Articles

RE-EXAMINING RUSSELL'S PARALYSIS: RAMIFIED TYPE-THEORY AND WITTGENSTEIN'S OBJECTION TO RUSSELL'S THEORY OF JUDGMENT

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It is well known that Russell abandoned his multiple-relation theory of judgment, which provided the philosophical foundations for *PM*'s ramified type-theory, in response to criticisms by Wittgenstein. Their exact nature has remained obscure. An influential interpretation, put forth by Sommerville and Griffin, is that Wittgenstein showed that the theory must appeal to the very hierarchy it is intended to generate and thus collapses into circularity. I argue that this rests on a mistaken interpretation of type-theory and suggest an alternative one to explain Russell's reaction.

I am very sorry to hear that my objection to your theory of judgment paralyses you. I think it can only be removed by a correct theory of propositions.
<div align="right">(Wittgenstein to Russell, 22 July 1913)</div>

INTRODUCTION

Russell's multiple-relation theory of judgment is generally recognized as providing epistemological foundations for the ramified theory of types at the heart of *Principia Mathematica*'s formal system. The multiple-relation theory offers a correspondence theory of truth which generates a hierarchical notion of truth (and falsehood), by

defining truth recursively on a base level of correspondence between atomic judgments and facts. Hence the judgment expressed by "Fa" will have truth of the lowest level in the hierarchy; that expressed by "$(x)Fx$" will have "second truth", since its truth is dependent on the first-order truth of Fx for all values of x. The judgment expressed by "$(\exists F)(x)Fx$" will, accordingly, have third-order truth, its truth being determined by the second-order truth of $(x)Fx$ for some value(s) of F.[1] In this way, the multiple-relation theory allows the hierarchy of orders, with its ability to avoid vicious-circle fallacies, to be admitted into the formal system of *Principia* without endorsing an ontology of propositions which would then need dividing into ontological obscurity if vicious-circle fallacies are to be avoided.[2]

Of course, variables carry type as well as order indices in *Principia*.[3] So, for example, "Fx", in the symbolism common in the literature, becomes "$^{1}F^{(0/0)}\,{}^{0}x^{0}$", where the superscript "1" shows the order of F, and the index in parentheses shows the order/type which any argument for F must comply with (as does x in the example), thereby displaying the type of F (the type of a variable must be exactly $n-1$ for arguments to a function of type n). A second-order function will also carry order and type indices to ensure that it does not take as argument any function of illegitimate type or order. For example,

$$^{2}\theta^{(1/(0/0))}\,(^{1}\varphi^{(0/0)}\,{}^{0}x^{0})$$

will be a properly type-, and order-, stratified formula with indices restored.[4]

In a number of influential articles, Nicholas Griffin (inspired by the

[1] See *PM*, I: 41–3, 45–7 (all page references are to the 2nd edition, 1925–27).

[2] As had been done on a largely *ad hoc* basis in the 1908 paper "Mathematical Logic as Based on the Theory of Types". See below.

[3] Though Whitehead and Russell famously chose to suppress the indices, favouring the device of "typical ambiguity".

[4] Notational variants abound. For example, Copi would write the above formula as "$^{2}\theta^{2}[^{1}\varphi^{1}(x)]$" (see Copi, *The Theory of Logical Types* [London: Routledge, 1971], p. 87); Church would have "$\theta^{2/2}(\varphi^{1/1}x^{0/0})$" (see Church, *Introduction to Mathematical Logic* [Princeton, NJ: Princeton U. P., 1956], p. 351, and "A Comparison of Russell's Solution to the Semantical Antinomies with that of Tarski", *Journal of Symbolic Logic*, 41 [1976]: 747–60, at p. 748).

work of Stephen Sommerville), has sought to elucidate Wittgenstein's famous yet obscure criticisms of the multiple-relation theory by arguing that Wittgenstein showed the incompatibility of the formal system outlined above with the multiple-relation theory which was intended to generate and philosophically justify it. This interpretation (hereafter, the Sommerville/Griffin interpretation) has the noticeable advantage of providing an explanation for the severity which Russell clearly attributed to the consequences of Wittgenstein's objection—Russell's response to Wittgenstein's criticism was to permanently cease work on the book which he had been composing (*Theory of Knowledge*), and to abandon the multiple-relation theory.[5] *Theory of Knowledge* was consequently not published until after Russell's death.[6] In what follows, I will suggest that the Sommerville/Griffin interpretation of Wittgenstein's criticisms, ingenious though it is, is incorrect. In particular, I will argue that the interpretation rests on a misunderstanding of the philosophical foundations of the formal system of *Principia*. The explication of this point requires a fairly substantial review of the origins of Russellian type theory in order to make plain the exact role played by the multiple-relation theory in generating the ramified type theory of *Principia*.

THE RAMIFIED THEORY OF TYPES

Ramified type-theory consists of two distinguishable hierarchies: the hierarchy of types, and the hierarchy of orders. It is well known that the

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[5] It certainly seems unlikely that Wittgenstein and Russell would have held the same views on what was important about the criticisms. Wittgenstein, for example, would not have been overly concerned by any incompatibility of the theory of judgment with the theory of types. Indeed Wittgenstein's own remarks on types are brief and enigmatic, suggesting little, if any, understanding of (or interest in) the sophisticated ramified theory at the heart of *Principia*. In what follows, I will focus on the impact of Wittgenstein's criticisms on Russell's philosophical views, rather than seeking to situate those criticisms in the wider context of Wittgenstein's early philosophy of logic. For more on those issues, see my "From Russell's Paradox to the Theory of Judgment: Wittgenstein and Russell on the Unity of the Proposition", *Theoria*, forthcoming.

