SUBSTITUTION AND THE THEORY OF TYPES

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Gregory Landini. *Russell's Hidden Substitutional Theory.* Oxford and New York: Oxford U.P., 1998. Pp. xi, 337. £69.95; US$92.00.

When, in 1902, Russell communicated to Frege the famous paradox of the class of all classes which are not members of themselves, Frege immediately located the source of the contradiction in his generous attitude towards the existence of classes. The discovery of Russell's paradox exposed the error in Frege's reasoning, but it was left to Whitehead and Russell to reformulate the logicist programme without the assumption of classes. Yet the "no-classes" theory that eventually emerged in *Principia Mathematica*, though admired by many, has rarely been accepted without significant alterations. Almost without exception, these calls for alteration of the system have been induced by distaste for the theory of types in *Principia*.

Much of the dissatisfaction that philosophers and logicians have felt about type-theory hinged on Whitehead and Russell's decision to supplement the hierarchy of types (as applied to propositional functions) with a hierarchy of orders restricting the range of quantifiers so as to obtain the ramified theory of types. This ramified system was roundly criticized, predominantly because it necessitated the axiom of reducibility—famously denounced as having "no place in mathematics" by Frank Ramsey, who maintained that "anything which cannot be proved without it cannot be regarded as proved at all".[1] Russell himself, who had admitted that the axiom was not "self-evident" in the first edition of *Principia* (*PM*, 1: 59[2]), explored the possibility of removing the axiom in the 1925 second edition. Ramsey's extrication and expulsion of the order part of the ramified hierarchy, along with the offending axiom, placated

[1] F. Ramsey, *The Foundations of Mathematics and Other Logical Essays*, ed. R. Braithwaite (London: Routledge, 1931), p. 28.

[2] All page references to *PM* are to the 2nd edition, 1925–27.

some of the opponents of type-stratified logic, but dissatisfaction remained. Even when confronted by only the simple theory of types, most remained unconvinced that Whitehead and Russell really had reduced mathematics to logic, rather than simply added, *ad hoc*, sufficient complexity to logic to provide it with the resources to express mathematical reasoning.

Russell's earlier *Principles of Mathematics* had presented the logicist thesis with extraordinary elegance. Logic was there presented as a universal science, applying indiscriminately to everything. This account of logic, reflected in the formal "doctrine of the unrestricted variable" (the requirement that the non-logical constituents of every Russellian proposition be treated as belonging to one all-inclusive logical type), provided the philosophical appeal of the logicist project. By contrast, type-restricted variables appear devoid of any philosophical appeal other than their formal ability to block the paradoxes.

Or so traditional interpretations of Russell's philosophical and mathematical logic have maintained. Gregory Landini's hugely important book is a determined, and largely successful, effort to expose this traditional picture as wholly inaccurate. According to Landini's interpretation, previous attempts to make sense of the development of the theory of types have run into a locked door. The key that is required to unlock that door is Russell's much neglected substitutional theory of classes and relations developed between 1905 and 1907.

It is perhaps unsurprising that the substitutional theory has been neglected until very recently. Russell's two most important papers on the theory, though both written in 1906, were not made widely available until 1973,[3] and several important manuscripts on the subject went unstudied until very recently. These fascinating manuscripts, forthcoming in Volume 5 of *The Collected Papers of Bertrand Russell*, are subjected to a detailed analysis by Landini, which results in a comprehensive commentary on the theory. This achievement alone would be of great merit, but Landini's study has more to offer than an explanation of the mechanics of the calculus of substitution.

Russell's Hidden Substitutional Theory divides into three sections, covering roughly the periods 1903–05, 1905–06, and 1906–10. The first section provides an exposition of the philosophy espoused in the *Principles*, extracting a Russellian calculus of propositions from the informal sketches made in that work. The discussion pays particular heed to the doctrine of the unrestricted variable,

[3] Namely, "On the Substitutional Theory of Classes and Relations" and "On 'Insolubilia' and Their Solution by Symbolic Logic". The former was withdrawn from publication by Russell after he found the system, as presented there, needed some improvements. The latter was published in French in 1906 under the title "Les Paradoxes de la Logique", *Revue de Métaphysique et de Morale*, 14: 627–50. Both papers are reprinted (in English) in Russell, *Essays in Analysis*, ed. Douglas P. Lackey (London: Allen and Unwin, 1973).

illustrating admirably Russell's reasons for making the doctrine the pivot on which any system of symbolic logic must turn. The connections between this syntactic doctrine and its foundations in the philosophical programme instigated by Russell and Moore in response to their neo-Hegelian predecessors is clearly explained in the first two chapters, alongside discussions of Russell's relation to Frege and Peano. Acknowledgement is also made of the influence on Russell of Cantor, Weierstrass and Dedekind. Chapter 3 is devoted to the impact of the 1905 theory of descriptions on the logic of the *Principles*, and this paves the way for the introduction of the substitutional theory in the second section of the book.

