RUSSELL'S NOTES ON FREGE'S *GRUNDGESETZE DER ARITHMETIK*, FROM §53

Bernard Linsky
Philosophy / U. of Alberta
Edmonton, AB T6G 2E5
bernard.linsky@ualberta.ca

This paper completes a series of three devoted to the notes that Russell made on reading Gottlob Frege's works beginning in the summer of 1902. Notes in the two previous papers were used in the preparation of Appendix A of *The Principles of Mathematics*, "The Logical and Arithmetical Doctrines of Frege". The bulk of the notes published here are on the formal proofs in *Grundgesetze der Arithmetik*, which begin at §53 and continue through the rest of Vol. 1. There is no mention of these notes in published works of Russell. Additional notes were made in 1903 when Vol. 2 arrived. Brief notes found in Russell's copy of *Grundgesetze* include some on *Die Grundlagen der Arithmetik* and preliminary notes on *Grundgesetze*. With material on the contradiction already published this completes the publication of Russell's notes on Frege in the Russell Archives.

This article presents a transcription of the notes on Gottlob Frege's *Grundgesetze der Arithmetik* in the Bertrand Russell Archives.[1] Five leaves of notes on the *Grundgesetze* and *Grundlagen der Arithmetik* found in Russell's copy of the former work are also printed here. With my "Russell's Marginalia in His Copies of Frege's

[1] In RAI 230.030420–F1. A missing folio 23 appears on the verso of folio 3 of the notes on Meinong in RAI 230.030450. The remainder of the notes on Frege are in RAI 230. 030420–F2. G. Frege, *Grundgesetze der Arithmetik*, 2 vols. (Jena: Verlag Hermann Pohle, 1893, 1903); repr. in 1 vol., Hildesheim: Georg Olms Verlagsbuchhandlung, 1962, with the same pagination. Partially transl. as *The Basic Laws of Arithmetic: Exposition of the System*, by Montgomery Furth (Berkeley and Los Angeles: U. of California P., 1967).

russell: the Journal of Bertrand Russell Studies n.s. 26 (winter 2006–07): 127–66
The Bertrand Russell Research Centre, McMaster U. ISSN 0036-01631

Works" and "Russell's Notes on Frege for Appendix A of *The Principles of Mathematics*", this completes the publication of all notes on Frege in the Archives.[2] The first twenty-two leaves of notes consist of a transcription of a number of theorems and proofs from the *Grundgesetze* into Russell's notation, beginning with the first theorem at §53 and continuing through to §144 on page 181 of the first volume, 57 pages short of the last section. (The final leaf of these notes was found on the verso of a leaf of notes on Alexius Meinong that Russell made in preparation for writing one of his reviews of Meinong's work.[3]) The next four leaves of notes list the principal results of the whole of Volume II, but appear to have been added later. The final three pages of "Other Notes on *Grundgesetze*, Vol. II" were found on a single folded leaf in Russell's copy of the *Grundgesetze* when Russell's library was received by the Archives.[4] Four additional half-leaves found in that copy contain notes on the *Grundlagen der Arithmetik*.[5] The bulk of the notes seem to have been written beginning in June 1902. The second volume reached Russell by February 1903, but after he had composed the appendix for *The Principles of Mathematics*.[6] Although there is no reference to the notes in

[2] "Russell's Marginalia in His Copies of Frege's Works", *Russell*, n.s. 24 (2004): 5–36, and "Russell's Notes on Frege for Appendix A of *The Principles of Mathematics*", *Russell*, n.s. 24 (2004): 133–72. The notes on *Grundlagen* are included here to complete the publication of the notes and because they were found in Russell's copy of *Grundgesetze*. Russell's notes on Frege's solution to the Contradiction, to be found in *Grundgesetze der Arithmetik* (*Gg*), Vol. 2, are published in *Papers* 4, Appendix I, pp. 607–19. In the spirit of completeness, I want to add here an overlooked typographical correction made by Russell in the margin of the *Nachwort* on the Contradiction in *Gg*, Vol. 2. On p. 258, col. 2, line 6, Frege's ξ is corrected to a β in the margin.

[3] Review of Meinong *et al.*, *Untersuchungen zur Gegenstandstheorie und Psychologie*, on which Russell was engaged in 1904–05 (*Papers* 4: 595ff.). Russell refers in these notes to Rudolf Ameseder's letter to him. That letter was probably Ameseder's letter of 3 Jan. 1905 (RA1 710.047044).

[4] The notes in Russell's *Grundgesetze* are RA2 220.148001c and RA2 220.148001b, described by Kenneth Blackwell and Carl Spadoni in B&S, p. 7.

[5] *Die Grundlagen der Arithmetik, eine logisch mathematische Untersuchung über den Begriff der Zahl* (Breslau: Verlag von Wilhelm Koebner, 1884). Translated as *The Foundations of Arithmetic: a Logico-Mathematical Enquiry into the Concept of Number*, trans. J. L. Austin (Oxford: Basil Blackwell, 1974).

[6] In a letter to Frege of 20 February 1903, Russell thanks Frege for the second volume and says that "Until now I have not been able to read the whole …" (Frege, *Philosophical and Mathematical Correspondence*, ed. B. F. McGuinness [Chicago: U. of Chicago P., 1980], p. 154).

Russell's correspondence, they can be dated from internal evidence, in particular by the notation he used at various times in this period.

I. BACKGROUND OF THE NOTES

It was only after substantially completing the *Principles* in 1902 that Russell studied the works of Gottlob Frege and discovered the extent to which Frege had anticipated his project of reducing mathematics to logic.[7] In the Preface to the *Principles*, Russell writes:

> In Mathematics, my chief obligations, as is indeed evident, are to Georg Cantor and Professor Peano. If I had become acquainted sooner with the work of Professor Frege, I should have owed a great deal to him, but as it is I arrived independently at many results which he had already established.
> (*PoM*, p. xviii)

After finishing the body of the *Principles* in May 1902, Bertrand Russell turned to a review of the literature on the subject with the intention of adding scholarly references in the proofreading process. This review began in June and included the works of Frege. On 16 June Russell wrote the famous letter to Frege, announcing the paradox and beginning a correspondence (*SLBR*, 1: 245–6). At the same time Russell studied papers by Frege and the logical works, *Begriffsschrift* and *Grundgesetze der Arithmetik.* The reading resulted in several changes to the *Principles* in proof, and the addition of Appendix A, "The Logical and Arithmetical Doctrines of Frege", which was completed in November 1902. In that Appendix Russell discusses only the introductory and philosophical issues in the *Grundgesetze* and remarks of the formal presentation that:

> In the *Grundgesetze der Arithmetik*, various theorems in the foundations of cardinal Arithmetic are proved with great elaboration, so great that it is often very difficult to discover the difference between successive steps in a demonstration. (*PoM*, p. 519)

The question arises of just how carefully Russell read Frege's technical work, and what impact it had on Russell's own project. It is clear that

[7] For the chronology of Russell's work on the *Principles*, and the study of Frege, see the Introduction and Chronology to *Papers* 3: xxxvi–xliii and xxxvii.

Russell did find that Frege had not only anticipated many of his results in the *Principles*, but also had much to teach him about carrying out the logicist programme. In the Preface to *Principia Mathematica*, in 1910, Whitehead and Russell write:

> In all questions of logical analysis, our chief debt is to Frege. Where we differ from him, it is largely because the contradictions showed that he, in common with all other logicians, ancient and modern, had allowed some error to creep into his premisses; but apart from the contradictions, it would have been almost impossible to detect this error. In Arithmetic and the theory of series, our whole work is based on that of Georg Cantor. (*PM*, 1: viii)

The notes transcribed here are among the results of Russell's reading of Frege in 1902 and show the care with which Russell attended to the logical details of the *Grundgesetze der Arithmetik*, as well as providing evidence of what Russell and Whitehead learned from Frege about "logical analysis".

II. CONTENT OF THE NOTES

The notes follow the principal theorems of the first volume of the *Grundgesetze*, beginning with Part A.[8] The main result of Part A is theorem 32, what has come to be known as "Hume's Principle": if there is a one-to-one function mapping u onto v, then the number of u's is the same as the number of v's.[9] Theorem 32 is the end of a series of lemmas, beginning with theorem 1: a is f if and only if a is in the course of values of f.

Russell transcribed these theorems into his own notation, using occasional borrowings from Frege, and copied selected lines from the proofs, again translating them. Russell's notation is generally adequate to translate Frege's, although around the notions of ancestral and number series he had to introduce some new defined expressions. Russell seems to have slipped in representing the logical structure of some of the lines of the

[8] The Parts and section numbers are: A (§§53–65), B (§§66–87), Γ (§§88–95), Δ (§§96–101), E (§§102–7), Z (§§108–13), H (§§114–19), Θ (§§120–1), I (§§122–57), K (§§158–71), and Λ (§§172–9).

[9] Actually "Hume's Principle" is generally taken to be the biconditional rather than just this one direction.

proof of theorem 1 on the first leaf of notes, but that error was not repeated, and the rest of the notes seem to be a fair, though sometimes selective, reporting of the proofs.

The main result of Part *B* is theorem 71: the successor relation is one to one. The proof occupies Frege from §66 to §87, over twenty-seven pages, but Russell devoted less than one leaf to it, merely stating the result and transcribing a few lines of the proof without citation. The proofs of various results about the natural numbers 0 and 1 and the successor relation following in Parts *Γ*, *Δ* and *E* are skipped, with just some of the results stated, again without citation.

With Part *Z* at §108 the notes are more careful, with theorems identified more clearly by number and with intermediate lines marked with the lower-case Greek letters that Frege uses for his lemmas. The principal result of *Z* is theorem 145, that no number follows itself in the number series. Here Russell notes Frege's definition of the ancestral of a relation, remarking that it is "giving a new view of mathematical induction". This remark clearly marks his appreciation of its role in giving a logical analysis of induction, by providing a way of defining the ancestral of the successor relation and thus the number series.[10] The next major theorem that Russell notes is 155, that "the number of finite numbers up to and including b is $b+1$". This is a key step in proving that every number has a successor, one of Peano's axioms. Russell does not indicate when Frege has proved each of Peano's axioms in Volume I, but Frege himself does not remark on this in the *Grundgesetze* either.[11]

With Part *I* Russell's notes are more detailed, copying and noting almost every line of proofs by theorem and section number. Part *I* is devoted to "The proof of various propositions about *Endlos*". "*Endlos*" is

[10] Russell's first appreciation of the definition of ancestral is included in his notes on the *Begriffsschrift*, transcribed in "Russell's Notes on Frege for Appendix A of the *Principles*", pp. 159–61.

[11] The "theory of finite numbers" is restricted to one chapter (XIV) in the middle of the *Principles*, and the work is not organized around proving them from logical principles, then constructing the rest of mathematics from that, as would be contemporary order. Russell only briefly states the axioms: "(1) 0 is a number. (2) If *a* is a number, the successor of *a* is a number. (3) If two numbers have the same successor, the two numbers are identical. (4) 0 is not the successor of any number. (5) If *s* be a class to which belongs 0 and also the successor of every number belonging to *s*, then every number belongs to *s*. The last of these propositions is the principle of mathematical induction" (p. 125).

Frege's name for the cardinal of the natural numbers. Russell writes this as α_0, using $\aleph_0$ in a shaky hand only in the last few leaves of notes written later than 1902. The proof of theorem 207 takes up seven leaves of notes. Russell's transcription has it that if the cardinal number of u is α_0 then there is a many-one relation R such that the ancestral of R is nonreflexive, u is included in the range of R, and there is some x such that the u's are the objects in the range of the ancestral of R starting with x. In other words, a set with the same cardinality as the natural numbers can be arranged in a series satisfying the Peano axioms.

One leaf then follows for the proof of theorem 263, "the converse of (207)". Richard Heck describes 263 as tantamount to proving that, "all 'simply infinite systems'—that is, structures which satisfy the Dedekind–Peano axioms—are isomorphic", a result proved less formally by Dedekind in *Was sind und was sollen die Zahlen?* in 1887.[12] Russell does not follow this proof in detail, remarking that several symbolizations are "approximate" and concluding that "the proof occupies 20 pages". The notes on Volume I end with §158 and §172 on folio 23 of the notes. This leaf reappeared as the verso of a leaf of notes on Meinong. When Russell received Volume II, he began the new notes with a remark on both §158 and §172 of Volume I at the top of a new leaf foliated again as 23, here indicated as 23_2. This is evidence that there was a gap between the writing of the notes, in addition to the evidence from the new notation in the second group.

What follow are three leaves of notes on Volume II which pick out the principal results and lemmas to the end of the volume.[13] One final, unnumbered leaf summarizes the axioms and rules of inference from §48 in Volume I, using a notation that places it later than the rest of the notes. The concluding "other notes" cover exactly the same portions of Volume II, but differ in many small points of notation. They are presumably the result of a first pass through the volume, and were rewritten to join the larger manuscript.

[12] See Richard Heck, "Definition by Induction in Frege's *Grundgesetze der Arithmetik*" in *Frege's Philosophy of Mathematics*, ed. William Demopolous (Cambridge and London: Harvard U. P., 1995), pp. 295–333, which is an extended discussion of Frege's proof of theorem 263.

[13] See Michael Dummett, *Frege: Philosophy of Mathematics* (Cambridge, Mass: Harvard U. P., 1991), pp. 285–91, for a summary of the technical contents of this section.

