REMARKS ON PEANO ARITHMETIC

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Russell held that the theory of natural numbers could be derived from three primitive concepts: number, successor and zero. This leaves out multiplication and addition. Russell introduces these concepts by recursive definition. It is argued that this does not render addition or multiplication any less primitive than the other three. To this it might be replied that any recursive definition can be transformed into a complete or explicit definition with the help of a little set theory. But that is a point about set theory, not number theory. We have learned more about the distinction between logic and set theory than was known in Russell's day, especially as this affects logicist aspirations.

In *Introduction to Mathematical Philosophy* Russell asserts that Peano "showed that the entire theory of the natural numbers could be derived from three primitive ideas and five primitive propositions in addition to those of pure logic."[1] He expressed the three primitive ideas with the terms

0, number, successor

and formulated Peano's axioms as follows:

(1) 0 is a number.
(2) The successor of any number is a number.
(3) No two numbers have the same successor.
(4) 0 is not the successor of any number.

[1] *IMP*, p. 5.

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(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.

He also said that "[t]he work of analysing mathematics is extraordinarily facilitated by this work of Peano's" (*ibid.*).

What immediately strikes one as strange in these remarks is that they omit mention of addition and multiplication. For what is "the theory of the natural numbers" without those two operations?

Relevant to the question Russell wrote:

> Suppose we wish to define the sum of two numbers. Taking any number m, we define $m+0$ as m, and $m+(n+1)$ as the successor of $m+n$. In virtue of (5) this gives a definition of the sum of m and n, whatever number n may be. Similarly we can define the product of any two numbers. (*Ibid.*, p. 6[2])

Russell had previously defined 1 as "the successor of 0". So, writing "the successor of 0" as "s0", Russell is suggesting that we can define $m+0$ as m and $m+(n+s0)$ as $s(m+n)$.

The second definition, $m+(n+s0) = s(m+n)$, involves a (minor and unimportant) slip on Russell's part, for that clause leaves $0+s0$ undefined. (It is clear that defining $m+0$ as m does not tell us how to define $0+s0$.) I *think* that what Russell must have had in mind was the definition of $m+sn$ as $s(m+n)$. In any case, the common contemporary formulation takes that form and runs as follows:

$$m+0 = m \text{ and } m+sn = s(m+n).$$

I propose to work with it. I am sure Russell would have had no objection.

Russell used the word "define". I think he was thereby expressing the thought that the formula

$$m+0 = m \text{ and } m+sn = s(m+n)$$

provides an alternative mode of expression for an idea or concept

[2] For a discussion of addition and other arithmetical operations, in which number theory is couched within a theory of classes, see *PM*, 2: 63–74.

antecedently available, that he was thinking of definition as *eliminative* definition.

The reason I think this is what he must have meant is that *otherwise* he would have regarded addition, and perhaps also multiplication, as an *additional* primitive idea.

But this is, I think, a conception which must be in error. For suppose that the "definition" Russell proposes actually served as a *definition* in the sense just alluded to. Then each phrase or formula in which the sign for addition occurred would be expressible in terms of the initial non-logical vocabulary. For example, by Russell's "defining" formula, the phrase "0+0" would be defined as "0", "0+s0" would be defined as "s0", etc. And the phrase "s0+s0" would be defined as "s(s0+0)" which in turn be defined as "ss(0+0)", since "s0+0" is defined as "s0". And then "ss(0+0)" would be defined as "ss0", since "(0+0)" is defined as "0". Thus, the sums

$$\begin{aligned} 0+0 &= 0 \\ s0+0 &= s0 \\ 0+s0 &= s0 \end{aligned}$$

would respectively come to

$$\begin{aligned} 0 &= 0 \\ s0 &= s0 \\ ss0 &= ss0 \end{aligned}$$

and similarly for all the other cases. That is, if addition is a *defined* notion in the sense that the sign for addition is eliminable in favour of the antecedently available defining terms, then the sign simply drops out and all that is left are the equations of the form

$$n = n$$

for numeral n.

The moral I draw from these reflections is this: addition is *not* definable in terms of the notion of *number*, *successor*, and 0.