In the second section, Landini presents a rigorous formulation of the substitutional calculus drawn from the (somewhat less rigorous) presentations in Russell's manuscripts. Drawing mainly from a 1905 manuscript[4] (which contains an axiomatization of the substitutional calculus) and from a system published by Russell around the same time,[5] Chapter 4 offers a complete reconstruction of the basic calculus of substitution. Proofs of some of the fundamental theorems that Russell left either unproved, or only stated with brief sketches of their proofs in the manuscripts, are provided. Some of these apparently simple theorems are trickier to prove than Landini's skillful presentation would have the unwary reader believe, and a careful study of them quickly produces an appreciation both of Russell's genius in devising the theory, and of the depth of study that Landini has given to it. Elements of the reconstructed system do draw on a knowledge of formal developments that it is controversial to attribute to Russell, though justification is given for their introduction by detailed textual analysis and argumentation. Some may still remain unconvinced, in places, that Russell himself saw these distinctions as clearly as Landini does. For example, a clear distinction between the object- and meta-language of the system is utilized throughout—a distinction commonly thought to be unavailable to Russell at this time. The justification for this has to be traced to Landini's argument, in an earlier chapter of the book, that evidence can be found in the *Principles* of an awareness of the distinction in Russell's response to Lewis Carroll's paradox (Landini, p. 45; see *PoM*, p. 35).

The remaining chapters in the second section of the book deal predominantly with the technical aspects of the substitutional theory. The basic idea behind the theory is that both the calculus of classes and (first- and higher-order monadic and polyadic) predicate calculi can be proxied[6] by the method

[4] Russell, "On Substitution", ms. dated 22 Dec. 1905 (RA1 220.010940б).

[5] Russell, "The Theory of Implication", *American Journal of Mathematics*, 28, (1906): 159–202. The paper was written in 1905.

[6] The term "proxy" is Landini's. The substitutional theory is a proxy of simple type-theory in

of substitution. Most importantly, however, the substitutional theory naturally proxies these calculi in their type-stratified forms. In other words, the logical grammar of the substitutional theory yields a system of logic equivalent to a predicate logic or calculus of classes stratified by a simple theory of types. In short, the substitutional theory appears to answer in one swift stroke almost all of the criticisms of Russell's logicism and, remarkably, does so with no ontological commitment whatsoever to either classes or propositional functions. All of the usual criticisms of type-theory are met completely if, like Landini, we understand the *real* theory of types simply to be the substitutional theory (p. 5). For the benefit of those unfamiliar with the theory, I will give a brief sketch of the basic ideas involved.

The theory demands an ontological commitment to propositions since these are essential to the theory's ability to recapture the advantages of the predicate and class calculi without the disadvantage of sharing their paradox-spawning ontologies. There are no distinctions in types of entities in the substitutional theory. In fact there is only one kind of entity supposed by the theory, hence there is only one type of variable: the "entity" or "individual" variable (thus the doctrine of the unresestericted variable is faithfully adhered to). There are, of course, no propositional functions in the theory, but, in their place, we find *matrices* of the form "*p/a*". Here *p* is called the *prototype* and *a* is called the *argument*. The heart of the system is the operation of substitution, whereby an entity is substituted for the argument in a matrix; for example, the substitution of *x* for *a* in *p*, which is written as "*p/a*’*x*".[7] This is to be read as "the *q* which results from the substitution of *x* for *a* in *p*". The formula "*p/a*’*x*! *q*" is to be read as, "*q* results from *p* by substituting *x* for *a* in all those places (if any) where *a* occurs in *p*".[8] An obvious consequence of the theory is that the matrices we now have in place of propositional functions are, in a very obvious way, "incomplete symbols". In the matrix *p/a*, *p* and *a* are both entities but *p/a* is not (the symbol "*p/a*" does not stand for a complete proposition but should be read as "the result of replacing *a* in *p* by …"). Russell's claim, however, is that matrices are best understood as classes (or, perhaps it is better to say, classes should be understood as being matrices and, hence, are incomplete symbols or "logical fictions"). With *p/a* understood as being a class, *x* is a member of *p/a* if and only if *p/a*’*x* is true.[9] Understanding classes in such a way has the in-

the sense that the formal grammar of substitution respects the type-distinctions that would apply to translations of substitutional formulas into a predicate calculus with type restrictions.

[7] On occasions Russell uses the alternative notation of "*p* (*x/a*)" which means the same as "*p/a*’*x*". See Russell, "On Some Difficulties in the Theory of Transfinite Numbers and Order Types" (1906), reprinted in *Essays in Analysis*, p. 155.