III. RUSSELL'S NOTATION[14]

The notation in the notes is based on that of Peano. Russell's additions to that notation by 1902 are presented in several articles, including two in Peano's journal, *Revue de Mathématiques*, which have been translated as "The Logic of Relations with Some Applications to the Theory of Series" and "General Theory of Well-Ordered Series".[15] The following selections from those articles include many of the symbols in the notes.

$*1\cdot 0$ *Primitive idea*: Rel = Relation

$\cdot 1$ $R \,\epsilon\, \mathrm{Rel} \,.\supset :\, xRy \,.=.\, x$ has the relation R with y.

$\cdot 21$ $R \,\epsilon\, \mathrm{Rel} \,.\supset.\, \rho = x \ni \{\exists y \ni (xRy)\}$ Df

$\cdot 22$ $R \,\epsilon\, \mathrm{Rel} \,.\supset.\, \breve{\rho} = x \ni \{\exists y \ni (yRx)\}$ Df

Note. If R is a relation, ρ can be called the *domain* of the relation R, that is to say, the class of terms which have that relation with a single term, or with several terms.…

$\cdot 31$ $R \,\epsilon\, \mathrm{Rel} \,.\, x \,\epsilon\, \rho \,.\supset.\, \breve{\rho}x = y \ni (xRy)$ Df

$\cdot 32$ $x \,\epsilon\, \breve{\rho} \,.\supset.\, \rho x = y \ni (yRx)$ Df

(*Papers* 3: 315)

Thus $\breve{\rho}x$ is the class of y such that R relates x to y, and ρx is the class of y such that R relates y to x. $\breve{\rho}$ standing alone signifies the range of R. For a relation S the domain and range will be σ and $\breve{\sigma}$, similarly for N (the successor relation) with ν and $\breve{\nu}$, and so on. The ι comes from Peano, where it is a function symbol applying to x to give ιx, the singleton class containing x. The inverted iota, ℩, later the symbol for a definite description, serves here as the inverse of ι. If x is the unique element of the singleton y, then ℩y is just x. Russell uses 1' for identity and 0' for

[14] As several of Russell's symbols involve single quotation marks, in what follows I will use symbols ambiguously as names for themselves, and allow the reader to sort out which are cases of use and which of mention.

[15] "The Logic of Relations with Some Applications to the Theory of Series", *Papers* 3: 310–49; also in *LK.* Published originally as "Sur la logique des relations avec des applications à la théorie des séries", *Revue de mathématiques*, 7 (1901): 115–48. Russell submitted this to Peano in March 1901. The second paper is "General Theory of Well-Ordered Series", *Papers* 3: 384–421. Originally published as "Théorie générale des séries bien-ordonnées", *Revue de mathématiques*, 8 (1902): 12–43.

non-identity:

$*4\cdot 1$ *Primitive idea*: 1' = identity

Note. This symbol is given the notation of Schröder. I do not use the symbol = for the identity of individuals, since it has another usage for the equivalence of classes, of propositions, and of relations.

$\cdot 2 \quad 1' \in \mathrm{Rel}$ Pp
$\cdot 3 \quad 0' = \sim 1'$ Df

(*Papers* 3: 318)

The notions of many-one (Nc→1) and one-one (1→1) relations are defined next:

$*5\cdot 1 \quad \mathrm{Nc}\rightarrow 1 = \mathrm{Rel} \cap R \ni \{xRy \,.\, xRz \,.\, \supset_x .\, y1'z\}$ Df
$\cdot 11 \quad 1\rightarrow\mathrm{Nc} = \mathrm{Rel} \cap \breve{R} \ni \{yRx \,.\, zRx \,.\, \supset_x .\, y1'z\}$ Df
$\cdot 2 \quad R \in \mathrm{Nc}\rightarrow 1 \,.=.\, \breve{R} \in 1\rightarrow\mathrm{Nc}$
$\cdot 3 \quad 1\rightarrow 1 = (\mathrm{Nc}\rightarrow 1) \cap (1\rightarrow\mathrm{Nc})$ Df

Note. Nc→1 is the class of many-one relations. The symbol Nc→1 indicates that, if we have *xRy*, when *x* is given, there is only one possible *y*, but that, when y is given, there is some cardinal number of *x*'s which satisfies the condition *xRy.* Similarly, 1→Nc is the class of the converses of many-one relations, and 1→1 is the class of one-one relations. (*Papers* 3: 319)

In these notes Russell uses ∩ for sentential conjunction in the notes with ∧ used for the notion of intersection it expresses in these earlier papers. $*5\cdot 1$ thus defines Nc→1 as the relations *R* such that if *x* is related by *R* to *y* and *z* then *y* is identical with *z*. In a new section on cardinal numbers, Russell defines the relation of similarity:

$*1\cdot 1 \quad u, v \in \mathrm{Cls} \,.\supset:\, u \ \mathrm{sim}\ v \,.=.\, \exists 1\rightarrow 1 \cap R \ni (u \supset \rho \,.\, \breve{\rho}u = v)$ Df

(*Papers* 3: 320)

This should be read as saying that if *u* and *v* are classes, then they are similar if and only if there is a one-to-one relation *R* such that *u* is included in the domain of *R* and the range of *R* is the whole of *v.* Gregory Moore reports that Russell used ⊃ for class inclusion as well as implication until March or April 1902, when he started to use ⊂ for class inclusion. Thus $u \supset \rho$ here means that u is included in ρ.[16] In the second paper, "General Theory of Well-Ordered Series", published in 1902, Nc'u the cardinal number of a class u, is defined as well as the relation of *being the cardinal number of*, Nc, from which it is derived:

$$*7\cdot 1 \quad u \in \mathrm{Cls} \,.\supset.\, \mathrm{Nc}'u = \mathrm{Cls} \cap v \ni (u \ \mathrm{sim}\ v) \qquad \mathrm{Df}$$

$$\cdot 11 \quad \mathrm{Nc} = \mathrm{Cls}'\mathrm{Cls} \cap w \ni \{\exists \mathrm{Cls} \cap u \ni (v \in w \,.=.\, u \ \mathrm{sim}\ v)\} \qquad \mathrm{Df}$$[17]

Nc is the relation which u bears to w when w is the class of classes v similiar to u, so Nc'u is the class of classes v which are similar to u. This is Russell's version of the "Frege–Russell definition" of cardinal number.[18] The notion is also described in the earlier paper, but not explicitly defined.[19]

Some of Russell's notation for relations has become standard:

> It is necessary to distinguish $R_1 \cap R_2$, which signifies the logical product, from R_1R_2, which signifies the relative product.... For example, *grandfather* is the relative product of *father* and *father* or of *mother* and *father*, but not of *father* and *mother*.
>
> $$\cdot 12 \quad R \in \mathrm{Rel} \,.\supset.\, R^2 = RR \qquad \mathrm{Df}$$
>
> (*Papers* 3: 316)

When Russell encountered Frege's definition of the ancestral of a relation, his notation for it was consequently "R^N". The rest of the notation is either defined in the notes or will be explained in the annotation as it appears.

In his notes on the *Grundgesetze* Russell uses Frege's numbering for the theorems (in parentheses to the right, as in (155) for theorem 155). Russell also follows Frege's annotation of lemmas by Greek letters in

[16] *Papers* 3: xiv. This notation makes sense. If $u \supset \rho$ means that if something is in u then it is in ρ, this is a way of saying that u is a subset of ρ. Russell reads $R \supset 0'$ as R is contained in diversity or irreflexive. $R \supset 1'$ will mean that R is reflexive.

[17] "General Theory of Well-Ordered Series", *Papers* 3: 408–9.

[18] Moore suggests that Russell was led to this definition by reading a paper by Peano from 1901 which rejected such a proposal. Peano had encountered the definition when writing his 1895 review of Frege's *Grundgesetze* (*Papers* 3: xxvii).

[19] "The Logic of Relations", *Papers* 3: 321.

parentheses: (α), (β), (γ), etc. The use of inference rules is marked with the symbols "$\supset$:" at the beginning of a line, sometimes only followed by the consequent in a series of conditionals which derive results from a repeated group of premisses as antecedents.

IV. RUSSELL'S REPRESENTATION OF FREGE'S NOTATION

Very little of Frege's notation appears in the notes. Russell either had notation ready or created it as needed. Some of Frege's symbols do appear, however. Folio 1 includes the content stroke in the subformula $-f(a) = b$ at line 9 and following. Line 18 of that leaf is a statement of Frege's first theorem, (1), which Russell symbolizes as: $f(a) = a \cap \grave{\epsilon} f(\epsilon)$, where $\cap$ expresses membership in a course of values. Folio 4, line 8 introduces Part *B*, – I f, that f, the successor relation ("following in the number series directly after") is many-one ("*eindeutig*"). Russell glosses this as $N \epsilon \mathrm{Nc}{\rightarrow}1$. Folio 2 simply transcribes Frege's definition of >, the "mapping" relation between concepts. In use, Russell translates it with his way of expressing the fact that a relation is one-one and "onto". Thus the main result of Part *A*, theorem 32:

$$v \cap (u \cap > {}^{-1}q) \rightarrow u \cap (v \cap > q) \rightarrow \mathrm{N}u = \mathrm{N}v$$

(with Frege's symbol for the converse of a relation replaced by $^{-1}$, and the conditional stroke replaced with an arrow) comes out as:

$$R \,\epsilon\, 1{\rightarrow}1 \,.\, u = \rho v \,.\, v = \breve{\rho} u \,.\supset.\, \mathrm{Nc}'u = \mathrm{Nc}'v.$$

Russell's version asserts that if there is a one-to-one mapping *R* with *u* as its domain and *v* as its range, then the cardinal number of *u* is equal to the cardinal number of *v*.

As Russell introduces new notation in the course of the notes, it can occur that some expressions from Frege have two different notations. Part *H* proves theorem 155 (again approximating Frege's font):

$$\cap\, (b \cap \cup' f) \rightarrow b \cap (N\,(b \cap \cup' f) \cap f)$$

Russell's initial representation of this is:

$$0(N^{N} \cup 1\text{'})b\ .\supset .\ \text{Nc'}(\nu^{N}b\ \nu\iota b)\ \breve{N}b$$

but this becomes

$$0N'b\ .\supset .\ bN(\text{Nc'}\nu'b)$$

at the end of the proof. $N^{N} \cup 1$' for the "weak ancestral" of the successor relation becomes N' by a convention adopted immediately after the first statement, but Nc'$(\nu^{N}b\ \nu\iota b)\ \breve{N}b$ becomes $bN(\text{Nc'}\nu'b)$ without remark.[20] Since N is the successor relation and so ν is its domain, by the abbreviation Russell adopts on folio 8, ν' will be its "weak" ancestral, i.e. precedes or is identical with. $\nu'b$, then, is the number series ending with b. (Russell glosses theorem 155 as "the number of finite numbers up to and including b is $b + 1$".)

Because of the limitation of fonts, even some of Russell's notation can only be approximated. The representations (ɟ, Ⅎ, ł, ≬, ð, ş⟩, ≚) of Russell's transcription of Frege's novel symbols on folios 23 to 25 are not exact.

V. TEXT OF THE NOTES

The notes from RA1 230.030420–F1 are on twenty-six leaves measuring 17.5 cm × 22.5 cm. The first twenty-five leaves are numbered 1 to 25 in the upper right-hand corner, and have "Frege" in the upper left-hand corner, except for 1, which has "Frege, Grundgesetze d. Arithmetik. p. 74 ff.", and 24 and 25, which have "Frege. Gg. Vol. II." to the left. The last leaf is unnumbered and has "Frege. Gg I. p. 61" in the upper left corner. The extra folio 23 from the notes on Meinong in file RA1 230.030450 is consistent with the first 22. Russell may have left it on a stack of notepaper and decided that two brief lines were not enough to justify using a whole leaf. When he received Volume II and started the next notes, he simply repeated the last two theorems of Volume I and started

[20] The term "weak ancestral" and others are taken from Furth's translation and Richard G. Heck Jr., "The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*", *Journal of Symbolic Logic*, 58 (1993): 579–601; reprinted with a Postscript in *Frege's Philosophy of Mathematics*, ed. William Demopolous (Cambridge and London: Harvard U. P., 1995), pp. 257–94. Page references are to the latter version.

with the new volume.

Notes are on the recto of each leaf, except for short notes on the verso of 3 and 9. The final three pages of notes comprise one folded leaf, the right- and left-hand sides of one side, and the left-hand side alone of the verso. They were found in Russell's copy of the *Grundegesetze* when it was received at the Russell Archives. Four foliated half-leaves of notes on *Grundlagen* were also found, with notes on the recto of each and two lines on the verso of the last.

The symbols used can date the main body of notes, based on Moore's presentation of "The Evolution of Russell's Logical Symbolism" in *Papers* 3 (pp. *xliii–xlvii*). Russell used Schröder's symbol 1' for identity between 1901 and 1904. From May 1902 to March 1903 Russell used "$a \vee b$ for the union of classes a and b, but $a \cup b$ for 'a or b' if a and b were propositions; similarly he used $a \wedge b$ for the intersection of classes a and b" (3: *xlv*). This practice seems to have been followed in the notes.

