THE ROOTS OF MODERN LOGIC

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I. Grattan-Guinness. *The Search for Mathematical Roots, 1870–1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel.* Princeton: Princeton U. P., 2000. Pp. xiv, 690. US$45.00.

Grattan-Guinness's new history of logic is a welcome addition to the literature. The title does not quite do justice to the book, since it begins with the prehistory of English work in algebraic logic, including the work of the French analytical school, and extends to just after Gödel's great incompleteness paper of 1931. The core of the book, though, is the philosophical and mathematical developments leading up to and immediately following from the work of Whitehead and Russell. Russell is at the heart of the book, and as a whole the history forms an important contribution to Russell studies.

Commentators on Russell have often confined themselves to a rather narrow historical perspective in which Russell is seen as the (problematic) heir of Gottlob Frege, and few other historical figures (other than Peano) enter into the picture. Grattan-Guinness corrects this historical imbalance by placing Russell in the much wider context of the development of mathematics on the continent, and in particular emphasizing strongly the key influence of Cantor on

Russell's work. Cantor has usually received short shrift from philosophers, as unlike Frege, he appears rather naive from the philosophical point of view.

The story proper begins in Chapter 2 with Lagrange's version of analysis in which the basic concepts were to be defined in terms of algebraic manipulation of power series. This lead to the founding of the Analytical Society where Babbage, Herschel and Peacock were active. It is in this English tradition of algebraic analysis that the pioneering work of De Morgan and Boole found its roots. Grattan-Guinness gives a detailed account of the work of both logicians, although his discussion of Boole's methods does not seem entirely adequate. The question is: what are we to make of Boole's puzzling insistence that expressions like $x + y$ are "uninterpretable", while he nevertheless manipulated them freely in his mathematical derivations? It is not correct to say that the addition sign can only link disjoint class symbols, since it is belied by Boole's formal practice. A possible solution has been suggested by Hailperin, in his book on Boole's logic, in which he proposes interpreting the "uninterpretable" expressions as denoting signed multisets. Oddly, Grattan-Guinness refers to Hailperin's work,[1] but elsewhere (p. 42) adopts the view that Boole's addition sign could only link disjoint classes. The chapter concludes with brief accounts of the work of Cauchy, Weierstrass and Bolzano.

The next chapter is a detailed account of the work of Cantor and his creation of *Mengenlehre*. The origins of set theory in the theory of trigonometrical series, and the ensuing discovery of transfinite ordinal and cardinal numbers, are described clearly and succinctly. In addition, the chapter contains an account of Dedekind's philosophy of arithmetic and Cantor's philosophy of mathematics. Cantor's philosophy is an uneasy blend of formalism, platonism and idealism, and has understandably aroused little enthusiasm among philosophers of mathematics, although Michael Hallett has recently studied it in detail. Grattan-Guinness emphasizes, though, Cantor's magnificent mathematical achievements, in defining and clarifying basic concepts such as measure, dimension and cardinality of sets.

Chapter 4 is a rather miscellaneous chapter, in which six partly independent, partly intertwined stories are told. It begins with developments in set theory in Germany and France up to the turn of the century, goes on to discuss American logic in the work of C. S. Peirce and his students, and continues the theme of algebraic logic with Schröder and his logic of relatives. The remainder of the chapter is given over to Frege, Husserl and Hilbert.

The section on Frege is one of the more idiosyncratic parts of the book. Grattan-Guinness distinguishes between Frege, "a mathematician who wrote in

[1] T. Hailperin, *Boole's Logic and Probability*, 2nd ed. (Amsterdam: North-Holland, 1986).

German, in a markedly Platonic spirit", and Fregé, "a philosopher of language and founder of the Anglo-Saxon analytic tradition", to whom most of the massive Frege industry is devoted (p. 177). He even (rather confusingly) devotes different entries in the index to Frege and Fregé. His own account is of course devoted to the first rather than the second figure; he does not give a lot of space to Frege's philosophy, and is impatient with the polemical writings, which he describes as "childish". The sections on Frege's logical work would have been better if adequate type-setting software had been used. The software used was so inadequate that one of the key expressions appearing in the *Begriffsschrift* is not reproduced at all, but instead is paraphrased in English (p. 181). Unfortunately, the transcription is wrong (the "if" at the beginning should be deleted). The chapter concludes with the (logically somewhat marginal) figure of Husserl, and Hilbert's early proof theory and model theory.