[6] Although Russell did publish the first six chapters of the book (dealing with issues other than the theory of judgment) in a series of articles for *The Monist* 1914–15. See editors' introduction to *Papers* 7; and E. R. Eames, *Bertrand Russell's Dialogue with His Contemporaries* (Carbondale and Edwardsville, Ill.: Southern Illinois U. P., 1989), pp. 143–4, for further details.

theory was developed in response to the Russell paradox concerning the class of all classes which are not members of themselves. Call this class w, then

$$(x)(x \in w \equiv x \notin x).$$

Universal instantiation to w then yields the contradictory

$$w \in w \equiv w \notin w.$$

Type-theory, applied to classes, will ensure that classes come with a fixed logical type. Individuals are of type 0, classes of individuals are of type 1, classes of classes are of type 2, and so on. In general a class may only have members taken from the level immediately below it in the hierarchy, so that a class of type n can only be significantly asserted to belong to a class of type $n+1$. Thus regimented, our previously contradictory statement is reduced to a nonsensical violation of type distinctions:

$$w^n \in w^n \equiv w^n \notin w^n.$$

In *Principia*, however, the type hierarchy is somewhat complicated by Whitehead and Russell's decision to abandon any ontological commitment to classes and introduce the symbols for classes only through contextual definitions such as *Principia*'s *20.01:

$$f\{\hat{z}(\psi z)\} \, . = : (\exists \phi) : \phi!x \, . \equiv_x . \, \psi x : f\{\phi!\hat{z}\} \quad \text{Df.}$$

The class symbol, "$\hat{z}(\psi z)$", Whitehead and Russell tell us, is merely a convenient shorthand: "$f\{\hat{z}(\psi z)\}$ is in reality a function of $\psi\hat{z}$, which is defined whenever $f\{\phi!\hat{z}\}$ is significant for predicative functions $\phi!\hat{z}$" (*PM*, 1: 188).[7] It is propositional functions, not classes, in other words, which are to be primarily divided into types.

We have seen, above, that the multiple-relation theory of judgment

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[7] There are two distinct uses of "^" in the notation of *Principia*. Symbols of the form "$\hat{z}(\psi z)$" are class abstracts. The use of a circumflex-capped variable preceded by a predicate variable, as in "$\phi!\hat{z}$", is used to refer to the functional part of the expression rather than the open formula as a whole. See *PM*, 1: 40.

and, in particular, the recursive definition of truth which is a component of it, generates the hierarchy of orders in *Principia*. How, then, is the hierarchy of types to be generated? In order to answer this question, it is necessary to look at the evolution of the theory of types from its earliest inception in 1906 to its final statement in *Principia*.[8]

In a paper read before the London Mathematical Society in 1905 and published in 1906, Russell suggested three approaches to solving the paradoxes plaguing the foundations of mathematics: the "zigzag theory", the "theory of limitation of size", and the "no-classes theory". In a note added to the published paper after its original reading, Russell remarked: "From further investigation I now feel hardly any doubt that the no-classes theory affords the complete solution of all the difficulties stated in the first section of this paper."[9] Russell's faith in the theory was expressed more fully in a paper read before the same society on 10 May 1906, entitled "On the Substitutional Theory of Classes and Relations". The paper did not make it into print at this time. Being dissatisfied with the substitutional theory in that form, Russell withheld the paper from publication while he sought to strengthen the theory in order to block various paradoxes which he had discovered in the system.[10] The paper which he eventually did publish on the theory was only published at the time in French[11] and, along with its predecessor, was not published in English until 1973.

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[8] Russell himself states in his *Autobiography* that he discovered the theory of types in 1906, after which, he says, "it only remained to write the book out" (*Auto.*, 1: 152). Russell had set forth an earlier doctrine of logical types in an appendix to *The Principles of Mathematics*, but this theory has no real bearing on the later type-theory, as we shall see.

[9] Russell, "On Some Difficulties in the Theory of Transfinite Numbers and Order Types", reprinted in *Essays in Analysis*, ed. Douglas P. Lackey (London: Allen and Unwin, 1973), p. 164.

[10] In a letter to Jourdain of 14 June 1906, Russell reported that he was working on extending the substitutional theory to general propositions in an effort to solve the Epimenides paradox (see I. Grattan-Guinness, *Dear Russell—Dear Jourdain: a Commentary on Russell's Logic, Based on his Correspondence with Philip Jourdain* [London: Duckworth, 1977], p. 89). Manuscript evidence (discussed below) also reveals that the basic substitutional theory of "On the Substitutional Theory of Classes and Relations" was prone to a contradiction unique to substitution.

[11] "Les Paradoxes de la Logique"; eventually published as "On 'Insolubilia' and Their Solution by Symbolic Logic" in *Essays in Analysis*.

Russell's substitutional theory is simply the "no-classes" theory he had suggested in his 1905 paper to the London Mathematical Society.[12] Unsurprisingly, considering the lack of publications in English at the time of its development, the importance of the substitutional theory for an accurate understanding of the development of Russell's philosophy between 1903 and 1910 has, until only very recently, been overlooked.[13] However, Russell's unpublished manuscripts from 1905 through to 1907 show him working extensively on the substitutional theory in various forms, and reveal its crucial role in the evolution of the theory of types. Russell was aware, very soon after the discovery of the paradox bearing his name, that a hierarchy of types would provide a formal solution to his troubles. What occupied him throughout the first decade of the twentieth century, however, was the search for a solution to the paradoxes which was not only formally acceptable, but also retained Russell's philosophical insights.

The extreme pluralism developed by Russell (largely under the influence of Moore) in response to his neo-Hegelian upbringing at Cambridge provided the philosophical framework for the logicist project of the 1903 *Principles of Mathematics*. Central to this view was Russell's insistence that the ultimate constituents of propositions are atoms—entities which are logically independent of one another and stand on an ontological par at the end point of analysis. In addition, Russell was strongly committed to a conception of logic as a universal science. Logic, for Russell, was uniquely characterized by the fact that it applies equally to whatever there is.[14] Hence the logical constants, in the *Principles*, are construed as one-or two-placed predicates ranging over all terms. So, for

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[12] The substitutional theory was one of several "no-classes" theories; *Principia* itself offers another.