[8] Russell, "On the Substitutional Theory of Classes and Relations", *Essays in Analysis*, p. 168.

[9] See Russell, "On the Substitutional Theory of Classes and Relations", *Essays in Analysis*, p.

valuable merit of disposing of the Russell paradox, for self-membership of classes now becomes impossible: the equivalent of a proposition such as $x \in x$ in the substitutional theory would require a matrix of the form $p/a\text{;}(p/a)$ which is simply nonsense (amounting to something like "the result of replacing *a* in *p* by the result of replacing *a* in *p* by …"). Russell developed the theory to an impressive level of sophistication, using dual substitutions (simultaneous substitutions of more than one constituent in a matrix) to proxy higher type functions (and classes of classes).

The details of the substitutional theory are explored by Landini in lavish detail in Chapters 4–7. These chapters, it must be said, are highly technical in places, and some readers lulled into a false sense of security by Landini's promise in the preface to "presuppose only a knowledge of predicate logic" (p. vi) may flounder here. As Landini states (*ibid.*), however, Chapters 6–7 can be passed over without loss of continuity by those who want to follow the philosophical and historical developments without immersion in the technicalities of substitution. This is not to say that these chapters are superfluous; far from it. In fact, a careful study of them yields some of the most rewarding parts of the book. It is in these chapters that Landini delivers on his promise to illustrate the feasibility of the substitutional theory as a foundational logic for mathematics. Previous discussions of the theory have all tended to dismiss the theory (after little study) as either a classic example of Russell's tendency to stumble into a hopeless mess of use/mention confusions, or as a theory that Russell quickly abandoned as inadequate for capturing the necessary principles of mathematics. Both views are utterly refuted by Landini's study. Charting the development of Russell's thought, manuscript by manuscript, Landini reveals an elegant system which is neither confused nor inadequate. A proof that any theorem of simple type-theory has a translation into a theorem of the substitutional theory is given (pp. 140–4) in Chapter 5. Chapter 6, dealing with the substitutional theory as a proxy of a theory of classes, shows how the natural numbers are to be defined in a substitutional language (pp. 148–9). Two alternative approaches to classes are pursued: one drawn from hints in Russell's "On the Substitutional Theory of Classes and Relations" and one from a 1906 manuscript.[10] Recognizing the superiority of the latter alternative, Landini devotes most attention to it, and the general theory of classes is given on these lines. Proofs in this system of the Peano postulates for arithmetic are provided in an appendix (pp. 299–313). The chapter concludes with a comparison of the substitutional approach to classes

172. It should be noted that, on Landini's interpretation, this informal explanation does not imply the presence of a truth-predicate in the substitutional calculus. A more precise definition, then, might be "$x \in p/a =_{\mathrm{df}} (\exists q)(p/a\text{;}x!q\,.\,\&\,.\,q)$".

[10] Russell, "On Substitution", ms. dated April–May 1906 (RA1 220.010950).

with the theory in *Principia* (also, of course, a "no-classes" theory). All of this raises an important question, however: if, as Landini demonstrates, the substitutional theory is not inadequate to the task of generating arithmetic, why was it abandoned? Landini's answer to this question comes in two parts.

In giving the first part of the answer, Landini points out that the substitutional theory lingered in Russell's work far longer than is commonly recognized. Indeed it is firmly in place at the centre of the formal system offered in Russell's 1908 masterpiece, "Mathematical Logic as Based on the Theory of Types",[11] though many have failed to notice it there. However, there is no escaping the fact that the notation of substitution, particularly in the context of class theory, poses a severe challenge to the patience of its user. As Landini concedes, "it is, in fact, tedious to the point of practical impossibility to work in the substitutional notation [without standard class symbols]" (p. 174). Faced with this practical impasse, Whitehead and Russell opted to present their system in a more conventional form. In "Mathematical Logic", however, Russell makes it plain that this is for practical reasons only, and functions should be understood as being derived, ultimately, from substitution. Furthermore, both authors were keen to include an appendix on the substitutional theory in *Principia* at this time. As Landini points out (pp. 174–5), the decision to relegate the theory to an appendix and retain the notation of predicates and classes would by no means have been a rejection of substitution but, rather, simply a practical decision taken as much for the benefit of their readership as for themselves. Furthermore, Landini's translation function, mentioned above, would have allowed the substitutional theory to lay claim to the foundations of the system, even with a type-stratified class calculus used to express the proofs: "Practically, all that is required is a type-indexed language of class symbols plus a translation manual" (p. 175). Why, then, did this appendix not appear in the 1910 publication of *Principia*? This brings us to the second part of Landini's answer and the third section of his book.

