discovered ...", or even better "The discoverer of...." In these cases Frege's replacement can be done in exactly the same way: "There was someone who discovered...." Therefore, it is easy to see that if someone looking for a way to eliminate the definite article had read Frege's treatment of the question, that person is likely to have arrived at the idea of a general method for making a similar replacement, in terms of existence and uniqueness, in all sentences containing the definite article in the framework of a description.

Also, it could perhaps be objected that Frege did not propose any actual "replacement", but only said that the fact that certain descriptive expressions have reference does not depend on their grammatical form, but on the truth of certain sentences which clearly show that the conditions of existence and uniqueness have been met. A further sign of this non-replacement view would be that when he proposed the second sentence, he did not add the final clause "died in misery". This is again true. However, Frege actually said that in "Whoever discovered the elliptic form of the planetary orbits died in misery", the reference of the grammatical subject depends on the truth of "There was someone who discovered the elliptic form of the planetary orbits." So, *if* we replace that grammatical subject ("whoever") for the second, equivalent sentence in the original example, we obtain: "There was someone who discovered the elliptic form of the planetary orbits and he died in misery." And this is all we need to see how easy it might have been to imagine a general method for eliminating the definite article in studying Frege's examples.

In sum, Russell's three claims are all present, although in different degrees. I think that (1) is undoubtedly present, and in a clearer way than in *1884a*, for here a unique sentence is proposed to meet both conditions (existence and uniqueness) at the same time. For similar reasons (2) is to some extent admitted, at least in so far as the true reference of the description concerned depends on the truth of the same sentence, so the equivalence with the definite article, still implicit in *1884a*, seems to be made more explicit. The "in use" element appears through Frege's rejection of the isolated description as lacking sense by itself in the grammatical form. This supposes additional coincidences with Russell, who described all isolated descriptions as meaningless and thought that the sentence defining the definite article exhibits the true logical, as opposed to the grammatical, form of the original sentence.[5]

Thus, (3) is partially admitted: if either of the two conditions (existence and uniqueness) fails, then the proposed new sentence (which is the conjunction of both conditions) should be rejected as false. As Frege does not state clearly how this would affect the original sentence, it could be thought that this sentence would also be false *if* it can be admitted as really *equivalent* to the one—clearly expressing existence and uniqueness—proposed to replace it. If not, the only clue is that the original sentence would contain, in this case, a merely apparent logical subject with no reference at all, which is not enough, in my opinion, to give a precise response to this question in Fregean terms.

The similarity increases in relationship to *1884a*, for here the falsehood of the whole sentence (which is equivalent to the assertion of the two sentences from *1884a*) is the explicit condition for rejecting the apparent reference of the description. Finally, the entire new sentence looks very like the usual version of Russell's replacement sentence, which in this case would read more or less like: "There was one and only one who discovered the elliptic form of the planetary orbits, and he died in misery", and this sentence is almost exactly the same as Frege's.[6]

I come now to the differences pointed out by Mosterín *1968a* (pp. 40–1), the only explicit account I know. This author mentions three differences, which I will consider in the same order. Firstly, we are told, the logical relation between the sentence whose subject is the description and the sentence asserting existence and uniqueness is *implication* in Russell and *presupposition* in Frege. However, I cannot see in Frege *1892a* the explicit statement of this particular relation. The only similar thing he says is that in the assertion of a sentence the logical subject is presupposed to actually designate something. But this seems only to state that in ordinary language proper names (simple or complex) are supposed to have some reference, which is not exactly the same as saying that the original sentence *presupposes* the reconstructed one stating existence and uniqueness. In addition, Frege by no means tells us about an explicit *logical* relation called presupposition exhibiting particular properties.

[5] There is, however, some ambiguity in this element, for when Russell explicitly introduced the expression "in use" in *PM*, he was speaking about the need for inserting the whole description into a sentence. But it is also possible to take this expression into consideration when we speak about the need for considering the definite article in the context of the description itself. In Frege we can find both elements, but it is necessary to distinguish them in order to avoid possible misunderstandings.