[13] Gregory Landini's work on the substitutional theory has performed an invaluable service in revealing the importance of this period in Russell's thought and, in particular, the significance of the substitutional theory to Russell's later type-theory. See his "New Evidence Concerning Russell's Substitutional Theory of Classes", *Russell*, n.s. 9 (1989): 26–42; "Reconciling *PM*'s Ramified Type Theory with the Doctrine of the Unrestricted Variable of the *Principles*", in A. D. Irvine and G. A. Wedeking, eds., *Russell and Analytic Philosophy* (Toronto: U. of Toronto P., 1993), pp. 361–94; *Russell's Hidden Substitutional Theory* (New York: Oxford U. P., 1998); and "Russell's Substitutional Theory", in Nicholas Griffin, ed., *The Cambridge Companion to Bertrand Russell* (Cambridge: Cambridge U. P., 2003), pp. 241–85.

[14] See, e.g., "Mathematics and the Metaphysicians", *ML*, p. 763; *Papers* 3: 366–7.

example, "⊃" stands for a dyadic relation and yields as well-formed formulae both "Frege ⊃ Russell" and "Socrates is mortal ⊃ Plato is mortal".[15] The first case, however, will automatically be false on the grounds that only propositions can actually imply one another.[16] In this way, Russell adheres to the so-called doctrine of the unrestricted variable in the *Principles*. The variables of pure logic range over all entities without restriction and all values of these variables are deemed to be of the same logical type. Such a view seems fundamentally opposed to the theory of types. Russell's substitutional theory, however, provides a means of reconciling the two approaches.

There is only one kind of variable present in the substitutional calculus: the unrestricted entity variable. The calculus is built on the operation of substitution whereby an entity is substituted into a "matrix" which consists of a further two entities (in the simplest instance). For example, a matrix, symbolized by "p/a", consists of a "prototype" (symbolized by "p"), and an "argument" (symbolized by "a"). In the interesting cases, the argument will be a constituent of the prototype, allowing substitutions such as the substitution of x for a in p, which is written "$p/a\,;x$". The formula "$p/a\,;x!q$" is to be read as "q results from p by substituting x for a in all those places (if any) where a occurs in p".[17] It should be noted that the formula "$p/a\,;x$" is to be read as "the q which results from the substitution of x for a in p" and is therefore a definite description to be defined contextually in accordance with the 1905 theory of descriptions. Russell first defines "$p/a\,;x$" as follows:

$$p/a\,;x = (\imath q)\,(p/a\,;x!q) \quad \text{Df.}[18]$$

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[15] More correctly, nominalizing braces should be used, as in "{Frege} ⊃ {Russell}", and "{Socrates is mortal} ⊃ {Plato is mortal}" which will be read as something like "Socrates's being mortal implies Plato's being mortal". See my "The Truth and Nothing But the Truth, Yet Never the Whole Truth: Frege, Russell and the Analysis of Unities", *History and Philosophy of Logic*, n.s. 24 (2003): 221–40, for further details.

[16] Hence, the inclusion of the antecedent clause "$p \supset p$" in any formula can be used to guarantee that "p" stands for a proposition as only a proposition can imply itself.

[17] "On the Substitutional Theory of Classes and Relations", *Essays in Analysis*, p. 168. In those cases where the argument is not a constituent of the prototype, the result of substituting an entity for the argument will leave the prototype unaltered. See Russell, "On Substitution", manuscript dated 22 Dec. 1905 (Russell Archives file 220.0109406), *12.14.

[18] "On Substitution" (1905), *12.12.

The status of "p/a'x" as a definite description having been made explicit, the following contextual definition can now be given:

$$\phi\{(\imath q)\ (p/a\text{'}x!q)\} = (\exists q)\ [(p/a\text{'}x!r \equiv_r r = q)\ \&\ \phi(q)] \quad \text{Df.}[19]$$

The importance of this point becomes clear when we turn our attention to the status of a matrix. If "p/a'x" stands for the result of substituting x for a in p, then the symbol "p/a" will be grammatically, as well as logically, incomplete. The expression literally means "the result of replacing a in p by … " ("On 'Insolubilia'", p. 201). It comes as something of a surprise, therefore, to find Russell advocating the view that "this shadowy symbol p/a represents a *class*".[20] However, it must be remembered that Russell's point is just that "[a] matrix has all of the formal properties of a class".[21] Having thus justified the sweep of Occam's razor, Russell can denounce classes altogether and make the signs which appear to stand for them literally incomplete symbols. Understanding "p/a" to stand for a class, the members of that class will simply be the values of "x" for which "p/a'x" is true. Crucially, this notion of a class leaves no possibility for self-membership of classes, whilst still maintaining a correlate of class membership of a class of classes (*ibid.*). The formation of a $\in$-cycle such as that required for the Russell paradox would seem to require something like the following (pseudo-)matrix:

$$p/a\text{'}(p/a)$$

which is quite clearly ungrammatical. In place of the Russell paradox we now have the obviously nonsensical "the result of replacing a in p by the result of replacing a in p by …".

Self-membership of classes is thus impossible in the (substitutional) no-classes theory. Classes, however, may still be members of classes of classes. For example, q/c will be a member of $p/(a, b)$ if $p/(a, b)$'(q, c) is true.[22] In this manner, the formal grammar of the substitutional calcu-

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[19] "On Substitution" (1905), *12.11, and "On 'Insolubilia'", *Essays in Analysis*, p. 201.

[20] Russell, "On the Substitutional Theory of Classes and Relations", manuscript dated 24 April 1906 (Russell Archives file 220.010910), fol. 9.

[21] "On 'Insolubilia'", *Essays in Analysis*, p. 201.

[22] The formula "$p/(a, b)$'(q, c)" is to be read "the result of substituting q for a and c

lus allows the formation of a hierarchy of *types*. The hierarchy arises naturally from the grammar of the system without depending on any *ad hoc* measures, and, furthermore, the distinctions are kept at the level of grammar; they do not require the postulation of uncomfortable ontological categories. The universe remains populated only by entities of the one inclusive type which the pluralism of the *Principles* had earlier demanded.