The short answer is that Russell discovered the substitutional theory to be inconsistent. A paradox, derived by a Cantorian diagonal argument, can be formulated in the substitutional system set out in the second section of Landini's book. This paradox, first unearthed from the Russell Archives by Landini,[12] takes centre stage in Chapters 8 and 9 of the book, as it did in Russell's manuscripts in 1906. Against the received wisdom that semantic paradoxes forced Russell into ramifying the theory of types, Landini holds that it was this "syntactic" paradox that led Russell, first to modify the substitutional theory

[11] See Russell, "Mathematical Logic as Based on the Theory of Types", *LK*, p. 77.

[12] See Landini, "New Evidence concerning Russell's Substitutional Theory of Classes", *Russell*, n.s. 9 (1989): 26–42.

(by abandoning his ontological commitment to general propositions), then to ramify it (in "Mathematical Logic"), and finally to abandon it altogether in favour of the "no-propositions" theory of *Principia.* Most controversially, Landini claims (in Chapter 10) that ramification in *Principia* is intended, not to ensure the avoidance of semantic paradoxes (though, of course, that is a subsidiary benefit), but to *retain* the doctrine of unrestricted variation that most hold it to have finally destroyed.

Chapter 8 covers the modifications that Russell made to the substitutional theory in "On Insolubilia". The salient difference, on Landini's interpretation, lies in Russell's decision to refuse general propositions entry into his ontology. Formally, this entails that no quantified formula can be nominalized to form a term in the substitutional calculus. The paradox is blocked but, without general propositions, so is much of the mathematics that the system is supposed to generate. Russell had no choice but to introduce a "reducibility" axiom that relaxed his stern line on general propositions by admitting that general statements, while not propositions, can sometimes be *equivalent* (co-extensional) to them. It is important to Landini's interpretation that this axiom is not, in any way, to be conflated with *Principia*'s (or "Mathematical Logic"'s) axiom of reducibility: on this interpretation, after all, there are no orders of propositions *to* reduce here.[13] Unfortunately, as Landini demonstrates, the axiom re-admits the paradox that general propositions were ousted to avoid (pp. 232–3).

It is the failure of "On Insolubilia" that Landini holds responsible for driving Russell into full-blown ramification of the substitutional theory in "Mathematical Logic". Furthermore, it is the failure of "On Insolubilia" to block the substitutional paradox, not the semantic paradoxes (which are adequately dealt with in "On Insolubilia"), that led to ramification. This is, of course, of enormous importance. If Landini is right that the "syntactic" substitutional paradox led to ramification, then Ramsey's suggestion, echoed by Quine and Gödel, that ramification is unnecessary for the mathematical logician who is unconcerned with semantics, is shown to be incorrect:

> When applied to "Mathematical Logic", Quine's quip that the ramified theory "calls for amputation" is misguided. Due to the p_0/a_0 [substitutional] paradox, ramification and the substitutional theory are connected at the chest, and it is not clear that amputation is possible without killing the patient. (Landini, p. 254)

The problem with the solution to this paradox in "Mathematical Logic", ac-

[13] This point was first noted by Nino Cocchiarella, "The Development of the Theory of Logical Types and the Notion of a Logical Subject in Russell's Early Philosophy", *Synthese*, 45 (1980): 71–115. Cocchiarella also noted the crucial distinction between "Mathematical Logic" and *PM* in the same article.

cording to Landini, is that ramification and substitution, joined at the chest though they are by the paradox, stand in direct ontological opposition to one another. The whole point of the substitutional theory was to avoid restricted variables by building type-distinctions into logical grammar. To then fall back on a hierarchy of orders which is attached to the theory without philosophical justification is simply to lose everything gained by substitution. Understandably dissatisfied with the situation, Russell abandoned substitution and opted for a very different solution to the problem in *Principia*.

There can be little doubt that Landini is correct in this account of the difference between "Mathematical Logic" and *Principia*: the latter abandons propositions in favour of the multiple-relation theory of judgment, and, as mentioned previously, substitution cannot survive without an ontology of propositions. Nor is Landini on overly controversial ground in suggesting that the multiple-relation theory is intended to preserve a form of unrestricted variation: it has been noted by several other commentators that Russell intended to preserve ontological simplicity by invoking the epistemological theory of judgment. What is controversial, however, is Landini's claim that Russell *succeeds* in the enterprise as well as his account of how this success is achieved.

Landini's claim is that the multiple-relation theory is but one component of a semantic enterprise that aims to write type-distinctions into the admissible interpretations of predicate variables in *Principia*. These semantic manoeuvres replace the syntactic limitations previously entailed by the grammar of substitution. Predicate variables are to be "internally limited" by their "conditions of significance" rather than being syntactically restricted (pp. 274–5). Obviously, on an objectual interpretation of these variables (an interpretation of them as taking entities as values), we would still have a type-stratified ontology. Landini's solution is that the semantics in question is a *nominalistic* one: what appear to be predicate variables ranging over entities in *Principia* are actually schematic letters whose values are statements.