The three leaves 23 to 25 have numerous changes to the notation, suggesting that they were composed later than folios 1 to 22. These include the use of a very awkwardly drawn $\aleph_0$ rather than α_0 for the cardinal of the set of natural numbers, a change in the direction of the quotation mark from Nc'v to Nc‘v, for example, and the accompanying adoption of Frege's notation with the smooth breathing accent over a variable to indicate classes, so that $\grave{s}(\ldots s \ldots)$ replaces $s \ni \{\ldots s \ldots\}$ as it would be in 1 to 22. In addition folios 23 to 24 differ stylistically, including page references in the left margin and the use of Frege's assertion sign for theorems, both practices missing in 1 to 22. These features place them in 1903.[21]

The next, unnumbered, leaf contains a number of symbols that date it later; the use of (x) for the universal quantifier, $\hat{x}$, $C\between x$, and $f!x$ for propositional functions. The last was not in use until 1905, in "On Fundamentals" (*Papers* 4: 359–413).

The leaves of notes found in Russell's copy of *Gg* seem to be the result of a first reading of Volume II. References to the same pages and definitions occur in the main body of notes, but with considerable alteration in the symbols. It is not possible to date the notes on *Grundlagen*,

[21] They are all in place in the notes on "Functions", *Papers* 4: 49–73.

although they use Russell's notation "NC induct" for finite cardinal numbers and so date from after Russell's adoption of Peano notation.

VI. RUSSELL'S NOTES

The notes are transcribed below, with each new leaf identifiable by the heading in the upper left-hand corner and a folio number 1 to 25 in the right, with the exception of the last, unnumbered leaf. Annotations are placed below a solid line keyed by angle-bracketed indices in the text. All comments in square brackets are Russell's. Editorial comments are also in angle brackets.

Frege, *Grundgesetze d. Arithmetik.* p. 74 ff. ⟨230.030420–F1 fol.⟩ 1

Proof of $R \in \cancel{\mathrm{Rel}}\ 1{\rightarrow}1 \,.\, u = \rho v \,.\, v = \breve{\rho} u \,.\supset.\, \mathrm{Nc}'u = \mathrm{Nc}'v$ ⟨22⟩
(p. 57) $\mathrm{Nc}'u = v \ni \{\exists\ 1{\rightarrow}1 \wedge R \ni (v = \rho u \,.\, u = \breve{\rho} v)\}$ Df [This cannot be an exact rendering of the definition, but it comes near it.] ⟨23⟩
(a) Proof of P, $R \in \mathrm{Rel} \,.\, v = \breve{\pi} u \,.\, w = \breve{\rho} v \,.\, S = PR \,.\supset.\, w = \breve{\sigma} u$ ⟨24⟩
$x \ni fx = x \ni gx \,.\supset:\, fa \,.\equiv.\, ga \,:.\, a = b \,.\, fa \,.\supset.\, fb \,:.$
$\supset: \sim\{f(a) = b\} \,.\, x \ni f(x) = x \ni g(x) \,.\supset.\, \sim\{g(a) = b\} :$ ⟨25⟩
$\supset: \exists g \ni \{x \ni f(x) = x \ni g(x) \,.\supset.\, g(a) = b\} \,.\supset.\, f(a) = b :$ ⟨26⟩
$\supset:.\, f(a) = b \,.\supset.\, \exists g \ni \{x \ni f(x) = x \ni g(x) \,.\supset.\, g(a) = b\} :$

---

[22] This is the main theorem of Part *A*, one direction of what has recently come to be known as "Hume's Principle": if there is a one-to-one relation *R* with domain *u* and range *v*, then the number of *u*'s is equal to the number of *v*'s. This series of theorems is completed with theorem 32, fol. 4, line 6.

[23] Russell notes that this is not "exact", but it seems as close as Russell's notation will allow, and his antecedent and consequent are each logically equivalent to Frege's.

[24] While (a) comes from the prefatory remarks in §53, with the next line Russell begins to follow the "*Aufbau*", or Construction, in §55.

[25] This line is Frege's β and in Russell's notation ought to have a subscripted g on the $\supset$, indicating universal quantification.

[26] This is γ. Russell has read $\neg\forall\mathfrak{g}\,[\grave{\epsilon} f(\epsilon) = \grave{\epsilon}\,\mathfrak{g}(\epsilon) \rightarrow \neg\, \mathfrak{g}(a) = b]$ as
$\neg\forall\mathfrak{g}\ \neg[\grave{\epsilon} f(\epsilon) = \grave{\epsilon}\,\mathfrak{g}(\epsilon) \rightarrow \mathfrak{g}(a) = b]$ and so translated it as
$\exists\mathfrak{g}\,[x \ni f(x) = x \ni \mathfrak{g}(x) \rightarrow \mathfrak{g}(a) = b]$ when it should read
$\exists\mathfrak{g}\,[x \ni f(x) = x \ni \mathfrak{g}(x) \,.\, \mathfrak{g}(a) = b]$. This mistake is repeated twice more, until it is silently corrected at 1: 12 (ζ). It is not repeated later in the notes.

$\supset :. \exists g \ni \{x \ni f(x) = x \ni g(x) \,.\supset.\, g(a) = b\} \,.\equiv.\, -f(a) = b \quad (\delta)$
$x \ni f(x) = x \ni g(x) \,.\supset_g.\, g(a) \sim= b : \supset . f(a) \sim= b :.$
$\supset : f(a) = b \,.\supset.\, \exists g \ni \{x \ni f(x) = x \ni g(x) \,.\, g(a) = b\} \quad (\zeta)$
$(\delta) \,.\, \zeta \,.\supset : \exists g \ni \{x \ni f(x) = x \ni g(x) \,.\, g(a) = b\} \,.\equiv.\, -f(a) = b :$
$\supset :. -f(a) = b \,.\equiv.\, f(a) = b : \supset : \exists g \ni \{x \ni f(x) = x \ni g(x) \,.$
$g(a) = b\} \,.\equiv.\, f(a) = b :.$
$\supset : \exists g \ni \{x \ni f(x) = x \ni g(x) \,.\, g(a) = b\} \,.\equiv.\, f(a) = b :$
$\supset : \exists g \ni \{x \ni f(x) = x \ni g(x) \,.\, g(a) = y\} \,.\equiv_y.\, f(a) = y :$
$\supset : f(a) = \iota y \ni \{\exists g \ni [x \ni f(x) = x \ni g(x) \,.\, g(a) = y]\} \,.$
$\supset : f(a) = a \cap \grave{\epsilon} f(\epsilon)$ (1) ⟨27⟩
[I do not understand why this is not an immediate result of the definition of $a \cap \grave{\epsilon} f(\epsilon)$]

Frege. ⟨fol.⟩ 2

$> q = (\alpha\, ; \beta) \ni [Q \,\epsilon\, \mathrm{Nc}{\to}1 \,.\supset.\, \exists(x, y) \ni \{xQy \,.\, x \,\epsilon\, \beta \,.$
$\supset_{x,y} .\, y \,\epsilon\, \alpha\}]$ Df ⟨28⟩
i.e. if β is any class chosen out of domain of Q, α is its correlative.
i.e. if we put R for Q, $\alpha = \breve{\rho}\beta$.
Thus $x > Ry \,.\equiv.\, x = \breve{\rho}y$ ⟨Diagram on right side: two regions, u and v, an arrow to a dot "R_u" in v from a dot in u, and a return arrow to a distinct point "$\breve{R}_u$" in u.⟩
$u = \rho v \,.\, v = \breve{\rho} u.$
$\equiv : x \,\epsilon\, u \,.\equiv_x.\, \exists v \wedge y \ni (xRy) : y \,\epsilon\, v \,.\equiv_y.\, \exists u \wedge x \ni (xRy) :$
$\equiv :. xRy \,.\supset : x \,\epsilon\, u \,.\equiv.\, y \,\epsilon\, v :.$
$\equiv :. R_u\breve{R}_u = S_u \,.\supset.\, \sigma_u = u : \breve{R}_u R_u = S_v \,.\supset.\, \sigma_v = v :.$
$\equiv : R_u\breve{R}_u = \epsilon_u \breve{\epsilon}_u \,.\, \breve{R}_u R_u = \epsilon_v \breve{\epsilon}_v$

[27] Theorem 1 is a fairly immediate result, compared with what is to come. Perhaps because of the quantifier error above Russell does not follow the proof. This line is in Frege's notation. Russell's would be: $f(a) \equiv a \,\epsilon\, x \ni f(x)$. A simple instance of (1) would put $x \ni (x \sim \epsilon\, x)$ for a and $(x \sim \epsilon\, x)$ for $f(x)$, producing the paradox. Russell does not remark on the paradox here or elsewhere in any of the marginalia, notes for Appendix A, or these notes on the *Grundgesetze*.

[28] The definition of $> q$ occurs at *Gg* §56, p. 76, between theorems 6 and 7.

Frege. ⟨fol.⟩ 3

To prove (p. 81ff.) $P\breve{Q} = \breve{Q}\breve{P}$ [29]

$f(x, d) \,.\equiv_x.\, g(x, d) : \supset .\, x \ni f(x, d) = x \ni g(x, d) :.$

$\supset :. f(x, y) \,.\equiv_{x, y}.\, g(x, y) : \supset .\, x \ni f(x, d) = x \ni g(x, d) :.$

$\supset :. f(x, y) \,.\equiv_{x, y}.\, g(x, y) : \supset .\, (x, y) \ni f(x, y) = (x, y) \ni g(x, y)$ (20)

$(x, y) \ni (yQx) = \breve{Q} \,.\supset :$ ⟨Note directed to "yQx": "This is not an exact rendering of Frege's meaning, but it is the best that can be done with Peano's notation."⟩ [30]

$aQr \,.\equiv.\, r\breve{Q}a :$ (21)

$\supset : F(aQr) \,.\supset.\, F(r\breve{Q}a)$ (22) [31]

$(21) \,.\supset : F(r\breve{Q}a) \,.\supset.\, F(aQr)$ (23)

$(23) \,.\supset : r\breve{Q}a \,.\supset.\, aQr :$

$\supset :. aQy \,.\supset_y.\, y \sim Pb : \supset : r\breve{Q}a \,.\supset.\, r \sim Pb :.$

$\supset :. aQy \,.\supset_y.\, y \sim Pb : \supset : rPb \,.\supset.\, r \sim \breve{Q}a :.$ (β)

$\cancel{\supset :. aQy \,.\supset_y.\, y \sim Pb : \supset :. b\breve{P}z \,.\supset_z.\, z \sim \breve{Q}a :.}$

$(\beta) \,.\, 23 \,.\supset :. aQy \,.\supset_y.\, y \sim Pb : \supset : b\breve{P}r \,.\supset.\, r \sim \breve{Q}a :.$

$\supset :. aQy \,.\supset_y.\, y \sim Pb : \supset : b\breve{P}z \,.\supset_z.\, z \sim \breve{Q}a :.$

$\supset :. \exists z \ni (b\breve{P}z \,.\, z\breve{Q}a) \,.\supset.\, \exists y \ni (aQy \,.\, yPb) :.$ (ϵ) [32]

$\exists y \ni (aQy \,.\, yPb) \,.\supset.\, \exists z \ni (b\breve{P}z \,.\, z\breve{Q}a)$ (κ) [Similarly proved]

$(\kappa) \,.\supset :. (\epsilon) \,.\supset : \exists y \ni (aQy \,.\, yPb) \,.\equiv.\, \exists z \ni (b\breve{P}z \,.\, z\breve{Q}a)$ (λ)

$(\epsilon) \,.\, (\lambda) \,.\supset : \exists y \ni (aQy \,.\, yPb) \,.\equiv.\, \exists z \ni (b\breve{P}z \,.\, z\breve{Q}a) :$

$\supset :. \exists y \ni (aQy \,.\, yPb) \,.\equiv.\, aQPb : \supset : aQPb \,.\equiv.\, \exists z \ni (b\breve{P}z \,.\, z\breve{Q}a) :.$

$\supset : aQPb \,.\equiv.\, \exists z \ni (b\breve{P}z \,.\, z\breve{Q}a) :$

$\supset : xQPy \,.\equiv_{x, y}.\, \exists z \ni (y\breve{P}z \,.\, z\breve{Q}a) :$ (o) [33]

---

[29] This is theorem 24, proved in §60.

[30] Frege defines a double course of values for the extension of the converse of a relation, as the result of two monadic operations, thus: $\dot{\alpha}\dot{\epsilon}(\alpha \cap (\epsilon \cap q)) = {}^{-1}q$.

[31] In the proof of theorem 22 Russell substitutes equivalent expressions in the context F, following Frege's rule, symbolizing Frege's "=" as "≡".

[32] Lemma ϵ. Russell correctly interprets the combination of quantifier, conditional and negation which he mistook on folio 1. He gets it right in the rest of the notes.

[33] Lemma o from *Gg* §60, p. 83. The final "a" should be "x". Russell mistakenly follows Frege's bound variables in the transcription, rather than his own.

⟨*verso of fol. 3*⟩ [34]

$\exists\ \mathrm{Nc}{\to}1 \wedge S \ni \{\sigma = m \,.\, \breve{\sigma} = m\,(i, \mathrm{Cls}'h)\}$

Given an association of *i* and Cls'*h* which is Nc→1, is *R* thence determinate?