The earliest manifestation of the ramified theory of types is located in Russell's 1908 paper "Mathematical Logic as Based on the Theory of Types" (reprinted in *LK*). Two important differences between this version of the theory and that found in *Principia* are often overlooked. First, "Mathematical Logic" explicitly endorses a version of the substitutional theory as a means of generating the hierarchy of types. Secondly, although the theory does introduce a hierarchy of orders, there is no mention whatsoever of the multiple-relation theory at this time.[23] By 1906, and possibly even before, Russell was aware of the necessity of restricting the range of a bound variable in a general proposition to exclude the possibility of the proposition itself becoming a value of its own bound variable and leading to contradiction. The substitutional theory was not immune to such problems, as mentioned above; contradiction emerges in the substitutional theory if general propositions, along with identity, are admitted without restrictions. In the 1906 man-

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for *b* in all those places, if any, where *a* and *b* occur in *p*". Such matrices can be used to proxy dyadic relations as well as classes of classes, depending on the status of the entities involved in the substitution. Russell sometimes uses the notation "*p*|(*a*/*b*)$\cdot$(*q*/*c*)" when he wishes to make it explicit that he is dealing with a class of classes.

[23] Indeed the multiple-relation theory cannot feature in the system of "Mathematical Logic" if the substitutional theory is to be invoked, for the two theories cannot coexist. The multiple-relation theory requires the abandonment of propositions as entities, whereas the substitutional theory depends on an ontology of propositions which can be both substituted into and themselves be substituends. This was first noticed by Nino Cocchiarella (see "The Development of the Theory of Logical Types and the Notion of a Logical Subject in Russell's Early Philosophy", *Synthese*, n.s. 45 [1980]: 95–6), and has been explored in detail by Landini ("Reconciling *PM*'s Ramified Type Theory with the Doctrine of the Unrestricted Variable of the *Principles*", in Irvine and Wedeking, pp. 361–94, and *Russell's Hidden Substitutional Theory*. Quine's particularly influential interpretation of type-theory misses this crucial disparity between the 1908 system and *Principia* (see Quine, *Set Theory and its Logic* [Cambridge, Mass.: Harvard U. P., 1963], p. 243).

uscript "The Paradox of the Liar", Russell gives the following sketch of what he calls "the fallacy which led to the abandonment of substitution before":[24]

$$\{p_0 = (\exists p, a) \: [(a_0 = \{p/a \, {}^\backprime b! q\}) \: \& \sim (p/a \, {}^\backprime a_0)]\}$$
$$\supset p_0/a_0 \, {}^\backprime \{p_0/a_0 \, {}^\backprime b! q\} \: \& \sim (p_0/a_0 \, {}^\backprime \{p_0/a_0 \, {}^\backprime b! q\}).$$

The contradiction is formulated by substituting the proposition $p_0/a_0 \, {}^\backprime b! q$ for a_0 in p_0 (as p_0 is defined in the antecedent clause above), which yields the proposition

$$(\exists p, a) \: [(\{p_0/a_0 \, {}^\backprime b! q\} = \{p/a \, {}^\backprime b! q\}) \: \& \sim (p/a \, {}^\backprime \{p_0/a_0 \, {}^\backprime b! q\})].$$

From the identity statement on the left-hand side of the conjunction, we can prove that $p_0 = p$ and $a_0 = a$, from which the contradictory consequent readily follows.[25]

Appealing to the substitutional theory as a method of generating the hierarchy of types required to block the Russell paradox would only be successful if the substitutional theory could be restored to consistency. In "Mathematical Logic" this requirement is met by the introduction of orders. Propositions are now regimented into orders in such a way as to outlaw the offending substitution as well as blocking the various semantic paradoxes. We therefore have a fully fledged *ramified* theory of types for the first time in Russell's work. Within this ramified theory, the hierarchy of types is to be generated more or less naturally by the grammar of substitution. As mentioned above, however, there is no sign of the multiple-relation theory as an explanation for the hierarchy of or-

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[24] "The Paradox of the Liar", manuscript dated Sept. 1906 (Russell Archives file 220.010930), fol. 72. I have replaced Russell's dot notation and rendered the substitutions in their single-line form. The paradox was first uncovered from Russell's manuscripts by Landini, "New Evidence concerning Russell's Substitutional Theory of Classes" (1989).

[25] See Russell, letter to R. Hawtrey, in Bernard Linsky, "The Substitutional Paradox in Russell's Letter to Hawtrey", *Russell*, n.s. 22 (2002): 150–9. The proof required to validate this last step is surprisingly complicated, requiring several lemmas. A proof can be found in Landini's reconstruction of the substitutional theory in *Russell's Hidden Substitutional Theory*, pp. 119–25. He also supplies a detailed formal derivation of the contradiction (p. 204).

ders. How, then, is this further division of propositions to be accounted for? Quite simply, it is not. Russell had come to accept that a ramified hierarchy was formally required for the adequate protection of his system, but he was at a loss to provide any grounds for accepting ramification beyond its ability to block paradoxes. "Mathematical Logic", written mainly for a mathematical audience, avoids the issue, laying out the formal system without delving into the philosophical justifications behind it. As Russell later put it in a paper intended to provide the justification behind the ramified theory: "as ['Mathematical Logic'] appeared in a mathematical journal, I was unwilling to devote more space to philosophical interpretations than appeared absolutely indispensable" (*Papers* 6: 4). When one reads this later paper, however, what one finds is not an elucidation of the more philosophical aspects of the 1908 system, but a presentation of the ramified theory of types as present in *Principia*, multiple-relation theory and all.[26]

Ramification now finds its justification in the rejection of propositions as entities outright and the appeal to the multiple-relation theory of judgment as a means of generating a hierarchy of senses of truth and falsehood which, in turn, yields a corresponding hierarchy of orders of judgments. Russell sets the hierarchy out as follows:

That the words "true" and "false" have many different meanings, according to the kind of proposition to which they are applied, is not difficult to see. Let us take any function $\phi\hat{x}$, and let ϕa be one of its values. Let us call the sort of truth which is applicable to ϕa, "*first* truth" … Consider now the proposition $(x) . \phi x$. If this has truth of the sort appropriate to it, that will mean that every value ϕx has "first truth". Thus if we call the sort of truth that is appropriate to $(x) . \phi x$ "second truth", we may define "$\{(x) . \phi x\}$ has second truth" as meaning "every value for $\phi\hat{x}$ has first truth", i.e. "$(x) . (\phi x$ has first truth)". Similarly, if we denote by "$(\exists x) . \phi x$" the proposition "ϕx sometimes", i.e. as we may less accurately express it, "ϕx with some value of x", we find that $(\exists x) . \phi x$ has second truth if there is an x for which ϕx has first truth; thus we may define "$\{(\exists x) . \phi x\}$ has second truth" as meaning "some value for $\phi\hat{x}$ has first truth", i.e. "$(\exists x) . (\phi x$ has first truth)". Similar remarks apply to falsehood.   ("The Theory of Logical Types", *Papers* 6: 9[27])

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[26] "The Theory of Logical Types". Indeed the paper *is*, in essence, Chapter 11 of the Introduction to *Principia* (both pieces also share the title).

[27] See also *PM*, 1: 42–3.

This, then, is intended to justify the assertion that judgments divide into a hierarchy of orders in accordance with the kind of truth attributable to them. But what of the justification for the hierarchy of types? As mentioned above, it is no longer possible, in the absence of an ontology of propositions, to make any appeal to the substitutional theory as a method of generating types. In *Principia*, Whitehead and Russell invoke the "direct inspection" argument to justify their claim that functions divide into a hierarchy of types.[28] According to this argument, the hierarchy of types follows directly from the very nature of functions. A function, they tell us, "is essentially an ambiguity, and …, if it is to occur in a definite proposition, it must occur in such a way that the ambiguity has disappeared, and a wholly unambiguous statement has resulted" (*PM*, 1: 47).

The idea here is reminiscent of Frege's construction of a hierarchy of "levels" of functions.[29] Frege's revolutionary logical theory modelled the analysis of propositional functions on mathematical functions. Taking functions to be essentially incomplete (or "unsaturated"), Frege held it to be impossible for a function to take, as argument, another function of the same level, on the grounds that a complete proposition would not result. First-level functions, for example, are thought of as essentially incomplete; the expression "Fx" is of the form " … is F " or, as Frege sometimes expresses it, "$F(\ )$". A truth-evaluable item will only result if the function is saturated by a "complete" object. Of course, however, an incomplete expression "Fx" can also be made into a truth-evaluable item through the binding of the variable, as in "$(x)Fx$". Frege therefore understands the quantifiers "$(x)(\ldots x)$" and "$(\exists x)(\ldots x)$" to be second-level functions; the only arguments they can meaningfully take are first-level functions. In this way, the hierarchy of levels can be seen as a direct consequence of the incompleteness of functions.

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[28] See *PM*, 1: 47–8.

[29] See, for example, Frege, "Function and Concept", reprinted in Frege, *Translations from the Philosophical Writings*, ed. P. Geach and M. Black, 3rd ed. (Oxford: Blackwell, 1980), p. 38, and "On Concept and Object", *ibid.*, pp. 50–1. Russell was certainly aware of the similarity between his account and Frege's. In a letter to Frege in 1902, Russell had discussed the possibility of avoiding the contradiction through a theory of types, and noted the equivalence with Frege's hierarchy (see Frege, *Philosophical and Mathematical Correspondence*, ed. B. F. McGuinness, trans. Hans Kaal [Oxford: Blackwell, 1980], p. 144).

There are differences between the Fregean and Russellian type-hierarchies. Frege's hierarchy is built on the kinds of metaphysical excesses which Russell was striving to avoid. Frege would not have objected to the imposition of type distinctions on his ontology; he explicitly states, in fact, that such distinctions are "not made arbitrarily, but founded deep in the nature of things" ("Function and Concept", in Geach and Black, eds., p. 41). Russell's project thus far, however, has been to avoid such ontological distinctions and to find a philosophical justification for type-stratified variables which does not require the admission of type-distinctions at the ontological level. Whether *Principia* succeeds in this venture or whether, as Quine has influentially argued,[30] Russell ultimately arrives at an ontology just as lavish as Frege's but with the burden of its platonic excesses put on to functions rather than classes, is a question that we will not pursue here. We can, however, note two things pivotal to our discussion: first, that the multiple-relation theory (despite its numerous other faults) provides a means of accounting for the hierarchy of orders without imposing such distinctions on any ontology of propositions; and, secondly, that the hierarchy of types is not (and never, in any incarnation of the ramified theory of types, was) generated by the multiple-relation theory.

According to Sommerville and Griffin, the ramified hierarchy falls prey to Wittgenstein's criticisms of the multiple-relation theory, not just because the multiple-relation theory is needed to generate a substantial portion of it, but also because Wittgenstein's criticisms show the multiple-relation theory and the theory of types to be incompatible with one another. We are now in a position to critically examine the validity of their interpretation.

OBJECTIONS TO THE MULTIPLE-RELATION THEORY: THE DIRECTION PROBLEMS

We saw above how the multiple-relation theory generates orders of truth and falsehood recursively so as to provide the foundations for the hierarchy of orders in ramified type-theory without reverting to the ontol-

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[30] See, for example, Quine, "Logic and the Reification of Universals", *From a Logical Point of View* (Cambridge, Mass.: Harvard U. P., 1961; 1st ed., 1953), pp. 122–3.

ogical luxuries which the 1908 ramification seemed to call for. Avoidance of those luxuries also depends on the ability of the theory to provide an ontologically acceptable account of the atomic judgments at the base level in the hierarchy. By making propositions "incomplete symbols"[31] Russell can maintain a healthy agnosticism regarding the ontological status of propositions. And if there are no propositional entities, there can be no division of those entities into orders. The ontological cutbacks required for the justification of the ramified theory demand, therefore, that Russell makes good on his claim that there really is no reason to believe in the existence of such things as propositions.