Of course, Landini's interpretation needs to be somewhat complicated if it is to preserve the doctrine of the unrestricted variable in *Principia* without committing essential elements of Russell's ontology to the flames. Individual variables (interpreted objectually) must be retained, for example, or the axiom of infinity would fail (p. 277). Nor should the presence of a nominalistic semantics be taken to illustrate an actual commitment to nominalism, as this would clearly contradict Whitehead and Russell's ontological commitment to universals (*ibid.*).[14] Landini concludes that we have to distinguish between individual and predicate variables. Only individual variables are genuine variables, and

[14] See, for example, *PM*, 1: 43.

these are unrestricted, ranging over universals and particulars (and, indeed, complexes composed of them); the "predicate variables", however, are dummy variables—metalinguistic letters whose values are well-formed formulas in the semantics. The difference between the sorts of variables is then simply a difference in the kinds of quantifiers used to bind them. Individual (genuine) variables are bound by an objectual quantifier; while a substitutional quantifier ranges over the predicate variables.[15] The result is that the values of the predicate variables will be well-formed formulas whose truth-conditions are determined in such a way as to respect order-restrictions by the recursive truth definition at the heart of Russell's new correspondence theory of truth (namely the multiple-relation theory).

This explains how Landini thinks the ramified theory of types is to be understood as a semantic theory rather than a syntactic one, but we still need justification for the claim that the semantics is nominalistic. In fact, though undeniably revisionist, Landini's claim does fit remarkably well with the text of *Principia.* All that Landini is doing, though in ingenious fashion, is offering an interpretation of *Principia* which takes seriously Whitehead and Russell's proclamation that they have expelled propositions from their ontology.[16] The standard interpretation of *Principia*, largely due to Quine,[17] is that the ontological cutbacks attempted in the work backfired on its authors. In attempting to rid themselves of the burden of classes, Whitehead and Russell had to quantify over propositional functions,[18] thus committing themselves to an even less desirable ontology of universals. Of course, an ontology of universals, though noxious to Quine, was perfectly palatable to the authors of *Principia.* The problem, however, comes with the requirement that propositional functions carry (suppressed) order/type indices. If propositional functions are interpreted in realist fashion, there seems no choice but to conclude that the ontology of *Principia* is type-stratified into obscurity. If, however, we follow Landini in refusing to disregard the absence of propositions from *Principia*'s ontology, then the realist interpretation of propositional functions can be resisted. Indeed, as Landini notes, it is hard to see how the realist interpretation can be maintained; a propositional function "differs from a proposition solely by the fact that it is ambiguous" in *Principia* (*PM*, 1: 38). Taking seriously the

[15] The modern substitutional interpretation of quantification (which Landini finds in *Principia*) should not be conflated with Russell's earlier substitutional theory. The former is concerned with the substitution of *expressions* in formulas; the latter with substitutions of *entities.*

[16] See, for example, *PM*, 1: 44.

[17] See, for example, Quine, "Logic and the Reification of Universals", *From a Logical Point of View* (Cambridge, Mass.: Harvard U.P., 1961; 1st ed., 1953), pp. 122–3.

[18] See *PM*, 1: *20.01.

idea of a no-propositions theory, Landini concludes that the only candidates for propositions and propositional functions in such passages are *linguistic* items: "Since there are no propositions in an ontological sense, it seems clear that a proposition is just an open wff [well-formed formula]" (p. 277). This, as Landini goes on to mention (*ibid.*), certainly comports with remarks in Russell's later writings,[19] though Landini does not note that most of these were written after Russell had come into contact with Wittgenstein (whose conviction that logic was essentially linguistic undoubtedly influenced Russell later on).

There are still more controversial components to Landini's interpretation. A seemingly obvious consequence, of course, is that the logical constants of *Principia* now have to be understood as modern statement connectives. This is a significant change from the view expressed in the *Principles*, where the doctrine of the unrestricted variable demanded that the logical constants be understood as predicates and relations. With no propositions, however, it seems that those relations have nothing to relate. This, at least, is how Landini views the situation (pp. 255, 258). However, the assumption made by Landini is not, I think, one we should automatically share. Landini takes it that, in the absence of an ontology of propositions, all talk of "propositions" in *Principia* is best understood as referring to well-formed formulas. Another alternative, however, which remains faithful to the 1910 philosophical introduction of *Principia*, is to interpret propositions as judgment complexes (as defined by the multiple-relation theory). Such complexes, according to Landini, are within the range of the genuine (entity) variables (p. 292). Furthermore, expressions for them can, presumably, be nominalized to form singular terms referring to them in line with the more traditional Russellian logic, since Landini interprets *Principia*'s propositions (statements) as disguised definite descriptions of them (p. 287). In other words, the fact that reference to propositions can be eliminated via contextual definitions need not imply that things cannot be predicated of them anymore than the contextual definition of a denoting phrase need imply that the non-existence of the present King of France means that baldness is not predicated in "The present King of France is bald".[20]

Landini offers several other arguments in support of his interpretation of *Principia*. A common error in previous interpretations, he maintains, has been the automatic assignment to the vicious-circle principle (VCP) of the role of justifying ramification. Having made this assumption, most interpreters have thought Russell was simply wrong in diagnosing this as the common source of

[19] See, especially, *Introduction to Mathematical Philosophy* (1919), pp. 155, 157.