[6] It can even be maintained that Frege implicitly anticipated Russell's distinction between primary and secondary occurrences, when he added its negation to the new sentence as follows: "Either whoever discovered the elliptic form of the planetary orbits did not die in misery or there was nobody who discovered the elliptic form of the planetary orbits" (*1892a*, p. 70).

On the other hand, Russell did not state *in 1905* that the relation in question is implication. He only spoke about "equivalent" sentences (in the sense that when one asserts the first, one is also asserting the second), and he adds that the proposed new sentence is an "interpretation" or "reduction" of the original one. It is true that the exact symbolic expression—missing in "On Denoting" (*1905c*)—is a *definition* (as it is stated in *1908a*) and therefore it is supposed that in some sense it states a mutual implication. But then there is no point in emphasizing the implication from the *definiendum* to the *definiens*, especially if we remember that Russell had available another, different notation for logical equivalence, and that he later insisted that his definitions were merely nominal.

Only in *Principia Mathematica* does the relation of implication appear in this context. There Russell clearly states that the original sentence "implies" the other three, in the sense that if any of them fails, then the original sentence is false (*PM*, 1: 68). As we have seen above, Frege did not precisely state this point, so that the difference is now undeniable. However, the important comparison concerns mainly the historical origin of Russell's theory of descriptions, i.e. his *1905c*, and as we have seen the similarity of Frege's views with this first article is closer, no matter how much Russell's ideas were modified later.

This leads us to the second difference mentioned by Mosterín. As a consequence of the first difference, we are told that in case the description fails in having an actual reference, the original sentence is false for Russell (for it implies a false sentence), but neither true nor false for Frege (for it presupposes a false sentence). In the former three paragraphs I have pointed out the difficulties of the two relations supposedly involved, but here I can add that Mosterín seems to depend rather on later developments than on the historical Russell and Frege themselves. Of course, I am thinking of Strawson *1950a*, who introduced for the first time a precise relation of presupposition in this context. In any case one can hardly explain historical differences between two authors by attributing to them precise distinctions proceeding from later developments.[7]

According to Mosterín's last difference, for Russell descriptions are incomplete symbols, to be eliminated through certain definitions, while for Frege they must be given the number 0 as their "reference", just as a (conventional) guarantee than even the most doubtful cases will not lack some reference. It is, of course, a real difference, but, as we have seen, the character of "incomplete symbols" for descriptions could easily be admitted by Frege, at least in Russell's precise sense according to which they have no meaning outside the context of a meaningful sentence.[8] In addition, the conventional reference is useful only within the context of very precise logical and mathematical needs, the only ones of interest for Frege. In any case, the use of the expression "incomplete symbol" in Russell is again later than the theory of descriptions *per se*, and involves many complicated arguments which cannot be easily compared with Frege's semantics (see my *1989a* for an analysis of these arguments).

At any rate, this possible parallelism seems to proceed from a similar view on definitions. In saying this, I mean that for both Russell and Frege there are simples and complexes, so that we can analyze the second only in terms of the first, which would be known to us by immediate intuition. In addition, both Frege and Russell thought that the task of constructing a logically ideal language that provides an account of the mathematical concepts in logical terms can—and must—be attempted, and this is precisely the result of the former belief, for logical primitives are to be understood as the simplest terms that a reductive analysis can reach.[9]

7 See my *1990a* for a discussion about the extent in which the relation of presupposition can be attributed to Russell. See also note 2.

8 Frege maintained a theory of "incomplete entities" in a more general sense, as Resnik *1965a* pointed out. This author clearly explained the relationship between Frege's difficulties for elucidating his notion of "function" as exhibiting an unsaturated character, as well as his need for admitting undefined primitives in his system (p. 338). However, I think that this has to be complemented with a global account of Frege's ontology in order to avoid the danger of thinking that this relationship involves the remaining primitive entities being unsaturated as well. Frege clearly stated that a function (as "the" is) cannot be regarded as an entity; it has not the status of a proper name, but it is only a *sign*: the *name* of a concept. According to Frege's ontology, there are only two *exclusive* classes of entities: objects and functions (which embrace concepts and relations), which are respectively designated by saturated expressions (names) and unsaturated ones (functional expressions).