The mechanism invoked by Russell to meet this requirement is well known, so I shall be brief in my description of it. According to Russell, an "incomplete symbol" is "a symbol which is not supposed to have any meaning in isolation, but is only defined in certain contexts" (*PM*, 1: 66). In the case of a proposition, however, the act of judgment is itself deemed sufficient for providing the context which will bestow a unified meaning on that which is judged.[32] Russell therefore seeks to explain propositional content without propositions by explaining the nature of judgment. He does so by understanding judgments (including beliefs, desires, understandings, and the other attitudes) as multiple relations obtaining between the judging subject and the elements of her judgments. As a simple example, *S*'s judgment that *aRb* may be characterized as:

$$J\{S, a, R, b\}.$$

In place of a single, complex proposition, we now have a set of objects which are united by the multiple-relation of judgment obtaining between them. The analysis was plagued from the outset, however, by repeated failures to account for the unity of the judgment. These problems have become known as the direction problems, of which there are two; the "narrow" and "wide" direction problems.

The narrow direction problem concerns the ordering of the constituents of the subordinate judgment in the whole judgment complex. Take, as an example, Othello's belief that Desdemona loves Cassio:

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[31] See *PM*, 1: 44.
[32] See *PM*, 1: 44–5.

$$B\{O, d, L, c\}.$$

We have a four-placed relation of belief obtaining between Othello, Desdemona, love, and Cassio. But the theory, as it stands, lacks a means of distinguishing Othello's belief that Desdemona loves Cassio from Othello's belief that Cassio loves Desdemona. In each case, we have the same relation relating the same objects but a different judgment in each. Russell attempted several solutions to the problem. In 1910 he had hoped to rely on the "sense" (or "direction") of the subordinate verb ("loves" in our example) to appropriately order its two objects.[33] By 1912, how-ever, he had realized that this was implausible; the subordinate verb is itself an object of the primary relation of the judgment and hence must be treated as a term of the complex, rather than what Russell calls a "relating relation". As Russell put it, "The relation 'loving', as it occurs in the act of believing, is one of the objects—it is a brick in the structure, not the cement" (PP_3, p. 74). Having accepted this, Russell called on the primary relation of the judgment-complex to provide the necessary order.

Russell's satisfaction with this position was short-lived. In the 1913 *Theory of Knowledge* manuscript, he again wrestles with the problem. The main shortcoming of this component of the theory is its appeal to some mysterious property of the judgment relation which will perform the ordering operation on the constituents of the judgment. How is this property to be explained and accounted for? In 1913 Russell accepted the implausibility of such a property of the judgment relation and decided to account for the order and unity of the judgment, not by appealing to the sense of the judgment relation, but by positing *logical forms* in judg-ments.[34] The idea here is that the introduction of the form of the judg-ment into the judgment-complex will ensure the required unity and order. Two immediate problems arise. First, there is a problem as to the status of logical forms. Russell seems unsure as to precisely what they are. On the one hand they appear to be complex constituents of judgment-complexes, but on the other hand they appear to demand special treat-ment compared to the other constituents of judgments. For example, in

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[33] See "On the Nature of Truth and Falsehood" in his *Philosophical Essays* (London: Longmans, Green, 1910); reprinted in *Papers* 6.

[34] See *Papers* 7: 116.

the judgment-complex

$$J\{S, a, R, b, xRy\}$$

(where "*xRy*" stands for the logical form of the subordinate complex), the logical form appears to be just another term in the complex. As it is itself apparently complex, we are faced with the threat of a regressive argument: invoking the logical form to explain the structure of the judgment will serve little purpose if we are now left wanting an explanation of the structure of the logical form. Russell's response is that, though at first glance the logical form may seem complex, it is in fact not capable of analysis into any simpler parts. It is, Russell, assures us, not so much a complex which *has* a structure, rather it *is* a structure—the structure of the judgment in question.[35]

The second problem is that the introduction of logical forms is not, by itself, a sufficient measure for safeguarding against the narrow direction problem. Knowing that *a*, *R*, and *b* are to be structured in accordance with the form *xRy* does not assist us in finding a way of distinguishing between *aRb* and *bRa*. In order to meet this requirement, Russell supplements his introduction of logical forms with the notion of *position* in a complex. The constituents of judgments are not only structured in accordance with a logical form, they must also have a definite position within that structure. All may now seem well, even if somewhat complicated, as regards the narrow direction problem. Wittgenstein, however, is credited with authorship of the "wide" form of the direction problem which, as we shall see, is even more harmful to the multiple-relation theory.

The wide form of the direction problem is a further consequence of a problem we have noted above. The multiple-relation theory requires that all constituents of a judgment-complex are to be suitable terms for the relation obtaining between them. As noted above, this requires that the subordinate relation feature as a term in the judgment, rather than as a relating relation. However, this requirement appears to leave the subordinate relation syntactically on a par with its objects. The resulting complex, stripped of any clues as to its content, will be of the form

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[35] See *ibid.*, p. 114.

$$J\{S, x_1, x_2, \ldots, x_n\}.$$

With no syntactic differences between the constituents of the judgment, nonsensical complexes will be just as admissible as the meaningful ones. In other words, whereas the narrow direction problem left us unable to distinguish between Othello's belief that Desdemona loves Cassio and Othello's belief that Cassio loves Desdemona, the wide direction problem makes it equally permissible for Othello to believe that Love "desdemonas" Cassio. As Wittgenstein incisively puts it in the *Tractatus*: "The correct explanation of the form of the proposition, '*A* makes the judgment *p*', must show it is impossible for a judgment to be a piece of nonsense. (Russell's theory does not satisfy this requirement)."[36]