⟨A diagram on the right consists of a circular region *i* with a dot inside, and an irregularly shaped region *h* with no interior markings.⟩

Frege. ⟨fol.⟩ 4

$$(20)\,.\,(o)\,.$$
$$\supset : (x, y) \ni (xQPy) = (x, y) \ni \{\exists z \ni (y\breve{P}z \,.\, z\breve{Q}x)\} :$$
$$\supset : (x, y) \ni \{\exists z \ni (x\breve{P}z \,.\, z\breve{Q}y)\} = \breve{P}\breve{Q} \,.\supset.\, (x, y) \ni (yQPx) = \breve{P}\breve{Q} :$$
$$\supset .\ (x, y) \ni (yQPx) = \breve{P}\breve{Q}.$$
$$\supset .\ \breve{QP} = \breve{P}\breve{Q}$$

Thence we arrive at $u = \rho v \,.\, v = \breve{\rho}u \,.\supset.\, \mathrm{Nc}'u = \mathrm{Nc}'v$ (32)

B. Proof of ⊢ I f [*i.e.* $N \in \mathrm{Nc}{\to}1$ **:** my Number 40] [35]

(a). Proof of $\cancel{R \in \mathrm{Rel} \,.\supset_R : x \sim= \breve{\rho}c \,.\, x \sim= \rho b \,.\supset_x :}$

$$\cancel{\sim\exists x \ni (x = \breve{\rho}c) \,.\, \sim\exists x \ni (x = \rho b) \,.\, w = \breve{\rho}z \,.\supset.\, z \sim= \rho w :.}$$
$$\cancel{\supset : c \sim\in z \,.\, b \sim\in w \,.\supset.\, \mathrm{Nc}'w \sim= \mathrm{Nc}'z}$$

$R \in \mathrm{Rel} \,.\supset_R : c \sim\in \rho \,.\, b \sim\in \breve{\rho} \,.\, w = \rho z \,.\supset.\, z \sim= \breve{\rho}w :.$ [36]

$$\supset : c \sim\in z \,.\, b \sim\in w \,.\supset.\, \mathrm{Nc}'w \sim= \mathrm{Nc}'z$$

$$\sim\{\sim(\alpha = c) \,.\supset.\, \epsilon = b : \supset .\, \sim\alpha Q\,\epsilon\}$$
$$\sim\{\sim\alpha Q\,\epsilon \,.\cup.\, \epsilon \sim= b \,.\, \alpha \sim= c\} \qquad \alpha Q\,\epsilon : \epsilon = b \,.\cup.\, \alpha = c$$

$$\sim\{\sim(\alpha \sim= c \,.\supset.\, \epsilon = b\,) \,.\supset.\, \sim\alpha Q\,\epsilon\}$$
$$\sim\{\alpha Q\,\epsilon \,.\supset:\, \alpha \sim= c \,.\supset.\, \epsilon = b\} \sim\{\alpha Q\,\epsilon \,.\, \alpha \sim= c \,.\supset.\, \epsilon = b\}$$

---

[34] This seems to be Russell's speculation. There will be more than one many-one relations between classes, of course.

[35] Russell stopped transcribing the proof with theorem 24, p. 83. Theorem 32 is proved at p. 86. His reference to "my Number 40" is unidentified.

[36] The main theorem of Part *B*, proved at p. 86.

$\sim\{\epsilon = b \,.\cup.\, \alpha = c \,.\cup.\, \alpha \sim Q\,\epsilon\} \quad \epsilon \sim= b \,.\, \alpha \sim= c \,.\, \alpha Q\,\epsilon$

$\exists R \ni (\breve{\rho}w = z \,.\, \rho z = w) \,.\supset.\, \exists R' \ni (b \sim\epsilon\, w \,.\, c \sim\epsilon\, z \,.$

$\supset .\, b \sim\epsilon\, \rho' \,.\, c \sim\epsilon\, \breve{\rho}' : \breve{\rho}'w = z \,.\, \rho' z = w)$

$R' \,\epsilon\, \text{Rel} \,.\supset :.\, \breve{\rho}'w \sim= z : \cup : \rho' z \sim= w : \cup : \exists \rho' - w : \cup : \exists \breve{\rho}' - z$

$R' \,\epsilon\, \text{Rel} \,.\supset : \rho' = w \,.\, \breve{\rho}' = z \,.\, w = \rho' z \,.\supset.\, z \sim= \breve{\rho}' w$

Frege. ⟨fol.⟩ 5

B(a). To prove

$c \sim\epsilon\, \rho \,.\, b \sim\epsilon\, \breve{\rho} \,.\, z = \rho w \,.\supset_R.\, w \sim= \breve{\rho}z :$

$\supset : c \sim\epsilon\, z \,.\, b \sim\epsilon\, w \,.\supset.\, \text{Nc'}w \sim= \text{Nc'}z$

i.e. $c \sim\epsilon\, z \,.\, b \sim\epsilon\, w \,.\supset.\, \text{Nc'}w \sim= \text{Nc'}z :$

$\cup : \exists R \ni (c \sim\epsilon\, \rho \,.\, b \sim\epsilon\, \breve{\rho} \,.\, z = \rho w \,.\, w = \breve{\rho}z)$

i.e. $\text{Nc'}w \sim= \text{Nc'}z \,.\cup.\, c\,\epsilon\, z \,.\cup.\, b\,\epsilon\, w \,.$

$\cup .\, \exists R \ni (c \sim\epsilon\, \rho \,.\, b \sim\epsilon\, \breve{\rho} \,.\, z = \rho w \,.\, w = \breve{\rho}z)$

(b) $c\,\epsilon\, v \,.\, b\,\epsilon\, u \,.\, \text{Nc}\; u - \iota b = \text{Nc}\; v - \iota c \,.\supset.\, \text{Nc}\; u = \text{Nc}\; v$

Γ. To prove $\breve{N} \,\epsilon\, \text{Nc}\to 1$ ⟨37⟩

⟨This section is deleted with a large X:⟩

(a) To prove $\cancel{v = \rho u}$ $u = \breve{\rho}v \,.\, uRb \,.\, R\,\epsilon\, 1\to\text{Nc} \,.\, cRm \,.\supset :.$

$x\,\epsilon\, u - \iota b \,.\sim\supset.\, x = m : y\,\epsilon\, v - \iota c \,.\sim$

$\supset .\, y = n : \equiv : \sim[\sim(y \sim= c \,.\supset.\, x = b) \,.\supset.\, y \sim Rx]$

i.e. $u = \breve{\rho}v \,.\, nRb \,.\, cRm \,.\, R\,\epsilon\, 1\to\text{Nc} \,.\supset :.$

$x\,\epsilon\, u - \iota b - \iota m \,.\, y\,\epsilon\, v - \iota c - \iota n \,.\equiv.\, \sim\{y \sim Rx \,.\cup.\, y = c \,.$

$\cup .\, x = b\} \,.\equiv.\, yRx \,.\, y \sim= c \,.\, x \sim= b$

⟨End of deleted section.⟩

Put $yRx \,.\, y \sim= c \,.\, x \sim= b \,.\equiv.\, yR'x.$ Df

$\cancel{u - \iota b - \iota m = \breve{\rho}'(v - \iota c - \iota n)}$

(a) To prove $u = \breve{\rho}v \,.\, R\,\epsilon\, \text{Nc}\to 1 \,.\, nRb \,.\, cRm \,.$

$\supset .\, u - \iota b - \iota m = \breve{\rho}'(v - \iota c - \iota n)$

(b) To prove $b\,\epsilon\, u \,.\, c\,\epsilon\, v \,.\, \text{Nc}\; u - \iota b \sim= \text{Nc}\; v - \iota c \,.$

---

[37] Part Γ (pp. 113–27) proves theorem 89, $\mathrm{I}^{-1}f$: the predecessor function is one-one, a step in proving Peano's axiom that no two numbers have the same successor. Russell uses "N" for the relation of a number to its successor.

$\supset . \text{Nc'}u \sim= \text{Nc'}v$

Propositions about 0. ⟨38⟩ (a) $x \ni f(x) \in 0 . \supset . \sim f(a)$
(b) $x \in a . \equiv_x . \Lambda : \supset . a \in 0$

$\text{Nc} - \iota 0 = \breve{\nu}$ ⟨39⟩

Propositions about 1. ⟨40⟩ $u \in 1 . \supset . \exists u \quad 0N1 \quad 0 \sim N0 \quad u \in 1 .$
$x, y \in u . \supset . x = y.$

$\exists u : x, y \in u . \supset_{x,y} . x = y : \supset . u \in 1$

Frege. ⟨fol.⟩ 6

$Z . 0N^N b . \supset . b \sim N^N b$ ⟨41⟩

(a) $x \sim N^N 0$ ⟨42⟩

First prove $bQ^N a : F(x) . yQx . \supset_{x,y} . F(y) : F(b) : \supset . F(a)$

$d \sim N^N d . dN^N a . \supset . a \sim N^N a$ and $0 \sim N^N 0$

R^N is defined as follows: [giving a new view of mathematical induction] ⟨43⟩

---

[38] These "propositions about 0" come from *Δ* which begins at §96 (p. 127) and runs to p. 131. Russell's (a) is theorem 95. His (b) is not Frege's theorem 97, but an approximation to it.

[39] The range of the successor relation is all numbers except for 0, a way of expressing Peano's axiom that 0 is not the successor of any number.

[40] Propositions about 1 occupy *E*, §102 to §107, pp. 131–6. Russell transcribes theorem 113: if the number of *u* is 1 then *u* is non-empty and theorem 110: 1 is the successor of 0. Russell's notation $0 \sim N0$—0 is not the successor of itself—is not a theorem in Frege. However, theorem 114 says that only 1 is a successor of 0, which combined with $0 \neq 1$ (theorem 111) proves that result. Then follow theorem 117: if *a* and *d* are in a class with number 1 then $a = d$, and theorem 121: if whenever *a* and *c* are in *u* then $a = c$, then the number of *u* is 1.

[41] The next two and a half leaves are devoted to Part *Z*, §108 to §113, pp. 137–44, proving theorem 145: no number which bears the ancestral of the successor relation to 0 also bears that relation to itself. That is, no natural number precedes itself in the number series.

[42] Folio 6 concludes with theorem 126; 0 is not preceded by any number.

[43] Russell remarks here on the connection between the ancestral of the successor relation and mathematical induction. Frege does not discuss induction, nor use it explicitly as a proof technique, although many times a property is shown to hold of all numbers by using the properties of ancestrals, which amounts to induction.

$aR^Nb \,.\!=\,:. \phi x \,.\, xRy \,.\, \supset_{x,y} .\, \phi y : aRx \,.\, \supset_x .\, \phi x : \supset_\phi .\, \phi b$ (1)

Proof of $a \sim N^N 0$:

$[(1) \,.\, \supset :: aR^Nb : \phi x \,.\, xRy \,.\, \supset_{x,y} .\, \phi y : aRy \,.$
$\supset_y .\, \phi y : \supset .\, \phi b$ (2) [123]

$b \sim \epsilon \breve{\rho} \,.\, \supset .\, d \sim Rb : \supset : dRb \,.\, \supset .\, b \,\epsilon \breve{\rho} : \supset : d \,\epsilon \breve{\rho} \,.\, dRb \,.$
$\supset .\, b \,\epsilon \breve{\rho}$:

$\supset : x \,\epsilon \breve{\rho} \,.\, xRy \,.\, \supset_{x,y} .\, y \,\epsilon \breve{\rho}$ (3)

$(2) \,.\, (3) \,.\, \supset :. aR^Nb : aRy \,.\, \supset_y .\, y \,\epsilon \breve{\rho} : \supset .\, b \,\epsilon \breve{\rho}$ (4)

$aRb \,.\, \supset .\, b \,\epsilon \breve{\rho} : \supset : aRy \,.\, \supset_y .\, y \,\epsilon \breve{\rho}$ (5)

$(4) \,.\, (5) \,.\, \supset .\, aR^Nb \,.\, \supset .\, b \,\epsilon \breve{\rho}$ (124)

$(124) \,.\, \supset : b \sim \epsilon \breve{\rho} \,.\, \supset .\, a \sim R^Nb$ (125)

$c \sim N0 \,.\, \supset .\, 0 \sim \epsilon \,\breve{\nu}$ (6)

$(6) \,.\, (125) \,.\, \supset .\, a \sim N^N 0$ (126)

Frege. ⟨fol.⟩ 7

Proof of $dNb \,.\, aN^Nb \,.\, \supset : aN^Nd \,.\, \cup \,.\, a1\text{'}d$ ⟨44⟩

$[\phi x \,.\, xRy \,.\, \supset_{x,y} .\, \phi y : \supset : \phi e \,.\, eRy \,.\, \supset_y .\, \phi y$:.

$\supset :. \phi x \,.\, xRy \,.\, \supset_{x,y} .\, \phi y : \supset : \phi e \,.\, eRm \,.\, \supset .\, \phi m$:.

$\supset :. aR^Ne \,.\, eRm : \phi x \,.\, xRy \,.\, \supset_{x,y} .\, \phi y : aRy \,.\, \supset_y .\, \phi y : \supset .\, \phi m$:.

$\supset :. aR^Ne \,.\, eRm : \phi x \,.\, xRy \,.\, \supset_{x,y} .\, \phi y : aRy \,.\, \supset_y .\, \phi y : \supset_\phi .\, \phi m$:.