9 Dummett (*1981a*, pp. 256–7) has denied the parallelism with regard to definability

In *1893a* Frege introduced a further, much more sophisticated theory of descriptions, especially devoted to the best way of introducing a representative of the definite article into his whole formal system. I will finish this section with a brief explanation of the essentials of the new theory, but only for the sake of completing the historical survey, as I think Russell was not influenced by it. The new theory is especially complicated because of the incorporation of the notion of *Werthverlauf,* or *course of values* $\grave{\varepsilon}\Phi(\varepsilon)$ of the function $\Phi(\xi)$, which is the ground of the whole construction.[10] Then Frege introduces a symbol of function, $\backslash\xi$, to replace the definite article of ordinary language, so that, as usual, we can transform conceptual words into proper names.

Two cases are to be distinguished, according to Frege: "1. If to the argument there corresponds an object Δ such that the argument is $\grave{\varepsilon}(\Delta = \varepsilon)$, then let the value of the function $\backslash\xi$ be Δ itself; 2. if to the argument there does not correspond an object such that the argument is $\grave{\varepsilon}(\Delta = \varepsilon)$, then let the value of the function be the argument itself" (*1893a*, §11). In this way $\backslash\grave{\varepsilon}(\Delta = \varepsilon) = \Delta$ is the True, and the reference of $\backslash\grave{\varepsilon}\Phi(\varepsilon)$ is the object falling under the concept $\Phi(\xi)$, in case there is one

by claiming that (i) Frege, unlike Moore, Russell and Wittgenstein, did not share "the fundamental theorem of logical analysis" (that every proposition admits only a unique and ultimate analysis into unanalysable constituents); (ii) only once did Frege speak of *logische Urelemente* (in *1906a*).

However, Dummett himself quotes a text by Frege (from *1892b*) where he denies that everything can be defined, in the same way as a chemist cannot decompose every substance. In this passage Frege wrote: "something logically simple is no more given us at the outset than most of the chemical elements are; it is reached only by means of scientific work"; then he added that once we reach it, the only thing we can do is "to lead the reader or hearer, by means of hints, to understand the word as is intended" (*1892b*, p. 42).

In addition, there exists a much earlier and more important place than the one from 1906 cited by Dummett, where Frege introduced the same doctrine; it is the introduction to the *Grundgesetze*: "It will not always be possible to give a regular definition of everything, precisely because our endeavour must be to trace our way back to what is logically simple, which as such is not properly definable. I must then be satisfied with indicating what I intend by means of hints" (*1893a*, §0, p. 32). This states, I think, the belief in *logische Urelemente,* as well as the belief in intuition as the only possible access to them, which constitutes a much stronger similarity with Russell than the supposed "theorem" mentioned by Dummett.

10 For the difficulties in that notion, see, for instance, Mosterín *1968a* (p. 43), and the introduction by Furth to Frege *1893a* (pp. xxxvii ff.).

and only one object falling under it, while in other cases $\backslash ἐ\Phi(\varepsilon)$ has the same reference as $ἐ\Phi(\varepsilon)$. So $\backslash ἐ\Phi(\varepsilon)$ always has a reference, "whether the function $\Phi(\xi)$ be not a concept, or a concept under which falls no objects or more than one, or a concept under which falls exactly one object" (*ibid.*).

It has been claimed that the former equivalence between a sentence containing a description and the two sentences exhibiting existence and uniqueness is made easier through the special symbol "$\backslash\xi$" (see Tichý *1988a*, pp. 120ff.). However, I think that Russell chose the former version (Frege *1892a*) as his conceptual starting point, so I do not need to devote further space to discussing the matter, or other differences that can be found in comparing this last stage with the former ones.

3. THE SYMBOLIC ELIMINATION OF "THE" IN PEANO AND RUSSELL

The Peanian origins of the symbols relevant to Russell's theory of descriptions have already been noted and sometimes explained (see, for instance, my *1988a* and *1991a*, Chap. 3). I will confine myself to recalling that they were the letter iota (ι) for the unit class, and the same letter inverted (℩), or denied ($\bar{\iota}a$), for the only member of this class, i.e. the definite article of ordinary language. Peano's ideas also evolved in three stages towards greater precision in the treatment of descriptions.