Chapter 5 is devoted to the work of Peano and his school. Grattan-Guinness gives a clear and interesting presentation of Peano's invention of "wallpaper mathematics" (as he refers to Peano's formulaic presentation of proofs). Interesting sidelights appear in the description of the oddly competitive attitudes of logicians at this time, each of them claiming to have the most convenient or most profound logical system (Schröder said that the Peanists were "still making use of sailing boats, while the steamships are already invented").

Chapters 6, 7, 8 and 9 form the core of the book, centred in the work of Russell and Whitehead, the foundational project culminating in *Principia Mathematica* and its subsequent demise. Chapter 6 is an account of Russell's work in logic from 1895 to 1903, starting with his idealist phase and his work on geometry, and ending in the confused situation when Russell published *The Principles of Mathematics* without giving a solution to the logical paradoxes that continued to plague him for the next five years. Chapter 7 continues the account with the theory of denoting, the substitutional theory of 1905–07, and the emergence of the ramified theory of types in 1907. These two chapters together are perhaps the first connected and detailed account of Russell's logical research at this time, including descriptions of the unpublished manuscripts, as well as Russell's interaction with logicians and mathematicians such as MacColl and Poincaré.

Chapter 8 is a more miscellaneous chapter, describing Whitehead's and Russell's transitions to philosophy, and the fate of the logicist enterprise in Britain, America, Poland and Austria. The emphasis here is more on philosophical movements inspired by Russell's ideas, rather than work in pure logic. Chapter 9 continues the history of logic into the 1930s, including Gödel's incompleteness theorem, and the work of Carnap and Quine.

Chapter 10 is a short retrospective survey, tracing various themes of the history and supplying a flow diagram of influences between mathematical logic, set theory and formalism. The last chapter is an appendix giving transcriptions of manuscripts, including letters to and from Russell to logicians such as Couturat, Veblen, Quine and Henkin.

The strongest parts of the book are those relating to Russell. Important developments in logic and foundations not closely related to Russell's work are often dealt with in a very cursory manner. The work of Brouwer, described as a "great mathematician, but ghastly philosopher", is dealt with only briefly. A more serious omission is that of the post-war development of the Hilbert school and metamathematics. Chapter 9, which covers Gödel's work, is the weakest in the book, and does not give a clear or accurate picture of the background leading up to his great theorem. Hilbert's metamathematical programme and the consistency problem that forms the background to Gödel's theorem are almost entirely omitted. Gödel's second incompleteness theorem is incorrectly stated on page 510. Some of the historical details in this chapter are wrong, too. Jacques Herbrand was killed in a climbing accident, not a skiing accident, as stated on page 550.

There are rather a lot of typographical errors in the book, and it would have benefitted from more careful copy-editing—for example the error *Megenlehre* appears repeatedly in the running headers in Chapter 3. I've already referred to the inadequacy of the software used for typesetting—this flaw is apparent throughout the book. For a writer on logic, Grattan-Guinness shows a surprising aversion to logical formulas; he refers repeatedly with apparent disdain to "symbolic wallpaper". The manuscripts on functions cited on page 382 as written in 1907 are in fact from 1903; the formulas displayed on that page are to be found in *Papers* 4: 53. The definition of ordered pair due to Kuratowski is incorrectly stated on page 421; the definition given is that of an unordered pair. The peculiar double use of ε on page 297 is not a use–mention confusion, as stated. It is simply an example of ambiguity, where the symbol functions as both a noun and a verb. This kind of overloading of symbols was typical of Russell's Peano-style practice of 1900.

In spite of the grumbles recorded above, this book is an excellent read. It is written in a relaxed and often humorous manner, and filled full of all kinds of amusing historical trivia and anecdotes. I recommend to the reader's attention Russell's description of his plump typist, Mrs. Kyle (p. 433). I was amazed to discover that the formidable textual scholar W. W. Greg wrote an essay in which he used the logic of relations to concoct a symbolic representation of the relations between earlier and later versions of a text. A footnote on page 542 refers us to some particularly juicy erotic correspondence between Russell and two of his lovers; the word in the text to which it is keyed should, I think, be not "carousing", but "lechery".