$\supset : aR^Ne \,.\, eRm \,.\, \supset .\, aR^Nm$ (133)

$aRm \,.\, \supset .\, aR^Nm : \supset : e = a \,.\, eRm \,.\, \supset .\, aR^Nm$:

$\supset : aR^Ne \,.\, \cup \,.\, a1\text{'}e : a \sim R^Ne : eRm : \supset .\, aR^Nm$ (β)

$(\beta) \,.\, (133) \,.\, \supset :. a(R^N \cup 1\text{'})e \,.\, eRm \,.\, \supset .\, aR^Nm$ (134)

$F\{a \sim R^Nm \,.\, \supset .\, m = a\} \,.\, \supset .\, F\{a(R^N \cup 1\text{'})m\}$ (135)

$(135) \,.\, \supset : aR^Nm \,.\, \supset .\, a(R^N \cup 1\text{'})m$: (136)

$(134) \,.\, (136) \,.\, \supset : a \sim (R^N \cup 1\text{'})m \,.\, eRm \,.$
$\supset .\, a \sim (R^N \cup 1\text{'})e$: (137)?

$\supset :. xRn \supset_x .\, a \sim (R^N \cup 1\text{'})x : mRn \,.\, eRm : \supset .\, a \sim (R^N \cup 1\text{'})e$:.

[44] This leaf begins with §112 devoted to first proving that if *d* immediately precedes *b* and *a* precedes *b* then *a* precedes or is identical with *d*, i.e. *a* "weakly" precedes *d*, theorem 143.

$$\supset :.\ xRn \supset_x .\ a \sim (R^N \cup 1')x : mRn :$$
$$\supset : xRm \,.\, \supset_x .\ a \sim (R^N \cup 1')x :.$$
$$\supset :.\ \exists x \ni \{xRm \,.\, a(R^N \cup 1')x\} \,.\, mRn \,.$$
$$\supset .\ \exists x \ni \{xRn \,.\, a(R^N \cup 1')x\} :.\ (\delta)$$
$$(\delta) \,.\, (123) \,.$$
$$\supset :.\ aR^Nb : aRx \,.\, \supset_x .\ \exists y \ni \{yRx \,.\, a(R^N \cup 1')y\} :$$
$$\supset .\ \exists x \ni \{xRb \,.\, a(R^N \cup 1')x\} \quad (138)$$
$$b1'a \,.\, \supset .\ a(R^N \cup 1')b \quad (139) \qquad (139) \,.\, \supset .\ a(R^N \cup 1')a \quad (140)$$
$$(138) \,.\, (140) \,.\, \supset : aR^Nb \,.\, \supset .\ \exists x \ni \{xRb \,.\, a(R^N \cup 1')x\} :$$
$$\supset : \sim \exists x \ni \{xRb \,.\, a(R^N \cup 1')x\} \,.\, \supset .\ a \sim R^Nb \quad (142)$$

Frege. ⟨fol.⟩ 8

$$a \sim (N^N \cup 1')d \,.\, dNb \,.\, cNb \,.\, \supset .\ a \sim (N^N \cup 1')c : \quad (\alpha)$$
$$\supset : a \sim (N^N \cup 1')d \,.\, dNb \,.\, \supset : xNb \,.\, \supset_x .\ a \sim (N^N \cup 1')x \quad (\beta)$$
$$(\beta) \,.\, (142) \,.\, \supset : a \sim (N^N \cup 1')d \,.\, dNb \,.\, \supset .\ a \sim (N^N)b :$$
$$\supset : dNb \,.\, aN^Nb \,.\, \supset .\ a(N^N \cup 1')d \quad (143)$$

Hence shortly $0(N^N \cup 1')b \,.\, \supset .\ b \sim N^Nb$ (145) ⟨45⟩

Proof of $0(N^N \cup 1')b \,.\, \supset .\ \mathrm{Nc}'(\nu^N b\ \nu\iota b)\breve{N}b$ ⟨46⟩
i.e. the number of finite numbers up to and including b is $b + 1$.
Put $R^N \cup 1' = R'$ Df

$$bN'd \,.\, bN^Na \,.\, dNa \,.\, \supset : bN^Na \,.\, \equiv .\ bN'd \quad (\beta)$$
$$dNa \,.\, 0N'a \,.\, \supset : bN^Na \,.\, \equiv .\ bN'd \quad (\eta)$$
$$dNa \,.\, 0N'a \,.\, \supset .\ \mathrm{Nc}\ \nu^Na = \mathrm{Nc}\ \nu'd$$
$$xN(\mathrm{Nc}'\nu'x) \,.\, 0N'x \,.\, \supset_x .\ xNy \,.\, \supset_y .\ yN(\mathrm{Nc}'\nu'x) \quad (150)$$
$$\phi x \,.\, aR'x \,.\, xRy \,.\, \supset_{x,y} .\ \phi y : \phi d \,.\, aR'd : \supset : dRz \supset_z .\ \phi z :.$$
$$\supset :.\ \phi x \,.\, aR'x \,.\, xRy \,.\, \supset_{x,y} .\ \phi y : aR'b \,.$$
$$\supset .\ \sim \phi b : aR'b \,.\, aR'd \,.\, dRb : \supset .\ \sim \phi d \quad (\gamma)$$
$$(\gamma) \,.\, 137 \,.\, \supset :.\ \phi x \,.\, aR'x \,.\, xRy \supset_{x,y} .\ \phi y : \exists\breve{\rho}'a \wedge z \ni (\phi z) :$$

---

45 No natural number follows itself in the number series (see Heck, pp. 277 and 284).
46 This is Part *H*, §114–§19, ending with theorem 155, which Russell correctly reports as "the number of finite numbers up to and including *b* is $b + 1$". This is the crucial step in proving that every number has a successor (see Heck, p. 275).

$$\supset : \exists\breve{\rho} \wedge \breve{\rho}'a \wedge y \ni (\phi y) :. \quad (151)$$

$$\supset :. aR'b : \phi x \,.\, aR'x \,.\, xRy \supset_{x,y} .\, \phi y : \phi a \,.\, aR'b \,.\, \sim\supset .\, \sim \phi b :.$$

$$\supset :. aR'b : \phi x \,.\, aR'x \,.\, xRy \supset_{x,y} .\, \phi y : \phi a : \supset .\, \phi b :. \quad (152)$$

$$\supset :. 0N'b \,.\, 0N(\mathrm{Nc}'\nu'0) \,.\supset .\, bN(\mathrm{Nc}'\nu'b) \quad (153)$$

$$0N(\mathrm{Nc}'\nu'0) \quad (154) \qquad 0N'b \,.\supset .\, bN(\mathrm{Nc}'\nu'b) \quad (155)$$

Frege. ⟨fol.⟩ 9

Propositions about α_0 [p. 150 ff.]. ⟨47⟩

$\alpha_0 = \mathrm{Nc}\ \nu$ Df

We have to prove $\alpha_0 \sim\epsilon\ \nu$, which follows from $\alpha_0 + 1 = \alpha_0$ and (145)

This results from $\breve{\nu}$ sim ν. The proof is as follows:

$$0 \sim N^N 0 \,.\supset : 0N^N a \,.\supset .\, a \sim= 0 : \supset : 0N'd \,.\, dNa \,.\supset .\, a \sim= 0 :$$

$$\supset :. a\breve{N}'0 \,.\, 0N'd \,.\, dNa \,.\supset : a\breve{N}'0 \,.\sim\supset .\, a = 0 :.$$

$$\supset :. 0N'a \,.\, 0N'd \,.\, dNa \,.\supset : a\breve{N}'0 \,.\sim\supset .\, a = 0 :.$$

$$\supset : 0N'd \,.\, dNa \,.\supset : a\breve{N}'0 \,.\sim\supset .\, a = 0 :.$$

$$\supset :. a\breve{N}'0 \,.\supset .\, a = 0 : \supset : dNa \,.\supset .\, 0 \sim N'd :.$$

$$\supset :. aN^N 0 \,.\, dNa \,.\supset .\, 0 \sim N'd :.$$

$$\supset :. dNx \,.\supset_x .\, x \sim \breve{N}^N 0 : \supset : dNa \,.\supset .\, 0 \sim N'd :.$$

$$\supset :. dNx \,.\supset_x .\, x \sim \breve{N}^N 0 : \supset : 0N'd \,.\supset .\, d \sim Na :.$$

$$\supset :. dNx \,.\supset_x .\, x \sim \breve{N}^N 0 : \supset : 0N'd \,.\supset .\, d \sim\epsilon\ \nu :.$$

$$\supset :. dNx \,.\supset_x .\, x \sim \breve{N}^N 0 : \supset : 0N'd \,.\supset .\, 0 \sim N'd :.$$

$$[(156). : b \sim\epsilon\ \nu \,.\supset .\, 0 \sim N'b]$$

$$\supset :. dNx \,.\supset_x .\, x \sim \breve{N}^N 0 : \supset .\, 0 \sim N'd :.$$

$$\supset :. yNx \,.\supset_x .\, x \sim \breve{N}^N 0 : \supset_y .\, 0 \sim N'y :. [\supset :. y \,\epsilon\, \breve{\nu}'0 \,.$$

$$\supset_y .\, \exists\, \breve{\nu}y \wedge \breve{\nu}'0 - \iota 0 :.]$$

[47] Russell skips the single-page Θ and begins here with Part *I*, §122 to §157, "Proof of various propositions about the number *Endlos*" [Endless], Frege's name for the number of numbers, in his notation: '$\widehat{\omega}$' approximately, in Russell's: "α_0". This is the cardinal number of the natural numbers, Cantor's $\aleph_0$. The rest of the notes on Volume I, pp. 9–22 are devoted to this section, ending with §144 on p. 179. Volume I continues to p. 251. These notes are more careful than what precedes. Beginning with §125 the section numbers are indicated as well as Frege's Greek names or theorem numbers for each line. Russell misses naming only one theorem, 164, between 162 and 207 at the end of §143.

$\supset :. \mathrm{N} \epsilon \mathrm{Nc}{\to}1 . \supset . \cancel{\breve{\nu}'0 = \nu(\breve{\nu}'0 - \iota 0)}\ \breve{\nu}(\breve{\nu}'0) = \breve{\nu}'0 - \iota 0 :.$

$\supset . (0) = \breve{\nu}'0 - \iota 0 \quad (162)$

⟨*verso of fol.* 9⟩

$pq . = :. p . \supset . q \supset r : \supset_r . r$ Df

$pq \supset s . \supset : p . \supset . q \supset s$

$pq \supset s . \equiv :: p . \supset . q \supset r : \supset_r . r :. \supset . s$

$p . \supset . q \supset s : pq : \supset . s$

Frege. §125. ⟨fol.⟩ 10

$R' \cap \sim R^N \supset 1' . \supset :. cR'd : xRd . \supset_x . c \sim R'x : \supset . d = c$ (α) ⟨48⟩

$d\breve{R}x . \supset_x . c \sim R'x : aRd : \supset . c \sim R'a :.$

$\supset :. d\breve{R}x . \supset_x . c \sim R'x : \supset : yRd . \supset_y . c \sim R'y$ (δ)

$(\alpha) . (\delta) . \supset :.$

$d\breve{R}x . \supset_x . c \sim R'x : \supset : cR'd . \supset . d = c :.$

$\supset :. d\breve{R}x . \supset_x . c \sim R'x : d\breve{R}'c : \supset . d = c :.$

$\supset :. d\breve{R}x . \supset_x . c \sim R'x : \supset . \sim (d\breve{R}'c . d \sim = c) :.$

$\supset :. y\breve{R}x . \supset_x . c \sim R'x : \supset_y : y \sim \breve{R}'c . \cup . y = c :.$

$\supset :. R \epsilon 1{\to}\mathrm{Nc} . \supset : \rho(\breve{\rho}'c - \iota c) = \breve{\rho}'c :.$ (163)

$\supset : \nu(\breve{\nu}'0 - \iota 0) = \breve{\nu}'0 :$

$\supset : \breve{\nu}(\breve{\nu}'0) = \breve{\nu}'0 - \iota 0 . \supset . \mathrm{Nc}\, \breve{\nu}'0 = \mathrm{Nc}\,(\breve{\nu}'0 - \iota 0) :$ (β)

$(\beta) . (162) . \supset . \mathrm{Nc}\, \breve{\nu}'0 = \mathrm{Nc}\,(\breve{\nu}'0 - \iota 0)$ (γ)

$\supset : \mathrm{Nc}\, \breve{\nu}'0 = \alpha_o . \supset . \mathrm{Nc}\,(\breve{\nu}'0 - \iota 0) = \alpha_o :$ (ϵ)

$\supset : 0 \epsilon \breve{\nu}'0 . \supset . \alpha_o N \alpha_o :$

$\supset . \alpha_o N \alpha_o$ (165)

$\supset . \alpha_o N^N \alpha_o .$ (166)

$(145) . (166) . \supset . 0 \sim N'\alpha_o$ (167) ⟨49⟩

Frege. ⟨fol.⟩ 11

---

[48] $\sim R^N \supset 1'$ means that R^N is not "included in identity", i.e. R^N is not reflexive.

[49] Theorem 167 proves that *Endlos* is not a natural number.