In *1897a* Peano introduced his fundamental definition of the unit class as the class such that all of its members are identical; in Peanian symbols, $\iota x = \overline{y\varepsilon}\,(y = x)$. Likewise he defined indirectly the unique member of such a class: $x = \bar{\iota}a \,.=.\, a = \iota x$. However, concerning the definability of the definite article, he added the important idea that every proposition containing it can be reduced to the form $\bar{\iota}a\,\varepsilon\, b$, and this, again, to the inclusion of the referred unit class in the other class ($a \supset b$), which already supposes the *elimination* of the symbol $\bar{\iota}$. Thus, Peano says, we can avoid identities whose first member contains this symbol (*1897a*, p. 215).[11]

[11] As an anonymous referee pointed out to me, one major difference between Peano and Russell's treatment of classes in the context of description theory is that, while for Peano descriptions combine a class abstract with the inverse of the unit class operator, for Russell the free use of class abstracts was not available due to the discovery of paradoxes.

In *1897b* we find the same explanation, but an important idea is added: it is necessary that there *exist* the class pointed out by the symbol "℩" (which for Peano meant that this class is not an empty class) and that it have a *unique* member; if these two conditions are not met, the symbol is *meaningless* (similar ideas can also be found in *1898a*, p. 196).[12] In addition, Peano offered several symbolic examples for the handling of the symbol and for the way in which—starting from the indirect definition quoted—it can be eliminated. One of these examples is very interesting, for it states a link between such an elimination and the problem of doubtful existence, and so is worth considering.

Peano starts from the symbolic definition of the greatest number of a class of real numbers, as the number such that there is no number of this class being greater than it. Then he adds that we must not infer from this definition the existence of that greatest number, and he proves it by transforming the original definition (applying the method from *1897a*) until he obtains another expression where the symbol in question (ι) has disappeared (*1897b*, pp. 268–9). Therefore we must admit that, for Peano, the elimination of the definite article is not only possible, but advisable, and that precisely in those cases where doubtful existence is involved. However, this does not yet mean that for him "℩" was equivalent to, and could be systematically replaced by, the two conditions upon which its full significance depends (existence and uniqueness).

This last step took place explicitly in *1900a*. There Peano starts from the above-mentioned definition in terms of the unit class, but then he adds a series of "possible" definitions (the ones allowing an alternative logical order), one of which offers this equivalence:

$$℩a\varepsilon b \,.=:\, \exists x \ni (a = \iota x \,.\, x \varepsilon b). \quad (\mathit{1900a}, \S 7, 05, \text{p. } 351)^{13}$$

Here we find the assertion that the only individual belonging to a unit

[12] To be more precise, Peano did not write literally that the mentioned expression is meaningless, but rather "nous ne donnons pas de signification à ce symbole si la classe *a* est nulle, ou si elle contient plusieurs individus" (*1897b*, p. 269). But I take it to be equivalent in practice, given that if we do not meet the two mentioned conditions, the symbol cannot be used at all.

[13] There are, however, other additional ways of eliminating the same symbols according to Peano, e.g. the following one, which is very similar and depends on the same hypothesis: $℩a\varepsilon b \,.=:\, a = \iota x \,.\supset_x.\, x\varepsilon b$ (*ibid.*).

class (a) such that it belongs to another class (b) is equal to the *existence* of exactly *one* element such that this element is a member of that class (b). In other words: "*the* only member of a belongs to b" is to be the same as "*there is* at least one x such that (i) the *unit* class a is equal to the class constituted by x, and (ii) x belongs to b" (or "the class of x such that a is the class constituted by x, and that x belongs to b, is not an empty class"). This seems to be equivalent to Russell's celebrated definition, although, of course, Peano spoke in terms of classes instead of propositional functions; that is to say, in terms of properties or predicates, which define classes (without forgetting that Peano often read the membership symbol as "is").