I do not, of course, mean to deny that a system whose non-logical vocabulary consists just of "number", "successor" and "0" cannot be *extended* to employ as well the sign "+", yielding the customary equations for sums. That most obviously *can* be done. All I mean to deny is that Russell's method (which is the common one) of extending the system to include that sign (adding the noted laws or principles for addition) is one which provides a *definition* for that sign in any sense which shows that the notion we wish it to express is one *already* expressible via the initial signs.

To this it might be replied that addition most certainly *is* defined by Russell's formula (or our slight emendation of that formula), for that formula constitutes a *recursive definition* of addition. I would, of course, agree that the formulas

$$m+0 = m$$
$$m+sn = s(m+n)$$

recursively define addition. They are even a paradigm of a recursive definition. But still, and this is the main point, they do not *define* addition in the sense in which, e.g.,

$$x \subseteq y \text{ iff } \forall z (z \varepsilon x \equiv z \varepsilon y)$$

defines the subset relation. For in this case the defined symbol "$\subseteq$" is eliminable without diminishment of the theory of sets to which it is added. But we have already seen that this is not so for the theory of the natural numbers. Addition is a part of that theory, and *were* its recursive definition a method for eliminating occurrences of "+", the theory *would* thereby be diminished.

Nothing in what I am writing turns on this or that use of the term "definition". In all its uses (relevant to this discussion), definitions are sentences or formulas. Of the two formulas just displayed, the second serves to show us how to eliminate the symbol "$\subseteq$" from all its occurrences in the formulas of set theory. The first does not provide a method for eliminating "+" from all its occurrences in the formulas of elementary arithmetic, at least not without conceptually impoverishing arithmetic as a mathematical theory. Set theory less "$\subseteq$" is not a diminished set theory. But arithmetic less "+" is hardly worthy of the name. There need be no harm in calling all the formulas in question *definitions*. We just need to see the differences, and not forget them when they make a difference.

The role of the formula giving the recursive definition of addition is not to eliminate occurrences of the sign for addition but to *decide* (establish) which term in the basic series of terms

$$0, s0, ss0, sss0, \ldots$$

is to be counted the sum of a pair of terms from that series (either the same term taken twice, or two terms each taken once). In this respect the formula is like an axiom.

From a more technical point of view, a recursive definition differs from other sentences we call definitions in not providing for the eliminability of the sign it introduces in its occurrences within the scope of quantifiers. So, for example, the recursive definition of addition does not provide for the elimination of "+" from even so simple a quantified formula as

$$\forall n\, \forall m\, (n+m = m+n).$$

This is not a *fault* in the recursive definition, but, since what I have just displayed is a fundamental law of arithmetic, any theory of the natural numbers less such formulas as this would no more be a theory of the natural numbers than a system less equations for sums would be an arithmetic of the natural numbers.

To this it might be replied that, as is well known, any recursive definition can be transformed into a complete or explicit definition with the help of a little set theory.

There is something right in this, but the way it has just been expressed can lead to misunderstanding.

What *is* well known (speaking now just of the case at hand) is that *set theory*, augmented with the symbols "0", "s0", "ss0", ..., contains a three-place predicate suitable for a definition of addition. The predicate for terms a, b, and c runs as follows:

$$<<a, b>, c> \varepsilon \bigcup \{k : <<x, 0>, x> \varepsilon k \;\&\; <<x, y>, z> \varepsilon k \supset <<x, s(y)>, s(z)> \varepsilon k\}$$

where the variables take instances only from the series "0", "s0", "ss0", ... The definition would run as follows:

$$a+b = c \equiv \langle\langle a, b\rangle, c\rangle \in \bigcup \{k : \langle\langle x, 0\rangle, x\rangle \in k \; \& \; \langle\langle x, y\rangle, z\rangle \in k \supset \langle\langle x, s(y)\rangle, s(z)\rangle \in k\}.$$

What gains an explicit definition is not addition *relative to* the theory of natural numbers, but addition *relative to* set theory augmented with numerical symbols.

The general point is this: a formula does not *on its own* define this or that term. It does that only relative to some given system or language. What is misleading is the vague thought that there is a technical device for getting an explicit definition from a recursive definition, as if a bit of technical ingenuity would enable us to "extract" the notion of addition from the ideas of zero and successor (and number). But no such extraction is at hand. Rather, we *switch systems* and then explicitly define "+" relative to the switched-to system.

I might put the matter as follows: that addition can be explicitly defined is a point about *set* theory, not about *number* theory.