To prove: $0N' \, \mathrm{Nc}'v \,.\, \alpha_0 = \mathrm{Nc}'u \,.\, \supset .\, \alpha_0 = \mathrm{Nc}'(u \vee v)$ ⟨50⟩

§127. $c \sim\epsilon\, w \,.\, a \,\epsilon\, w \,.\supset.\, a \sim= c :\supset: c \sim\epsilon\, w \,.\, a \,\epsilon\, w \,.$

$$\supset .\sim(a \,\epsilon\, w \,.\supset.\, a = c):$$

$$\supset :.\sim(a \,\epsilon\, w \,.\supset.\, a = c) \,.\, a \,\epsilon\, w \,.\, c \sim\epsilon\, w \,.$$

$$\supset : a \,\epsilon\, w \,.\equiv.\, a \,\epsilon\, w \,.\, a \sim= c :.$$

$$\supset :.\, c \sim\epsilon\, w \,.\supset: a \,\epsilon\, w \,.\equiv.\, a \,\epsilon\, w \,.\, a \sim= c :.$$

$$\supset :.\, c \sim\epsilon\, w \,.\supset: x \,\epsilon\, w \,.\equiv_x.\, x \,\epsilon\, w \,.\, x \sim= c :.$$

$$\supset : c \sim\epsilon\, w \,.\supset.\, \mathrm{Nc}'w = \mathrm{Nc}'w - \iota c :$$

$$\supset :.\, c \sim\epsilon\, w \,.\supset: n = \mathrm{Nc}'(w - \iota c) \,.\supset.\, n = \mathrm{Nc}'w \qquad (168)$$

$$(69) \,.\supset: \mathrm{Nc}\,(w - \iota c) = m \,.\, c \,\epsilon\, w \,.\, \mathrm{Nc}'w \sim= n \,.\supset.\, m \sim Nn :$$

$$\supset : \mathrm{Nc}\,(w - \iota c) = m \,.\, c \,\epsilon\, w \,.\, mNn \,.\supset.\, \mathrm{Nc}'w = n : \qquad (169)$$

$$\supset : m = \mathrm{Nc}\,(w - \iota c) \,.\, c \,\epsilon\, w \,.\, mNn \,.\supset.\, \mathrm{Nc}'w = n : \qquad (\alpha)$$

$$\supset : m = \mathrm{Nc}\,(w - \iota c) \,.\, c \,\epsilon\, w \,.\, mNn \,.\supset.\, n = \mathrm{Nc}'w \qquad (170)$$

$$(165) \,.\, (170) \,.\supset: \alpha_0 = \mathrm{Nc}\,(w - \iota c) \,.\, c \,\epsilon\, w \,.\supset.\, \alpha_0 = \mathrm{Nc}'w \qquad (\alpha)$$

$$(\alpha) \,.\, (168) \,.\supset: \alpha_0 = \mathrm{Nc}\,(w - \iota c) \,.\supset.\, \alpha_0 = \mathrm{Nc}'w \qquad (171)$$

$$\supset : a \,\epsilon\, v \,.\, a \sim= c \,.\supset.\, a \,\epsilon\, v - \iota c :$$

$$\supset : a \,\epsilon\, v - u \,.\, a \sim= c \,.\supset.\, a \,\epsilon\, v - u - \iota c : \qquad (\alpha)$$

$$\supset :.\, a \sim\epsilon\, u \,.\supset.\, a \,\epsilon\, v - \iota c : \cancel{a \,\epsilon\, v - u \,.\supset.\, a \sim= c} : a \sim\epsilon\, u \,.$$

$$\supset .\, a \,\epsilon\, v : a \sim= c :\supset: a \,\epsilon\, v - u - \iota c \,.\equiv.\, a \,\epsilon\, v - u \,.\, a \sim= c \qquad (\beta)$$

$$a \,\epsilon\, v - \iota c \,.\supset.\, a \sim= c :$$

$$\supset :.\, a \sim\epsilon\, u \,.\supset.\, a \,\epsilon\, v - \iota c :\supset.\, a \sim\epsilon\, u \,.\supset.\, a \sim= c :.$$

$$\supset :.\, c \sim\epsilon\, u : a \sim\epsilon\, u \,.\supset.\, a \,\epsilon\, v - \iota c :\supset.\, a \sim= c :.$$

$$\supset :.\, a \sim\epsilon\, u \,.\supset.\, a \,\epsilon\, v : c \sim\epsilon\, u : a \sim\epsilon\, u \,.\supset.\, a \,\epsilon\, v - \iota c :$$

$$\supset : a \sim\epsilon\, u \,.\supset.\, a \,\epsilon\, v : a \sim= c \qquad (\epsilon)$$

Frege. ⟨fol.⟩ 12

§127 continued.

$$a \,\epsilon\, v - \iota c \,.\supset.\, a \,\epsilon\, v :$$

$$\supset :.\, a \sim\epsilon\, u \,.\supset.\, a \,\epsilon\, v - \iota c :\supset: a \sim\epsilon\, u \,.\supset.\, a \,\epsilon\, v \qquad (\zeta)$$

---

[50] The next four leaves of notes, 11 to 14, are devoted to §127, the proof of theorem 127, which occupy pp. 155 to 160 of *Gg.* Russell transcribes it line by line. Theorem 127 states that the cardinal of the union of a finite set v with u of cardinality $\aleph_0$ is $\aleph_0$. Frege describes this as "Wenn Endlos die Anzahl eines Begriffes ist und wenn die Anzahl eines andern Begriffes endlich ist, so ist Endlos die Anzahl des Begriffes *unter den ersten oder unter den zweiten Begriff fallend*" [When *Endlos* is the number of one concept and the number of another concept is finite, then *Endlos* is the number of the concept *falling under the first concept or under the second*].

$$(\epsilon) \,.\, (\zeta) \,.\, \supset :.\, c \sim\epsilon u : a \sim\epsilon u \,.\, \supset \,.\, a \,\epsilon\, v - \iota c : \supset : a \sim\epsilon u \,.$$
$$\supset \,.\, a \,\epsilon\, v : a \sim= c :. \quad (\eta)$$
$$(\eta) \,.\, (\beta) \,.\, \supset :: c \sim\epsilon u \,.\, \supset :.\, a \sim\epsilon u \,.\, \supset \,.\, a \,\epsilon\, v - \iota c : \equiv : a \sim\epsilon u \,.$$
$$\supset \,.\, a \,\epsilon\, v : a \sim= c \quad (\theta)$$
$$(\theta) \,.\, (77) \,.\, \supset :: c \sim\epsilon u \,.\, \supset :.\, a \sim\epsilon u \,.\, \supset \,.\, a \,\epsilon\, v - \iota c :$$
$$\equiv : a \,\epsilon\, x \ni (x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v) \,.\, a \sim= c :: \quad (\iota)$$
$$\supset :: c \sim\epsilon u \,.\, \supset :.\, x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v - \iota c : \equiv_x : x \,\epsilon\, y \ni (y \sim\epsilon u \,.$$
$$\supset \,.\, y \,\epsilon\, v) \,.\, x \sim= c :: \quad (\kappa)$$
$$\supset :.\, c \sim\epsilon u \,.\, \supset : \mathrm{Nc}\text{'}x \ni (x \sim\epsilon u \,.$$
$$\supset \,.\, x \,\epsilon\, v - \iota c) = \mathrm{Nc}\text{'}x \ni (x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v : x \sim= c) :. \quad (\lambda)$$
$$\supset :.\, c \sim\epsilon u \,.\, \supset : \alpha_0 = \mathrm{Nc}\text{'}x \ni (x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v - \iota c) \,.$$
$$\supset \,.\, \alpha_0 = \mathrm{Nc}\text{'}x \ni (x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v : x \sim= c) :. \quad (\mu)$$
$$(\mu) \,.\, (171) \,.\, \supset :.\, c \sim\epsilon u \,.\, \supset : \alpha_0 = \mathrm{Nc}\text{'}x \ni (x \sim\epsilon u \,.$$
$$\supset \,.\, x \,\epsilon\, v - \iota c) \,.\, \supset \,.\, \alpha_0 = \mathrm{Nc}\text{'}x \ni (x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v) \quad (\nu)$$
$$B \sim B \supset A \,.$$
$$\supset :.\, B : \sim B \,.\, \supset \,.\, C : \supset : \sim B \,.\, \supset \,.\, A :. \quad (\xi)$$
$$\supset :.\, B : \sim B \,.\, \supset \,.\, A : \sim B \,.\, \supset \,.\, C : \supset : \sim B \supset A \,.\equiv.\, \sim B \supset C \quad (o)$$
$$(\xi) \,.\, (o) \,.\, \supset :.\, B \,.\, \supset : \sim B \supset A \,.\equiv.\, \sim B \supset C \quad (\pi)$$
$$a \,\epsilon\, v - \iota c \,.\, \supset \,.\, a \,\epsilon\, v :$$
$$\supset :.\, a \,\epsilon\, v \,.\, \supset \,.\, a \,\epsilon\, v - \iota c : \supset : a \,\epsilon\, v - \iota c \,.\equiv.\, a \,\epsilon\, v :. \quad (\rho)$$
$$\supset :.\, a \sim= c \,.\, \supset : a \,\epsilon\, v - \iota c \,.\equiv.\, a \,\epsilon\, v :. \quad (\sigma)$$
$$\supset :.\, a \sim\epsilon u \,.\, c \,\epsilon\, u \,.\, \supset : a \,\epsilon\, v - \iota c \,.\equiv.\, a \,\epsilon\, v :. \quad (\tau)$$
$$\supset :: a \sim\epsilon u \,.\, c \,\epsilon\, u \,.\, \supset :.\, a \sim\epsilon u \,.\, \supset : a \,\epsilon\, v - \iota c \,.\equiv : a \sim\epsilon u \,.$$
$$\supset \,.\, a \,\epsilon\, v :: \quad (\upsilon)$$

Frege. ⟨fol.⟩ 13

§127 continued.

$$(\pi)(\upsilon) \,.\, \supset :: c \,\epsilon\, u \,.\, \supset :.\, a \sim\epsilon u \,.\, \supset : a \,\epsilon\, v - \iota c : \equiv : a \sim\epsilon u \,.$$
$$\supset \,.\, a \,\epsilon\, v :: \quad (\phi)$$
$$\supset :: c \,\epsilon\, u \,.\, \supset :.\, x \sim\epsilon u \,.\, \supset : x \,\epsilon\, v - \iota c : \equiv_x : x \sim\epsilon u \,.$$
$$\supset \,.\, x \,\epsilon\, v :: \quad (\chi)$$
$$\supset : c \,\epsilon\, u \,.\, \supset \,.\, \mathrm{Nc}\; x \ni (x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v - \iota c) = \mathrm{Nc}\; x \ni (x \sim\epsilon u \,.$$
$$\supset \,.\, x \,\epsilon\, v) : \quad (\psi)$$
$$\supset :.\, c \,\epsilon\, u \,.\, \supset : \alpha_0 = \mathrm{Nc}\; x \ni (x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v - \iota c) \,.$$
$$\supset \,.\, \alpha_0 = \mathrm{Nc}\; x \ni (x \sim\epsilon u \,.\, \supset \,.\, x \,\epsilon\, v) \quad (\omega)$$
$$(\nu) \,.\, (\omega) \,.\, \supset : \alpha_0 = \mathrm{Nc}\; u \vee v \vee \iota c \,.\, \supset \,.\, \alpha_0 = \mathrm{Nc}\; u \vee v \; (\alpha')$$

[I introduce this abbreviation for convenience. B.R.]

$(\alpha') \,.\, (75) \,.\, \supset :.\, d = \mathrm{Nc}'w \,.\, \supset_w .\, \alpha_0 = \mathrm{Nc}\,(u \vee w) : d = \mathrm{Nc}\,v - \iota c :$
$\supset .\, \alpha_0 = \mathrm{Nc}\,u \vee v \quad (\gamma')$

$(\gamma') \,.\, (159) \,.\, \supset :.\, d = \mathrm{Nc}'w \,.\, \supset_w .\, \alpha_0 = \mathrm{Nc}\,(u \vee w) : a =$
$\mathrm{Nc}\,v \,.\, dNa \,.\, c \,\epsilon\, v : \supset .\, \alpha_0 = \mathrm{Nc}\,u \vee v :. \quad (\delta')$

$\supset :.\, d = \mathrm{Nc}'w \,.\, \supset_w .\, \alpha_0 = \mathrm{Nc}\,u \vee w : a = \mathrm{Nc}\,v \,.\, dNa \,.\, \alpha_0 \sim$
$= \mathrm{Nc}\,u \vee v : \supset .\, c \sim\epsilon\, v :. \quad (\epsilon')$

$\supset :.\, d = \mathrm{Nc}'w \,.\, \supset_w .\, \alpha_0 = \mathrm{Nc}\,u \vee w : a = \mathrm{Nc}\,v \,.\, dNa \,.\, \alpha_0 \sim$
$= \mathrm{Nc}\,u \vee v : \supset .\, v = \Lambda \quad (\zeta')$

$\mathrm{Nc}'v = 0 \,.\, a = \mathrm{Nc}'v \,.\, \supset .\, d \sim Na \quad (\theta')$

$(\zeta') \,.\, (\theta') \,.\, \supset :.\, d = \mathrm{Nc}\,w \,.\, \supset_w .\, \alpha_0 = \mathrm{Nc}\,u \vee w : dNa \,.\, \alpha_0 \sim$
$= \mathrm{Nc}\,u \vee v \,.\, a = \mathrm{Nc}\,v : \supset .\, d \sim Na :.$

$\supset :.\, d = \mathrm{Nc}\,w \,.\, \supset_w .\, \alpha_0$
$= \mathrm{Nc}\,u \vee w : dNa \,.\, a = \mathrm{Nc}\,v : \supset .\, \alpha_0 = \mathrm{Nc}\,u \vee v :. \quad (\lambda')$

$\supset :.\, x = \mathrm{Nc}\,w \,.\, \supset_w .\, \alpha_0 = \mathrm{Nc}\,(u \vee w) : xNy \,.\, y = \mathrm{Nc}\,v : \supset_{x,y,v} .\, \alpha_0$
$= \mathrm{Nc}\,u \vee v \quad (\mu')$

$(144) \,.\, (\mu') \,.\, \supset :.\, 0N'(\mathrm{Nc}\,v) : 0 = \mathrm{Nc}'w \,.\, \supset_w .\, \alpha_0 = \mathrm{Nc}\,(u \vee w) :$
$\supset : \mathrm{Nc}\,v = \mathrm{Nc}\,w \,.\, \supset_w .\, \alpha_0 = \mathrm{Nc}\,(u \vee w) \quad (\nu')$

$a \,\epsilon\, u \vee v \,.\, \supset :.\, a \,\epsilon\, u \,.\, \supset : a \,\epsilon\, u \vee v \,.\, \equiv .\, a \,\epsilon\, u \quad (\xi')$

$a \sim\epsilon\, v \,.\, \supset : a \,\epsilon\, u \vee v \,.\, \equiv .\, a \,\epsilon\, u \quad (\pi')$

$(94) \,.\, (\pi') \,.\, \supset :.\, \mathrm{Nc}\,v = 0 \,.\, \supset : a \,\epsilon\, u \vee v \,.\, \equiv .\, a \,\epsilon\, u \quad (\rho')$

Frege. ⟨fol.⟩ 14

§127 continued.