Peano was completely aware of the importance of this device as a way to reduce the definite article to logical terms, i.e. to *eliminate* it, as a result of which the symbol would cease to be primitive. That is why he added that the above definitions "expriment la P[proposition] $\iota a \varepsilon b$ sous une autre forme, où ne figure plus le signe ι; puisque toute P contenant le signe $\iota\, a$ est réductible à la forme $\iota a \varepsilon b$, où b est une Cls, *on pourra éliminer le signe* ι *dans toute P*" (*1900a*, p. 352; my emphasis). Therefore, the general belief according to which the symbol "ι" was necessarily primitive and indefinable for Peano is wrong.

Before making more explicit the parallelism with Russell's theory, I have collected some different possible objections against this rather strong claim, in order to discuss them. I think that all of these objections are either misconceived or simply have no force with regard to my main claim as stated in the two previous paragraphs. However, I take them into consideration because they have been proposed by several people who read earlier versions of this paper and, consequently, could be proposed by others.

(1) It is true that the symbol "ι" has disappeared, but in the *definiens* we still can see the symbol of the unit class, which would refer somehow to the idea that is symbolized by "ιx", so the descriptor has not been really eliminated. The answer is very simple: for Peano there were at least two forms of defining this symbol with no need for using the letter iota (in any of its forms).

First, by directly replacing ιx by its value: $y \ni (y = x)$, as defined above. Making the replacement explicit, we have:

$$\iota a \varepsilon b \,.\!=: \exists x \ni \{a = y \ni (y = x) \,.\, x \varepsilon b\},$$

which expresses the same idea in a way where *any* reference to the letter iota has disappeared. We can read now "*the* only member of a belongs to b" as the same as "*there is* at least one x such that (i) the *unit* class a is equal to all the y such that $y = x$, and (ii) x belongs to b" (or "the class of x such that they constitute the class of y, and that they constitute the class a, and that in addition they belong to the class b, is not an empty class"). Thus, the full elimination underlay the mentioned definition, although Peano, in lacking philosophical goals, had no interest in making this point explicit.

Second, by pointing out that in the "hypothesis" preceding the quoted definition it is clearly stated that the class "a" is defined as the unit class in terms of the *existence* and identity of all of their members (i.e. *uniqueness*):

$$a \,\varepsilon\, \mathrm{Cls} \,.\, \exists a : x, y \,\varepsilon\, a \,.\supset_{x,y} .\, x = y : b \,\varepsilon\, \mathrm{Cls} \,.\supset : \ldots \qquad (\textit{Ibid.})$$

This is why "a" is equal to the expression "ιx" (in the second member).

The objection could still be maintained by insisting that since "a" can be read as "the *unit* class", Peano did not really achieve the elimination of the idea he was trying to define and eliminate, as it is shown through the occurrence of these words in some of the readings proposed above. However, as I will explain below, the hypothesis preceding the definition only states the *meaning* of the symbols which are used in the second member. Thus, "a" is stated as "an existing unit class", which has to be understood in this way: "'a' stands for a non-empty class such that all of its members are identical." Therefore, we can replace "a", wherever it occurs, by its meaning, given that this interpretation works as only a purely nominal definition, i.e. a convenient abbreviation.

However, the actual substitution would lead us to rather complicated expressions,[14] and given Peano's usual way of working (which can be

[14] Starting from this idea, we can interpret the definition as stating that "$\iota a \varepsilon b$" is only an abbreviation for the *definiens* and dispensing with the conditions stating existence and uniqueness in the hypothesis, which have been incorporated to their new place. Thus, the new hypothesis would contain only the statement of "a" and "b" as being classes, and the final entire definition *could* be something like the following:

$$a, b \,\varepsilon\, \mathrm{Cls} \,.\supset \therefore \iota a \varepsilon b \,.\!=: \exists x \varepsilon (\left[\{\exists a \varepsilon [w, z \,\varepsilon\, a \,.\supset_{w,z} .\, w = z]\} = \{y \varepsilon (y = x)\}\right] .\, x \varepsilon b),$$

characterized as the constant search for shorter and more convenient formulas), it is quite understandable that he preferred to avoid it. In fact, the operation is by no means necessary, for the symbolic expression above was already enough to obtain the full elimination of the descriptor. We must not forget that the important thing is not the *intuitive* and superficial similarity between the symbols "℩*a*" and "ι*x*", caused simply by the appearance of the letter iota in both cases, or the *intuitive* meaning of the words "the unit class", but the conditions under which these expressions have been introduced in the system, which were completely clear and explicit in the first definition.[15]