WITTGENSTEIN'S CRITICISMS: THE SOMMERVILLE/GRIFFIN INTERPRETATION

Griffin[37] raises the question of why this criticism would have such devastating effects on Russell. Why not just accept that it is possible to believe nonsense? Griffin, following Sommerville, suggests that it is the incompatibility of the multiple-relation theory with the theory of types which leaves Russell "paralysed" in the face of Wittgenstein's objection. Sommerville and Griffin place great importance on the manner in which Wittgenstein first presented his objection to the multiple-relation theory in a letter to Russell dated June 1913:

I can now express my objection to your theory of judgment exactly: I believe that it is obvious that, from the prop[osition] "*A* judges that (say) *a* is in the Rel[ation] *R* to *b*", if correctly analyzed, the prop[osition] "*aRb* . ∨ . ~ *aRb*" must follow directly without the use of any further premiss. This condition is not fulfilled by your theory.[38]

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[36] Wittgenstein, *Tractatus Logico-Philosophicus*, trans. D. F. Pears and B. F. McGuinness (London: Routledge, 1961), 5.5422.

[37] Griffin, "Russell on the Nature of Logic (1903–1913)", *Synthese*, n.s. 45 (1980): 176–8; and "Wittgenstein's Criticism of Russell's Theory of Judgment", *Russell*, n.s. 5 (1985): 144.

[38] Wittgenstein, *Cambridge Letters*, ed. B. F. McGuinness and G. H. von Wright (Oxford: Blackwell, 1995), p. 22.

Sommerville and Griffin take this to be an implicit reference to *Principia*'s *13.3. The dyadic analogue of this proposition, they point out, will be:

$$\vdash (aRb \lor \sim aRb) \supset \{(xRy \lor \sim xRy) \equiv [(x = a \ \& \ y = b) \\ \lor (x = a \ \& \ y \neq b) \lor (x \neq a \ \& \ y \neq b)]\}.$$

The antecedent clause simply gives the conditions of significance for *aRb*. According to Sommerville and Griffin, however, securing such conditions will be deeply problematic within the context of the multiple-relation theory. Wittgenstein's point is that, if the judgment of nonsense is to be avoided, then the judgment-complex *J*{*S, a, R, b*} should lead directly to the tautological *aRb* ∨ ~*aRb* without the need for further stipulations. Russell's theory, it is true, does not meet this requirement. Such an inference is only plausible if we have a guarantee that, for example, *a* and *b* are individuals, *R* is a relation of the appropriate type and order, and so on. Nonetheless, one is tempted to think that Wittgenstein's objection, though drawing attention to the painfully cumbersome mechanics of the theory of judgment, is less harmful than Russell's reaction suggests. Why can we not just accept that such stipulations are required? Griffin maintains that Wittgenstein will not allow us to make such stipulations for good reason:

Because to make them would require further judgments. We are trying to analyze what is supposed to be the simplest kind of elementary judgment. But to do so would seem to involve us in yet further judgments. Moreover the further judgments required are of an extremely problematic character. For to judge that *a* and *b* are suitable arguments for a first-order relation is to make a judgment of higher than first-order. Yet, as Russell makes quite clear in *Principia* (pp. 44–6), higher-order judgments are to defined cumulatively on lower-order ones. Thus we cannot presuppose second-order judgments in order to analyze elementary judgments.[39]

In short, Griffin interprets Wittgenstein's objection as pointing out that, in order to make the stipulations needed for the correct analysis of the judgment, we would first need to invoke type-theoretic distinctions to

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[39] "Wittgenstein's Criticism of Russell's Theory of Judgment" (1985), p. 144.

guarantee the sense of the judgment. But this turns the whole structure on its head, for type-theory is supposed to emerge from the multiple-relation theory of judgment, not provide the foundations for it. Hence to invoke type-theory in order to evade the problem only serves to make the problem more acute. Specifying the types and orders of the constituents of the judgment under analysis can only be done by making further judgments which will, by definition, be of higher order than the order of those elements we are seeking to specify. As Griffin rightly points out, this would be in direct contradiction of the recursive procedure for the establishment of orders given in *Principia*.

Ingenious though the Sommerville/Griffin interpretation of Wittgenstein's criticisms is, I do not find it convincing. The account of ramified type-theory given above showed that the type part of the ramified hierarchy has no significant connection with the multiple-relation theory of judgment (which was shown to be responsible purely for the order part of the hierarchy). The kinds of type distinctions that Wittgenstein suggests are called for in order to prohibit nonsensical pseudo-judgments such as Othello's belief that Love desdemonas Cassio, do not require the multiple-relation theory for their generation.

Griffin holds that the direct inspection argument is insufficient, as it stands, for furnishing Russell with the kinds of type-distinctions that Wittgenstein's objection shows to be needed if the constraint of meaningfulness is to be met. Only when supplemented by the multiple-relation theory, he argues, can the notion of acquaintance provide the necessary distinctions. His argument (following Sommerville's[40]) is that we cannot be acquainted with type *differences*: "[E]very act of acquaintance with a logical object is of a different logical type to an act of acquaintance with a different logical object."[41] Griffin concludes that no single act of acquaintance can take in more than one logical object; hence the very idea of acquaintance with type differences between two or more logical objects must be out of the question. Having decided that

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[40] Stephen Sommerville, "Wittgenstein to Russell (July 1913): 'I Am Very Sorry to Hear … My Objection Paralyses You'", in *Language, Logic and Philosophy; Proceedings of the 4th International Wittgenstein Symposium* (Vienna: Holder-Pichler-Tempsky, 1981), pp. 186–7.

[41] Griffin, "Russell's Multiple Relation Theory of Judgment", *Philosophical Studies*, n.s. 47 (1985): 243.

acquaintance will not help, Griffin takes it that the weight of providing the necessary distinctions will be forced back onto the act of judgment: "In Russell's epistemological system of 1910–1913, the type distinctions required for judgment could *only* be obtained by means of prior judgment" (*ibid.*).