[20] Sadly, Russell's 1913 manuscript *Theory of Knowledge* was abandoned at just the point where he was about to discuss molecular judgments, so the answer to this question of interpretation is not an easy one to give. See editors' introduction to *Papers* 7.

all the paradoxes. Having rejected the realist interpretation of propositional functions in *Principia*, however, Landini holds the VCP to be simply a consequence of the recursive definition of truth. Rather than being a solution of the paradoxes, therefore, the VCP is, as Russell said when he first accepted it, a condition that must be met by any philosophically convincing solution to the paradoxes.[21] Similar remarks in Russell's manuscripts lend support to Landini's intepretation,[22] though other interpretations which rejected the VCP as a solution to the paradoxes without positing a nominalistic semantics for *Principia* would find equal support. In other places, Landini's interpretation is likely to shock anyone familiar with the traditional readings of *Principia*'s formal grammar. Standard accounts of predicativity and circumflexion, for example, are turned on their heads, as Landini argues that "all and only predicate variables are predicative" (p. 264) and that "circumflexion was not a term forming operator in *Principia*" (p. 265). Whether or not one shares Landini's interpretation of *Principia* in full, however, his best argument for taking seriously Whitehead and Russell's rejection of propositions is wholly convincing. If propositions did feature in *Principia*, the multiple-relation theory would be utterly superfluous and, furthermore, there would have been no reason to abandon the substitutional theory. Only the wholesale rejection of propositions can explain the differences between "Mathematical Logic" and *Principia*.

The interpretation of *Principia* is, without question, the most controversial component of *Russell's Hidden Substitutional Theory*. It is certainly not its only controversial element, however. Landini's insistence that ramification was forced on Russell by logical paradoxes rather than semantic ones will almost certainly provoke objections from several quarters, and not without justification. Though Landini's discovery of the substitutional paradox certainly casts new light on this issue, it is doubtful that Russell was more than remotely aware of the distinction between logical and semantic paradoxes before it was pointed out by Ramsey. In the introduction to the second edition of the *Principles*, for example, it is Ramsey to whom Russell credits the discovery, and Russell also seems to acknowledge the superiority of "Ramsified" type-theory over his own: "This renders possible a great simplification of the theory of types, which, as it

[21] See Russell, "On 'Insolubilia' ", *Essays in Analysis*, p. 205.

[22] In "The Paradox of the Liar", ms. dated Sept. 1906 (RA1 220.010930), for example, Russell remarks that the Liar paradox shows that a proposition about a set cannot be a member of that set: "This impossibility cannot, however, be simply decreed because of the paradox; we must find some reason in the nature of the propositions which shows that the impossibility subsists" (fol. 2). He goes on to express reservations about blocking the paradox through either abandonment of propositions, or introduction of orders, because "it is difficult to express either in a form with anything to commend it except the solution of paradoxes" (fol. 7).

emerges from Ramsey's discussion, ceases wholly to appear unplausible or artificial or a mere *ad hoc* hypothesis designed to avoid the contradictions."[23] Later, in *My Philosophical Development*, Russell again refers to Ramsey's efforts at solving the paradoxes, and remarks: "But during the years before the publication of *Principia Mathematica*, I did not have the advantage of these later attempts at a solution, and was left virtually alone with my bewilderment."[24] Russell then seems to explicitly acknowledge that he had not seen the distinctions that Ramsey later did: "There were older paradoxes, some of them known to the Greeks, which raised what seemed to me similar problems, though writers subsequent to me considered them to be of a different sort. The best known of these was the one about Epimenides, the Cretan, who said that all Cretans were liars" (*ibid.*). Passages such as these strongly suggest that Russell was as much driven by semantic paradoxes (the Epimenides at any rate) as by the substitutional paradox, and that the distinction between the two was one that evaded his attention.