$(77) \,.\, (\rho') \,.\, \supset :.\, \mathrm{Nc}\,v = 0 \,.\, \supset : u \vee v = u :. f \quad (\sigma')$

$\supset :.\, \mathrm{Nc}\,v = 0 \,.\, \supset : x \,\epsilon\, u \vee v \,.\, \equiv_x .\, x \,\epsilon\, u \quad (\tau')$

$(\tau') \,.\, (96) \,.\, \supset :.\, \mathrm{Nc}\,v = 0 \,.\, \supset .\, \mathrm{Nc}\,(u \vee v) = \mathrm{Nc}\,u :. \quad (\nu')$

$\supset :.\, 0 = \mathrm{Nc}\,v \,.\, \supset .\, \mathrm{Nc}\,(u \vee v) = \mathrm{Nc}\,u :. \quad (\phi')$

$\supset :.\, \alpha_0 = \mathrm{Nc}\,u \,.\, 0 = \mathrm{Nc}\,v \,.\, \supset .\, \alpha_0 = \mathrm{Nc}\,(u \vee v) :. \quad (\chi')$

$\supset :.\, \alpha_0 = \mathrm{Nc}\,u \,.\, \supset : 0 = \mathrm{Nc}\,v \,.\, \supset_v .\, \alpha_0 = \mathrm{Nc}\,(u \vee v) \quad (\psi')$

$(\psi') \,.\, (\nu') \,.\, \supset :.\, 0N'(\mathrm{Nc}\,v) \,.\, \alpha_0 = \mathrm{Nc}\,u \,.\, \supset : \mathrm{Nc}\,v = \mathrm{Nc}\,w \,.$
$\supset_w .\, \alpha_0 = \mathrm{Nc}\,(u \vee w) :. \quad (\omega')$

$\supset :.\, 0N'(\mathrm{Nc}\,v) \,.\, \alpha_0 = \mathrm{Nc}\,u \,.\, \mathrm{Nc}\,v = \mathrm{Nc}\,v \,.$
$\supset .\, \alpha_0 = \mathrm{Nc}\,(u \vee v) :.$

$\supset :.\, 0N'(\mathrm{Nc}\,v) \,.\, \alpha_0 = \mathrm{Nc}\,u \,.\, \supset .\, \alpha_0 = \mathrm{Nc}\,(u \vee v) \quad (172)$

Frege. ⟨fol.⟩ 15

(c) Proof of $\alpha_0 = \mathrm{Nc}\, u \,.\supset.\, \exists R \ni \{R \in \mathrm{Nc}{\to}1 \,.\, R^N$
$\supset 0' \,.\, u \subset \rho \,.\, \exists x \ni (\breve{\rho}'x = u)\}$ ⟨51⟩

§129.

$$\breve{R} \epsilon\, \mathrm{Nc}{\to}1 \,.\supset.\, \breve{R}N \in \mathrm{Nc}{\to}1 \,.\supset:\, R \in 1{\to}1 \,.\supset.\, \breve{R}NR \in \mathrm{Nc}{\to}1 \quad (\beta)$$

$$(18)\,.\,(\beta)\,.\supset:\, \breve{\rho}u = v \,.\, \rho v = u \,.\, R \in 1{\to}1 \,.\supset.\, \breve{R}NR \in \mathrm{Nc}{\to}1 \quad (173)$$

$$d\breve{R}b \,.\, bSc \,.\supset.\, d\breve{R}Sc :\supset: d\breve{R}b \,.\, bSc \,.\, cRe \,.\supset.\, d\breve{R}SRe : \quad (174)$$

$$\supset: d\breve{R}b \,.\, bSc \,.\, d \sim \breve{R}SRe \,.\supset.\, c \sim Re : \quad (\alpha)$$

$$\supset: d\breve{R}b \,.\, bSc \,.\, \sim\exists x \ni (d\breve{R}SRx) \,.\supset.\, c \sim Re : \quad (\beta)$$

$$\supset: d\breve{R}b \,.\, bSc \,.\, \sim\exists x \ni (d\breve{R}SRx) \,.\, cRe \,.\supset.\, e \sim\in u : \quad (\gamma)$$

$$\supset: d\breve{R}b \,.\, bSc \,.\, \sim\exists x \ni (d\breve{R}SRx) \,.\supset:\, cRy \,.\supset_y.\, y \sim\in u :. \quad (\delta)$$

$$(\delta)\,.\,(8)\,.\supset:.\, R \in \mathrm{Nc}{\to}1 \,.\breve{\sigma}'m \subset \rho \,.\, \breve{\rho}(\breve{\sigma}'m) \subset u \,.$$
$$d\breve{R}b \,.\, bSc \,.\, \sim\exists x \ni (d\breve{R}SRx) \,.\supset.\, c \sim\in \breve{\sigma}'m \quad (\epsilon)$$

$$0N'b \,.\, bNc \,.\supset.\, 0N'c :$$

$$\supset: 0N'b \,.\, bNc \,.\supset.\, c\breve{N}'0 :$$

$$\supset: c \sim \breve{N}'0 \,.\, bNc \,.\supset.\, 0 \sim N'b : \quad (\eta)$$

$$(\epsilon)\,.\,(\eta)\,.\supset:.\, R \in \mathrm{Nc}{\to}1 \,.\, \breve{\nu}'0 \subset \rho \,.\, \breve{\rho}(\breve{\nu}'0) \subset u \,.\, d\breve{R}b \,.$$
$$\sim\exists x \ni (d\breve{R}NRx) \,.\, bNc \,.\supset.\, 0 \sim N'b :. \quad (\theta)$$

$$\supset:.\, R \in \mathrm{Nc}{\to}1 \,.\, \breve{\nu}'0 \subset \rho \,.\, \breve{\rho}(\breve{\nu}'0) \subset u \,.\, d\breve{R}b \,.$$
$$\sim\exists x \ni (d\breve{R}NRx) \,.\, 0N'b \,.\supset.\, b \sim \mathrm{Nc} :. \quad (\iota)$$

$$\supset:.\, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \,.\supset.\, b \sim\in \nu :. \quad (\kappa)$$

[51] The seven leaves from 15 to 21 are devoted to the proof of theorem 207, stated in §128, p. 160, and proved finally in §143 on p. 178. Frege describes this theorem as: "Wenn Endlos die Anzahl eines Begriffes ist, so koennen die unter diesen Begriff fallenden Gegenstände in eine unverzweigte Reihe geordnet werden, die mit einem bestimmten Gegenstande anfängt und, ohne in sich zurückzukehren, endlos fortläuft" [When *Endlos* is the number of a concept, then the objects falling under this concept can be ordered in an unbranching series, which begins with a given object and proceeds endlessly, without returning]. Russell's version has it that if the cardinal number of u is α_0 then there is a many-one relation R such that the ancestral of R is included in "other", u is included in the range of R, and there is some x such that the u's are the objects in the range of the ancestral of R starting with x.

$\supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ b \in \nu \,.\supset .\, 0 \sim N'b :. \quad (\lambda)$

$(\lambda) \,.\, (156) \,.\supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \,.\supset .\, 0 \sim N'b \quad (\mu)$

$(\mu) \,.\, (22) \,.\supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \,.\supset .\, b \sim \breve{N}'0 :. \quad (\nu)$

$\supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \sim \exists x \ni (d\breve{R}NRx) \,.\supset : d\breve{R}y \,.$

$\supset_y .\, y \sim \breve{N}'0 :. \quad (\xi)$

$(\xi) \,.\, (8) \,.\supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \,.\supset .\, d \sim \in u \quad (175)$

Frege. ⟨fol.⟩ 16

§130. We have to prove

$\breve{R} \in \mathrm{Nc}{\to}1 \,.\, TQ \supset T \,.\supset .\, \breve{R}TR\breve{R}QR \supset \breve{R}TR$

§131.

$x\breve{P}m \,.\, mTc \,.\, cPa \,.\supset .\, x\breve{P}TPa :$

$\supset :. x\breve{P}m : bQy \,.\supset_y .\, mTy : bQc \,.\, cPa : \supset .\, x\breve{P}TPa :. \quad (\alpha)$

$\supset :. b = e \,.\, x\breve{P}m : eQy \,.\supset_y .\, mTy : bQc \,.\, cPa : \supset .\, x\breve{P}TPa :. \quad (\beta)$

$(\beta) \,.\, (78) \,.\supset :. P \in 1{\to}\mathrm{Nc} \,.\, d\breve{P}b \,.\, ePd \,.\, x\breve{P}m : eQy \,.$

$\supset_y .\, mTy : bQc \,.\, cPa : \supset .\, x\breve{P}TPa \quad (\gamma)$

$\supset :. P \in 1{\to}\mathrm{Nc} \,.\, x \sim \breve{P}TPa \,.\, ePd \,.\, x\breve{P}m : eQy \,.$

$\supset_y .\, mTy : cPa \,.\, bQc : \supset .\, d \sim \breve{P}b :.$

Hence by (15), after transformations,

$\breve{R} \in \mathrm{Nc}{\to}1 \,.\, TQ \supset T \,.\supset .\, \breve{R}TR\breve{R}QR \supset \breve{R}TR \quad (176)$

Next $x(\breve{P}QP)^N y \,.\, P \in 1{\to}\mathrm{Nc} \,.\supset .\, x\breve{P}Q^N Py \quad (177)$

§133.

$\breve{R} \in 1{\to}\mathrm{Nc} \,.\, u \subset \breve{\rho} \,.\, \rho u \subset \breve{\nu}'0 \,.\, x\breve{R}NRx \,.\supset .\, x \sim \in u \quad (178)$

§134. We have to prove our series coextensive with u. Proof follows.

§135. $y \sim \in u \,.\, y = a \,.\supset .\, a \sim \in u :$

$\supset : y \sim \in u \,.\, P \in \mathrm{Nc}{\to}1 \,.\, nPy \,.\, nPa \,.\supset .\, a \sim \in u : \quad (\alpha)$

$\supset :. y \sim \in u \,.\, P \in \mathrm{Nc}{\to}1 \,.\, nPy \,.\supset : nPx \,.\supset_x .\, xPu :. \quad (\beta)$

$\supset :. P \in \mathrm{Nc}{\to}1 \,.\, v \subset \pi \,.\, \breve{\pi}v \subset u \,.\, y \sim \in u \,.\, nPy \,.$

$\supset .\, n \sim \in v \quad \cancel{(\gamma)}\ (179)$

$\cancel{(\gamma) \,.\, (18) \,.\supset .}\ mR'n \,.\supset .\, n\breve{R}'m :$

$\supset : a = m \,.\, aR'n \,.\supset .\, n\breve{R}'m : \quad (\alpha)$

$\supset : \breve{P} \in \mathrm{Nc}{\to}1 \,.\, x\breve{P}a \,.\, x\breve{P}m \,.\, aR'n \,.\supset .\, n\breve{R}'m : \quad (\beta)$

$\supset : \breve{P} \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, n \sim R'm \,.\, x\breve{P}m \,.\, aR'n \,.\supset.\, x \sim \breve{P}a :$ (γ)

$\supset :. \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \supset : yR'n \,.\supset_y.\, x \sim \breve{P}y :.$ (δ)

$\supset : \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \supset .\, x \sim (\breve{P}R')n$ (ϵ)

Frege. ⟨fol.⟩ 17

§135 continued.