(2) The supposed elimination is a failure, for (i) it depends upon Peano's confusion of class membership and class inclusion, so that (ii) a singleton class (℩*a*) and its sole member (℩*x*) are not clearly distinct notions; it follows that (iii) "*a*" is both a class and, according to the interpretation of the definition, an individual (iv), as is shown by joining the hypothesis preceding the definition and the definition itself. This multiple objection is very interesting because it can be taken as proceeding from the received view on Peano, according to which his logic not only falls short of strict logical standards, but also contains some important confusions here and there. However, the four points can easily be shown to be mistaken. (Incidentally, I think this could have been recognized with pleasure by Russell himself, who always thought of Peano and his school as being strangely free of logical confusions and mistakes.)

First, it can hardly be said that Peano confused membership and inclusion, given that it was he himself who introduced the distinction in 1889 through his symbol "ε" (previously to, and therefore independently of, Frege). If the objection means (which is rather unlikely) that Peano would admit the symbol for membership as taking place between two classes, it is true that this was the case when he used it to indicate the meaning of some symbols, but only through the reading "is" (e.g.

which could be read as " '*a* and *b* being classes, "*the* only member of *a* belongs to *b*" is to be the same as "*there is* at least one *x* such that (i) 'there is at least one *a* such that for every *w* and *z* belonging to *a*, *w* = *z*' is equal to 'the *y* such that *y* = *x*', and (ii) *x* belongs to *b*' ", where *both* the letter iota and the words "the unit class" have disappeared from the *definiens*.

[15] There is a well-known similar example in the apparent vicious circle of Frege's famous definition of number.

"$k \,\varepsilon\, K$" as "*k* is a class"; see also the hypothesis from above for another example). But this by no means involves confusion with inclusion, as it is shown by the fact that Peano soon added four definite properties precisely distinguishing both notions, which made it possible for him, as for Russell himself, to preserve the useful and convenient reading "is" (see my *1991a*, Chap. 3, §1.3).

Second, "℩*a*" does *not* stand for the singleton class. Peano stated with full clarity that "℩" ($\bar{\iota}$) makes sense only before *individuals*, and "ι" before *classes*, no matter which particular symbols we use for these notions. Thus, "ι*a*", like "ι*x*", both have to be read as "the class constituted by ...", and "℩*a*" as "the only member of *a*". Therefore, although Peano, to my knowledge, never used "℩*x*" (probably because he always thought in terms of classes), had he done so its meaning, of course, would have been exactly the same as "℩*a*", with no confusion at all.

Third, "*a*" stands for a class because it is so stated in the hypothesis, although it can represent an individual *when preceded by the descriptor, and together with it*, i.e. when both constitute a new symbol as a whole. Here Peano's habit could perhaps be better understood by interpreting it in terms of propositional functions, and then by seeing "℩*a*" as being somewhat similar to ϕx, no matter what reasons of convenience led him to prefer symbols generally used for classes ("*a*" instead of "*x*"). There is little doubt that this makes a difference with Russell. It could even be said that while, for Peano, the inverted iota is the symbol for an operator on classes, which leads us to a new term when it flanks a term, for Russell it was only a part of an "incomplete symbol". I am not sure about Peano's answer to this, but at any rate for him the descriptor could be eliminated only *in conjunction* with the rest of the full expression "℩$a \,\varepsilon\, b$", so that the most relevant point of similarity again can be found in Peano.

Last, there is no problem when we join the original hypothesis and the definition:

$$a \,\varepsilon\, \mathrm{Cls} \,.\, \exists a : x, y \,\varepsilon\, a \,.\supset_{x,y} .\, x = y : b \,\varepsilon\, \mathrm{Cls} \,.\supset \therefore$$
$$\text{℩}a \,\varepsilon\, b \,.=:\, \exists x \ni (a = \iota x \,.\, x \,\varepsilon\, b),$$

as I have pointed out in the interpretation contained in the last part of the reply to objection (1).