In fact, Landini, in accordance with his belief that the paradoxes of substitution were the real incentive for ramification, does not pay a great deal of attention to the Epimenides: "Curiously, this paradox has been thought to have played a pivotal role in the historical development of ramification. But we shall see that in the context of substitution, its role is quite minor" (p. 201). Landini then proceeds to quickly assimilate the Epimenides to the Propositional Liar. He then briefly surveys a treatment of the paradox in one of Russell's 1905 manuscripts before dismissing it as largely irrelevant to the substitutional theory and moving on to the syntactic contradictions.[25] The "Statement" (as opposed to "Propositional") Liar is discussed in Chapter 8 (on Landini's interpretation, Russell rejected general propositions in favour of statements in "On 'Insolubilia'", it will be recalled) in some detail (pp. 220–3). Landini illustrates how the paradox is to be solved, but the Statement Liar is not the Epimenides.

Landini's relegation of the Epimenides is somewhat puzzling. The 1905 manuscript that he discusses[26] is dated June of that year. The substitutional theory was only in its most prototypical form at this time. Furthermore, it is in a later manuscript, "The Paradox of the Liar", that Russell works through the possibilities of constructing a hierarchy of orders. Admittedly, it is in this manuscript that Russell refers to the substitutional paradox as "the fallacy which led to the

[23] *PoM*, "Introduction to the Second Editon", p. xiv.

[24] *MPD*, p. 77.

[25] Landini does return briefly to the Propositional Liar in Chapter 9, though only to illustrate how the hierarchy of orders in "Mathematical Logic" blocks it. See pages 251–2.

[26] "On Fundamentals", *Papers* 4: 359–413.

abandonment of substitution before",[27] and is often concerned with that paradox. However, the discussion is primarily driven by the Epimenides and variants of the Liar.[28] Furthermore, in a letter to Philip Jourdain quoted by Landini (p. 213), Russell explicitly states that the modifications made to the substitutional theory in "On 'Insolubilia'" are needed "in order to solve the *Epimenides*".[29]

Landini's reasons for attributing only a minor role to the Epimenides perhaps stem from his aforementioned assimilation of the paradox to the Liar. In fact, the Epimenides (or Cretan Liar) takes an interestingly different form to the Propositional Liar. The latter (usually attributed to Eubulides) takes the simple form whereby a person says "the proposition I am asserting is false" or, even more simply, "I am lying". The Epimenides, however, arose when Epimenides, himself a Cretan, asserted that "all Cretans are Liars". What separates the two is that the Cretan Liar is only paradoxical on the added assumption that all other propositions asserted by Cretans were indeed lies. It is in this latter form that Russell often presented the paradox.[30] This difference in structure is not trivial. Landini argues at one point (p. 201) that the Propositional Liar is avoided in the calculus of substitution by appeal to the following theorem:[31]

$$\text{(T4.20)} \quad \vdash (a)(a \neq \{(\exists p)(p = a \,.\, \& \,.\, {\sim}p)\}).$$

Dealing only with the simple form of the Liar, Landini is envisaging the paradox in the way Russell presented it in "On 'Insolubilia'":

$$(\exists p)(\text{I now assert } p \,.\, \& \,.\, {\sim}p).$$

The Epimenides, however, requires a more complex formulation, such as (taking ϕ to be the property of being asserted by a Cretan):

$$(\exists p)(((p = \{(\forall q)(\phi q \supset {\sim}q)\}) \,\&\, \phi p) \,\&\, (\forall q)((\phi q \,\&\, (q \neq p)) \supset {\sim}q)).$$

Formulated thus, the Epimenides paradox is derivable in simple type-theory. A

[27] "The Paradox of the Liar", fol. 72.

[28] Discussion of the Liar paradoxes is also frequently returned to in the April–May 1906 manuscript "On Substitution".

[29] Russell to Jourdain, 14 June 1906, in Grattan-Guinness, ed., *Dear Russell—Dear Jourdain* (London: Duckworth, 1977), p. 89.

[30] See, for example, Russell, "Mathematical Logic", *LK*, p. 59.

[31] I here follow Landini in using braces to signify the nominalizing transformation of wffs, dropping the convention where no confusion results.

simpler formulation, suggested by Myhill,[32] assumes that the set E of all propositions asserted by Cretans contains only one proposition, namely that of Epimenides to the effect that all propositions contained in E are false. Within the simple theory of types (with indices suppressed for convenience), we can legitimately have:

$$(p \in E) \equiv (p = (\forall q)(q \in E \equiv \sim q)),$$

which is also sufficient to generate the paradox. Translation into the language of substitution is straightforward:

$$(p_0 \in p/a) \equiv (p_0 = (\forall x)(x \in p/a \supset \sim x)).$$

Even if formal obstacles to the Epimenides can be overcome, however, Landini holds the substitutional calculus to be free from this threat on the grounds that the substitutional calculus is devoid of semantic or epistemological predicates of the sort needed to formulate the paradox (p. 201). However, it is not clear that the Epimenides can, or indeed, should be responded to in this way. At root, p_0 simply asserts the condition that any entity must meet to be a member of a particular class (i.e., for the result of that entity's substitution for the argument in a certain matrix to be true). On a realist interpretation of propositions, such conditions must obtain or not obtain independently of whether or not they are asserted, believed, etc. Indeed, Russell seems to have been aware of this. After discussing the possibility of solving the Liar by appeal to psychological features of the presentation of the paradox, he writes: "From this discussion I conclude that, so far as appears, the introduction of psychological considerations serves no purpose in solving the paradox of the liar. It remains to seek out some logical theory by which the vicious self-reference may be avoided."[33] In his discussion of the propositional paradox outlined in Appendix B of the *Principles* (a paradox which also infects the substitutional theory, as Landini demonstrates on pp. 202–3), Landini rejects the interpretation of this paradox as a semantic paradox. Although that paradox seems from Russell's description of it to make use of an unrestricted truth-predicate, Landini demonstrates that the paradox in no way depends on it. Rather, the paradox again exploits Cantor's diagonalization technique. The result is paradoxical, again, only on the basis of a realist interpretation of propositions, as can be seen if we recall the structure of that argument. First imagine a list of all

[32] John Myhill, "A Refutation of an Unjustified Attack on the Axiom of Reducibility", in G. W. Roberts, ed., *Bertrand Russell Memorial Volume* (London: Allen & Unwin, 1979), pp. 81–90.

[33] Russell, "The Paradox of the Liar", fol. 6.

classes of propositions. For every such class *M* there will be a corresponding proposition asserting the truth of all classes contained in *M*, and this proposition may or may not be a member of *M*. Now let *W* be the class of all such propositions which are not members of their correlated classes. We now form a proposition *r* asserting the truth of all members of *W*. Is *r* a member of *W*? Each alternative leads to its contradictory. For one who shares Russell's realism regarding propositions, the result is obviously bewildering: for any class of propositions we naturally suppose there to be a correlated proposition asserting the truth of all propositions contained in the class. Yet the paradox demonstrates that there can be no 1–1 correlation between all such classes and propositions.

Landini's derivation of the paradox in the substitutional theory proceeds from the introduction of the following substitution:

$$(\exists p, a)(a \text{ in } p \,.\, \& \,.\, (z)(p/a\text{'}z!\{(\exists r, c)(z = \{(y)(r/c\text{'}y \,.\!\supset\!.\, y)\} \;\&\; {\sim}(r/c\text{'}z))\})$$

which yields the proposition (*z*) which asserts the truth of all members of its correlated class (*r/c*) and is not a member of that class. The Epimenides is almost a mirror image, yielding a proposition asserting the falsity of all propositions contained in its correlated class and yet apparently also a member of that class. Hence in place of the resultant[34] of the above substitution, we will have:

$$(\exists r, c)(z = \{(y)(r/c\text{'}y \,.\!\supset\!.\, {\sim}y)\} \;\&\; (r/c\text{'}z)).$$

The problem here is far more obvious than in the case of the Appendix B paradox, but the similarity in structure is evident and, surely, unlikely to have escaped Russell's notice. The paradox shows, as the substitutional paradox and Appendix B paradox do, that the assumption of propositions as logical objects turned out to introduce problems just as severe as (and remarkably similar to) those introduced by the assumption of classes. This is not, of course, to suggest that the Epimenides is not a semantic paradox. Nor is it to belittle the role of the substitutional paradox in driving Russell to ramification. What I do think it shows, however, is that Russell was interested in the Epimenides and variants of the Liar because he thought they shared important features with the other paradoxes of propositions. This in turn, I think, should make us wary of thinking that Russell clearly perceived the differences between the logical and

[34] The *resultant* of a substitution is the entity that results from the substitution of an argument in a matrix, for example, *q* in $p/a\text{'}x!\, q$.

semantic paradoxes and of the distinction between logical and semantic considerations generally. And, obviously, if we doubt this element of Landini's thesis, then it is very hard to accept his interpretation of *Principia*, as it relies so heavily on that distinction. Of course, as Landini convincingly argues, there is not (contrary to what many have claimed) anything in Russell's philosophy of logic that excludes the possibility of his embracing this distinction, and Russell's logic of propositions certainly does not require an object language truth-predicate. The fact that such distinctions were not excluded by Russell's philosophy, however, is not sufficient to show that he did in fact either recognize or embrace them.

Notwithstanding the obstacles confronting Landini's interpretation of *Principia*, obstacles that any thesis as provocative as this is almost guaranteed to meet, *Russell's Hidden Substitutional Theory* is a remarkable work of scholarship and philosophy. Although the technical nature of the subject makes this undoubtedly a difficult book, this is unlikely to be a problem for those interested in this area of Russell's philosophy. It is without question the most important monograph on Russell's logic of at least the last decade, and a book that all who are interested in this aspect of Russell's thought or in the logicist philosophy of mathematics in general ought to read.