Griffin's recognition of the distinction between types and orders, though welcome, is not fine-grained enough to do justice to *Principia*'s formal grammar. "Acts of acquaintance" do not divide into types in the sense that acts of judgment divide into orders in *Principia*'s explanation of ramified type-theory. Judgments divide into orders corresponding to the kind of truth (or falsehood) applicable to them; acquaintance (as a mental occurrence independent of, and more primitive than, judgment) is not something to which we can attribute a truth-value, and hence the justification for its division into a hierarchy is unfounded.

With this in mind, it is clear that the kinds of distinctions Griffin's Wittgenstein demands for the avoidance of nonsense are type, rather than order, distinctions. The wide direction problem shows that some distinction in type must be made between, say, a dyadic relation and its referents and relata; the relational status of the relation must survive analysis if we are to fortify the theory of judgment against the possibility of admitting nonsense. There is no reason to share Griffin's insistence that such distinctions will rely on the very theory of judgment they have been invoked to serve. Types are fixed independently of the generation of orders by the multiple-relation theory, and there does not seem to be any reason for assuming that types cannot also be fixed prior to the generation of orders.[42] Hence the kinds of distinctions which Wittgenstein calls for seem to be adequately provided for. Why, then, was Wittgenstein's objection so devastating for Russell?

A more likely explanation for Russell's "paralysis" in the face of Wittgenstein's objection emerges from a more general observation of the different conceptions of the nature of logic held by the two. The central and defining feature of Russell's attitude towards logic is his insistence on the universality of logic; the view that logic applies equally and equivalently to all things. This, we saw earlier, is a constant theme throughout

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[42] The question to what extent the type and order indices of a given term are independent of one another is a separate issue: all that matters here is that orders are generated by the multiple-relation theory; types are not.

his struggle with the set-theoretic and semantic antinomies and is a governing principle behind the multiple-relation theory. The purpose of the multiple-relation theory, reflecting this principle, is to generate orders of judgment without positing corresponding *ontological* divisions. Every logical subject is to be treated as an ontological equal. Wittgenstein's objection to the multiple-relation theory, however, exposes this enterprise as a failure.

Judgments, for Russell, are no more linguistic than the propositions they have been invoked to replace. Judgments, like propositions before them, are composed of entities, not words. Unlike propositions, however, judgments are not themselves entities over and above the entities which are their constituents. In this way, Russell intended the multiple-relation theory to preserve the ontological simplicity that his conception of logic demanded; type distinctions were to be kept distinct from ontological categories. Throughout this entire period in Russell's thought, he was a staunch realist. Universals were just as welcome in his ontology as particulars and, indeed, were necessary constituents of judgment complexes.[43] The price of being admitted into Russell's ontology, however, is that all members of that ontology should, ideally, be of the same logical type. This ambition, we have seen, is maintained after the adoption of type theory, and places on the multiple-relation theory the requirement that all constituents of judgments stand on an equal ontological footing. Wittgenstein shows, however, that this analysis is inadequate for the treatment of universals (attributes in intension). Wittgenstein's criticism of the multiple-relation theory is wide-reaching in its consequences, for it shows that Russell's insistence on treating properties and relations

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[43] Russell was aware in 1903 that the presence of a universal ("verb", as he put it) in a proposition was a necessary condition for the unity of the proposition (though not a sufficient one). Hence his analysis then allowed for universals to have a "two-fold nature" whereby they can occur as either "verb" or "verbal noun": "The two-fold nature of the verb, as actual verb and as verbal noun, may be expressed, if all verbs are held to be relations, as the difference between a relation in itself and a relation actually relating" (*PoM*, §54). The doctrine is echoed in the multiple-relation theory's treatment of the subordinate relation as a term. Passages such as this have led many to suggest that Russell's analysis of propositional content, constantly plagued by the problem of the unity of the proposition, is inferior to Frege's analysis of propositional content into complete and incomplete parts. For a defence of Russell on this point, and a dissenting view of Frege's achievement, see my "The Truth and Nothing But the Truth, Yet Never the Whole Truth: Frege, Russell and the Analysis of Unities" (2003).

in intension as logical subjects is misguided (at least within the confines of the logical theory he advocated between 1910 and 1913).[44] The Sommerville/Griffin interpretation, then, is partly correct in so far as it recognizes that the severity that Russell attached to Wittgenstein's criticisms hinged on the consequences it raised for the theory of types. Those consequences were not that the criticisms showed that the multiple-relation theory had to make an implicit appeal to the order distinctions it was intended to generate. Rather they were that the multiple-relation theory, if it was to be successful, required the imposition of the type part of the ramified hierarchy onto Russell's ontology in exactly the way that the multiple-relation theory had been intended to help him avoid.

Russell's "paralysis" is unsurprising. The problems raised by Wittgenstein's criticisms are not restricted to pointing out an inadequacy in Russell's structural analysis of judgment. Russell's insistence on the non-linguistic nature of judgments means that the very situation he strove so hard to avoid in his solution of the paradoxes now becomes unavoidable if the multiple-relation theory is to be maintained. The type distinctions that Wittgenstein shows to be required for the adequate regimentation of judgments will be ontological distinctions. In Othello's belief that Desdemona loves Cassio it is Desdemona, Cassio, and the relation of love which Othello's belief places him in a multiple-relation to, rather than the linguistic items which represent each constituent. Were such type distinctions to be accepted, which it seems they must if the multiple-relation theory is to survive, then the very purpose of the multiple-relation theory (namely, to generate orders without different types of entities) will have been thwarted before the theory can get off the ground. Russell was left with no choice but to abandon *Theory of Knowledge* and to cut the formal edifice of *Principia* loose from its doomed epistemological moorings.[45]

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[44] Ramsey later commented, in a letter to Wittgenstein, that: "Of all your work [Russell] seems now to accept only this: that it is nonsense to put an adjective where a substantive ought to be which helps him in his theory of types" 20 Feb. 1924; *Cambridge Letters*, p. 197).