There are other, minor objections as well.

(3) If, as it seems, "a" is affected by the quantifier in the hypothesis, then it is a variable which occurs both free and bound in the formula (if it is a constant, no quantifier is needed). I am not sure about the possible reply by Peano himself. Perhaps he did not always distinguish with present standards of clarity between the several senses of "existence" (or related differences) involved in his various uses of quantifiers,[16] but in principle there is no problem when a variable appears both bound and free in the same expression, although in different occurrences. At any rate, I cannot see how this could affect my main claim; the important thing here is to recognize the *fundamental* similarities between the elimination of the descriptor in Peano and Russell. However, in the several readings I have proposed above I hope to have clarified a little the role of "∃" in Peano.

(4) Russell rejected definitions under hypothesis, therefore he would have rejected the Peanian definition of the descriptor. Of course, we must admit that Russell (like Frege) rejected this kind of definition, but this took place especially in the context of the unrestricted variable of *Principia*.[17] Besides, he himself used this kind of definition for a long period once he mastered Peano's system. It was because he interpreted these definitions as Peano did, i.e. merely as a device for fixing the meaning of the letters used in the relevant symbolic expressions. Thus, when for instance one reads, after whatever symbolic definition, things like "'x' being ..." or "'y' being ...", this would really be a definition under hypothesis, but, of course, only because the meaning of the symbols used always has to be determined somehow. Anyway, there is no point in continuing the discussion of this objection, given that it is hardly relevant to my main claim. Even if Peano's original elimination of the descriptor does not work because of its taking place in the framework of a merely "conditional" definition, the force of his original insight could well have influenced Russell; at any rate, it is worth knowing in itself.

[16] I would like to recall here that it was Peano himself who discovered the distinction between bound and free variables (which he respectively called "apparent" and "real"), and probably—and independently of Frege—also the existential and universal quantification (see my *1988a* and *1991a* for a detailed account of both achievements).

[17] And also in previous stages from 1906 onwards, through the (finally unsuccessful) attempt at a substitutional theory based upon propositions, with no classes and no propositional functions.

(5) Peano could hardly have thought that he was capable of eliminating the descriptor, for he continued to use the symbol and his whole system depended on it as a primitive idea.[18] The only additional reply is that only reasons of convenience can explain the retaining of a symbol in a system in cases where the symbol can be defined, i.e. eliminated. (After all, Russell himself continued to use the descriptor after its elimination by means of his theory of descriptions.) But, as we have seen, there is no doubt Peano thought that the descriptor could easily be eliminated from propositions.

(6) The reduction mentioned, even if it really took place, was by no means followed by the philosophical framework which made Russell's theory of descriptions one of the most important logical successes of the century. Thus, Peano did not realize the importance of the elimination. This last point can hardly be denied, but Peano's goals were very different from Russell's, so I think that to point out a "lack" like this makes little sense from a historical point of view.

In any case, I cannot help being convinced that none of these objections seems to have any force against my main claim: that the elimination of the descriptor was present in Peano with essentially the same symbolic resources as in Russell. This is equivalent to the first two claims at the beginning of this paper: (1) Peano clearly stated the conditions of existence and uniqueness as providing the true significance of the descriptor; and (2) he had enough symbolic techniques for dispensing with it, including those required for constructing a definition in use.[19]

[18] In his *1966a* (p. 659), Professor Quine wrote that "ι" was a primitive and indefinable idea in Peano. However, now that we have exchanged several letters concerning an earlier version of this article, I must say he has changed his mind. His letter to me of 11 October 1990 contains the following passage: "I am happy to get straight on Peano on descriptions. I checked your reference and I fully agree. *Peano deserves all the credit for it that has been heaped on Russell* (except perhaps for Russell's elaboration of the philosophical lesson of contextual definition)" (my emphasis). As for the sense in which the philosophical consequences of the elimination of the descriptor were not very important for Peano, I have faced the problem in my reply to objection (6).

[19] For according to him the descriptor cannot be defined in isolation, but only in the context of the class (a) from which it is the only member (ιa), and also in the context of the class (b) from which that class is a member, at least to the extent that the class a is included in the class b, although this supposes no confusion between membership and inclusion; see the second point of my reply to objection (2) above. I think this is just the right interpretation of the whole expression "$\iota a \varepsilon b$".

As for (3), we have a few relevant passages, but the clearest one occurs in *1897b* (p. 269), as I pointed out above. There we can read that " $\bar{\iota}a$" is meaningless if the conditions of existence and uniqueness are not fulfilled. Thus, even the third claim was made by Peano. Perhaps under certain different interpretations of Peano's devices it could be shown that his elimination of the descriptor was not exactly equivalent (in the technical sense) to Russell's. Yet even if so, I think that from the historical viewpoint, which means to do justice both to Peano and Russell, it is important to know that Peano had these resources at his disposal, and that they may have influenced Russell.

The parallelism is therefore complete, but before finishing this paper I want to insist on my main claims by resorting now to one of Russell's manuscripts from 1905, "On Fundamentals" (*1905b*).[20] First, we find there a definition stated in terms similar to Peano's, and with almost exactly the same symbolic resources:

$$\phi`\imath`u \,.=:\, (\exists y): y \,\varepsilon\, u : z \,\varepsilon\, u \,.\supset_z .\, z = y : \phi`y. \qquad (\textit{1905b, Papers } 4\text{: } 384)$$

Second, the later improvement of this definition was precisely in the sense of making clearer that, although the method of propositional functions was preferable to the one of class membership, the symbolic expression of the conditions of existence and uniqueness was preserved. Even the idea—also coming from Peano—according to which we cannot define the expression " $\imath a$" alone, but always in the context of a class (which in Russell became the form of propositional functions), appears here.

The first appearance of Russell's definition, under the form which was adopted as final, took place, not in "On Denoting", but in a letter to Jourdain of 3 January 1906:

$$\psi(\imath x)\,(\phi x) \,.=.\, (\exists b): \phi x \,.\equiv_x .\, x = b : \psi b.$$

(Grattan-Guinness *1977a*, p. 70)[21]

[20] For a fuller study of this manuscript, see my *1992a*.

[21] There is, however, a previous occurrence of this definition in the manuscript "On Substitution" (*1905d*), written in December 1905, with only slight symbolic differences. I am indebted to Gregory Landini for the historical point.

However, we can see the heritage from Peano in a clearer way if we compare the definition with the version for classes in the same letter:

$$\psi(\imath`u) \,.=:\, (\exists b): x \,\varepsilon\, u \,.\equiv_x .\, x = b : \psi b.$$

Finally, I am not accusing Russell of plagiarism. I only affirm that *some of the ideas and devices* which are important for the eliminative definition of the descriptor were already present in Frege and Peano, including the conceptual and symbolic resources, and that these works are ones that Russell had studied in detail before his own theory was formulated in 1905.[22]

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[22] According to that, all other influences must be regarded as secondary. Concerning Meinong's influence, for Russell the principle of subsistence disappears as a *consequence* of the eliminative construction of the definite article, which was a result of the new semantic monism. Russell's later attitude to Meinong as a "main enemy" was only a comfortable recourse (see, however, Griffin *1977a*). As for Bôcher, Russell himself admitted some influence from his nominalism (in his *1906a*). In fact, Bôcher *1904a* described mathematical objects as "mere symbols" (p. 122), and he advised Russell to follow this line of work in a letter of April 1905 (only two months before Russell's key idea): "the 'class as one' is merely a symbol or *name* which we choose at pleasure" (quoted by Lackey [Russell *1973a*, pp. 130–1]). Finally, for MacColl it is necessary to mention his *1905a*, which appeared in January 1905, where he spoke of "symbolic universes", which include things like round squares (p. 308), and also spoke of "symbolic existence". Russell published his *1905a* as a direct response to this author, and there we can see some conclusions from the unpublished manuscripts, although still by solving peculiar cases in a Fregean context (see my *1990a*). I agree with Ivor Grattan-Guinness that MacColl was an important part of the context of Russell's ideas on denoting (personal communication), but I have no room here to devote to the matter.

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