$(\epsilon) \,.\, (179) \,.\supset :. \breve{P} \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, \breve{\rho}'m \subset \pi \,.$

$\breve{\pi}(\breve{\rho}'m) \subset u \,.\, y \sim \epsilon\, u \,.\, x\breve{P}m \,.\, nPy \,.\supset.\, \sim x\breve{P}R'n :.$ (ζ)

$\supset :: \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots :. \supset : zPy \,.\supset_z.\, \sim x\breve{P}R'z ::$ (η)

$\supset : \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \supset .\, \sim x\breve{P}R'Py$ (θ)

$\supset : \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \, x\breve{P}R'Py \,.\, x\breve{P}m \,.\supset.\, y \,\epsilon\, u$ (ι)

$(\iota) \,.\, (22) \,.\supset :. \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \, mPx \,.\supset.\, y \,\epsilon\, u$ (κ)

$aR'e \,.\, eRm \,.\supset.\, aR'm :$

$\supset : xR'y \,.\, yRz \,.\supset_{x,y,z}.\, xR'z :$ (λ)

$(\lambda) \,.\, (176) \,.\supset : P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\supset.\, \breve{P}R'P\breve{P}RP \supset \breve{P}R'P$ (μ)

$(\mu) \,.\, (144) \,.\supset : x(\breve{P}RP)'y \,.\, P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, x\breve{P}R'Px \,.\supset.\, x\breve{P}R'Py$ (ν)

$\nu \,.\, 174 \,.\supset : x(\breve{P}RP)'y \,.\, P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, x\breve{P}m \,.\, mR'm \,.\, mPx \,.$

$\supset .\, x\breve{P}R'Py :$ (ξ)

$\xi \,.\, 22 \,.\, 140 \,.\supset : x(\breve{P}RP)'y \,.\, P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, mPx \,.\supset.\, x\breve{P}R'Py$ (180)

$\kappa \,.\, 180 \,.\supset : P \epsilon 1{\rightarrow}1 \,.\, \rho'm \subset \pi \,.\, \breve{\pi}(\rho'm) \subset u \,.\, mPx \,.\, x(\breve{P}RP)'y \,.$

$\supset .\, y \,\epsilon\, u :$ (α)

$\supset : P \epsilon 1{\rightarrow}1 \,.\, \rho'm \subset \pi \,.\, \breve{\pi}(\rho'm) = u \,.\, \pi u = \rho'm \,.\, mPx \,.$

$x(\breve{P}RP)'y \,.\supset.\, y \,\epsilon\, u$ (181)

§137. $e = x \,.\supset.\, x(\breve{P}RP)'e :$

$\supset : P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, mPx \,.\, mPe \,.\supset.\, x(\breve{P}RP)'e :$ (α)

$\supset :. P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, mPx \,.\supset : mPy \,.\supset_y.\, x(\breve{P}RP)'y :.$ (β)

$\supset :. P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, v \subset \pi \,.\, \breve{\pi}v \subset u \,.\, mPx \,.\supset : mPy \,.$

$\supset_y .\, x(\breve{P}RP)'y$ (182)

$d \sim \epsilon\, \pi \,.\supset.\, d \sim Pa :$

$\supset :. d \sim \epsilon\, \pi \,.\supset : dPy \,.\supset_y.\, y \sim \epsilon\, u :.$ (β)

$\supset :. P \epsilon \mathrm{Nc}{\rightarrow}1 \,.\, v \subset \pi \,.\, \breve{\pi}v \subset u \,.\, d \sim \epsilon\, \pi \,.\supset.\, d \sim \epsilon\, v$ (183)

Frege. ⟨fol.⟩ 18

§137. continued.

$174 \,.\, \supset :.\, e\breve{P}d \,.\, dRa \,.\, aPc \,.\, \supset \,.\, e\breve{P}RPc :.$

$\supset :.\, dPe \,.\, dRa \,.\, aPc \,.\, \supset \,.\, e\breve{P}RPc :.$ (184)

$184 \,.\, 137 \,.\, \supset :.\, x(\breve{P}RP)'e \,.\, e\breve{P}d \,.\, dRa \,.\, aPc \,.\, \supset \,.\, x(\breve{P}RP)'c :.$ (α)

$\supset :.\, dPy \,.\, \supset_y \,.\, x(\breve{P}RP)'y : dPe \,.\, dRa \,.\, aPc : \supset \,.\, x(\breve{P}RP)'e :.$ (β)

$\supset :.\, \ldots\ldots\ldots \quad x \sim (\breve{P}RP)'c \,.\, dRa \,.\, aPc : \supset \,.\, d \sim Pe :.$ (γ)

$\supset :.\, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad .\, \supset \,.\, d \sim \epsilon\, \pi :.$ (185)

$\supset :.\, P \,\epsilon\, \mathrm{Nc}{\to}1 \,.\, \breve{\rho}'m \subset \pi \,.\, \breve{\pi}(\breve{\rho}'m) \subset u : dPy \,.\, \supset_y$

$.\, x(\breve{P}RP)'y : x \sim (\breve{P}RP)'c \,.\, dRa \,.\, aPc : \supset \,.\, d \sim \breve{R}'m :.$ (α)

$\supset :.\, P \,\epsilon\, \mathrm{Nc}{\to}1 \,.\, \breve{\rho}'m \subset \pi \,.\, \breve{\pi}(\breve{\rho}'m) \subset u : dPy \,.\, \supset_y \,.\, x(\breve{P}RP)'y$

$: mR'd \,.\, dRa \,.\, aPc : \supset \,.\, x(\breve{P}RP)'c :.$ (γ)

$\supset :.\, mR'n \,.\, \ldots\ldots\ldots\ldots\ldots\ldots \quad : mPy \,.\, \supset_y \,.\, x(\breve{P}RP)'y : \supset : nPz$

$.\, \supset_z \,.\, x(\breve{P}RP)'z :.$ (ϵ)

$(\epsilon) \,.\, 182 \,.\, \supset :.\, \ldots\ldots\ldots \quad .\, mPx \,.\, \supset : nPy \,.\, \supset_y \,.\, x(\breve{P}RP)'y :.$ (ζ)

$\supset :.\, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad nPy \,.\, \supset \,.\, x(\breve{P}RP)'y :.$ (η)

$\supset :.\, n\breve{R}'m \,.\, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad y\breve{P}n \,.\, \supset \,.\, \ldots :.$ (θ)

$\supset :.\, x \sim (\breve{P}RP)'y \,.\, \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad .\, \supset \,.\, n \sim \breve{R}'m :.$ (ι)

$\supset :.\, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \supset : y\breve{P}z \,.\, \supset_z \,.\, z \sim \breve{R}'m :.$ (κ)

$(\kappa) \,.\, 8 \,.\, \supset :.\, P \,\epsilon\, 1{\to}1 \,.\, \breve{\rho}'m = \pi u \,.\, \breve{\pi}(\breve{\rho}'m) = u \,.\, mPx \,.\, y \,\epsilon\, u \,.$

$\supset \,.\, x(\breve{P}RP)'y$ (186)

Frege. ⟨fol.⟩ 19

§138. We wish now to restrict the field of $\breve{P}NP$ to u

$.\, xRy \,.\, y \,\epsilon\, u \,.\, = \,.\, x\, {}^{u}Ry$ Df ⟨52⟩

§139. $x\,{}^{u}Ry \,.\, \supset \,.\, y \,\epsilon\, u \,.\, xRy :$ (187)

$\supset : x\,{}^{u}Ry \,.\, \supset \,.\, xRy :$ (188)

$\supset : R \,\epsilon\, \mathrm{Nc}{\to}1 \,.\, eRd \,.\, e\,{}^{u}Ra \,.\, \supset \,.\, d = a :$ (α)

$\supset : R \,\epsilon\, \mathrm{Nc}{\to}1 \,.\, e\,{}^{u}Rd \,.\, e\,{}^{u}Ra \,.\, \supset \,.\, d = a :$ (β)

[52] Frege's notation is $u \simeq q$, where q is the relation and u the restriction.

$\supset : R \in \mathrm{Nc}\!\rightarrow\!1 \,.\, \supset .\, {}^{u}R \in \mathrm{Nc}\!\rightarrow\!1$ (189)

$189 \,.\, \supset : \breve{P}NP \in \mathrm{Nc}\!\rightarrow\!1 \,.\, \supset .\, {}^{u}(\breve{P}NP) \in \mathrm{Nc}\!\rightarrow\!1$ (α)

$(\alpha) \,.\, 173 \,.\, \supset : P \in 1\!\rightarrow\!1 \,.\, u = \breve{\rho}v \,.\, v = \rho u \,.\, \supset .\, {}^{u}(\breve{P}NP) \in \mathrm{Nc}\!\rightarrow\!1$ (190)

$187 \,.\, \supset : d\,{}^{u}Ry \,.\, \supset .\, y \in u \,.\, dRy :$

$\supset : d\,{}^{u}Ry \,.\, \supset .\, y \in u$ (191)

$\supset : y \sim\in u \,.\, \supset .\, d \sim {}^{u}Ry$ (192)

$\supset : y \sim\in u \,.\, \supset .\, y \sim\in {}^{u}\breve{\rho}$ (α)

$\alpha \,.\, 125 \,.\quad \supset : y \sim\in u \,.\, \supset .\, x \sim {}^{u}R^{N}y :$ (193)

$133 \,.\, 188 \,.\, \supset .\, xR^{N}d \,.\, d\,{}^{u}Ra \,.\, \supset .\, xR^{N}a :$ (α)

$\supset : xR^{N}y \,.\, y\,{}^{u}Rz \,.\, \supset_{y,\,z} .\, xR^{N}z :$ (β)

$\beta \,.\, 123 \,.\, \supset :.\, x\,{}^{u}R^{N}y : x\,{}^{u}Rz \,.\, \supset_{z} .\, xR^{N}z : \supset .\, xR^{N}y$ (γ)

$131 \,.\, 188 \,.\, \supset .\, {}^{u}R \supset R^{N}$ (ϵ)

$\gamma \,.\, \epsilon \,.\, \supset .\, {}^{u}R^{N} \supset R^{N}$ (194)

$194 \,.\, \supset .\, {}^{u}R^{N} \cap 1' \supset R^{N} \cap 1' \,.$

$\supset .\, \sim R^{N} \cup 0' \supset \sim ({}^{u}R^{N} \cap 1')$ (195)

$178 \,.\, 195 \,.\, \supset : \breve{P} \in \mathrm{Nc}\!\rightarrow\!1 \,.\, u \subset \breve{\pi} \,.\, \pi u \subset \breve{\nu}'0 \,.$

$\supset .\, {}^{u}(\breve{P}NP)^{N} \supset 0'$ (196)

Frege. ⟨fol.⟩ 20

§140. We have to prove ${}^{u}(\breve{P}NP)$ generates an endless series.

§141. $a \in u \,.\, dRa \,.\, \supset .\, d\,{}^{u}Ra$ (197)

$137 \,.\, 181 \,.\, \supset :.\, P \in 1\!\rightarrow\!1 \,.\, \pi u = \breve{\nu}'0 \,.\, \breve{\pi}(\breve{\nu}'0) = u \,.\, 0Px \,.$

$x(\breve{P}NP)'d \,.\, d\breve{P}NPa \,.\, \supset .\, a \in u :.$ (α)

$\alpha \,.\, 197 \,.\, \supset :.\, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \,.\, \supset .\, d\,{}^{u}(\breve{P}NP)a$ (198)

$198 \,.\, 186 \,.\, \supset :.\, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad d \in u \ldots \supset \ldots$ (α)

$\supset .$ etc. $\supset :.\, \ldots \ldots \ldots \quad \exists u \wedge z \ni (y \ni z\breve{P}NPy = \Lambda) \,.\, \supset .\, 0 \sim Px$ (199)

§143.

$130 \,.\, \supset .\, xR'y \,.\, y \sim= x \,.\, \supset .\, xR^{N}y$ (200)

$136 \,.\, 194 \,.\, 200 \,.\, 139 \,.\, \supset : x\,{}^{u}R'y \,.\, \supset .\, xR'y$ (201)

$201 \,.\, 181 \,.\, \cancel{130 \,.\, 135 \,.}\, \supset : P \in 1\!\rightarrow\!1 \,.\, \pi u = \breve{\nu}'0 \,.\, \breve{\pi}(\breve{\nu}'0) = u \,.$

$0Px \,.\, x({}^{u}\breve{P}NP)'y \,.\, \supset .\, y \in u$ (202)

$130 \,.\, 135 \,.\, \supset : F\{\text{truth of } aR'c\} = F(aR'c)$ (203)

$aR\breve{}'a$ (204)

$F(\mathrm{Nc}\ \breve{\nu}'0) = F(\alpha_o)$ (205)

Put $N_P = \breve{P}NP'$ Df

$137 \,.\, 198 \,.\, \supset : P \in 1\to 1 \,.\, \pi u = \breve{\nu}'0 \,.\, \breve{\pi}(\breve{\nu}'0) = u \,.\, 0Px \,.\, x\,{}^uN_P{}'d \,.$

$xN_P{}'d \,.\, dN_P\, a \,.\, \supset \,.\, xN_P{}'a :$ (α)

$\alpha \,.\, 152 \,.\, \supset : \ldots\ldots\ldots\ldots\ldots\ldots \quad xN_P{}'y \,.\, x\,{}^uN_P{}'x \,.\, \supset \,.\, x\,{}^uN_P{}'y$ (γ)

$\gamma \,.\, 140 \,.\, \supset : \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \supset \,.\, \ldots$ (δ)

$\delta \,.\, 186 \,.\, \supset : \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad y \in u \,.\, \supset \,.\, x\,{}^uN_P{}'y :$ (ϵ)

$\supset :. \; x\,{}^uN_P{}'y \,.\, \supset \,.\, y \in u : P \in 1\to 1 \,.\, \pi u = \breve{\nu}'0 \,.\, \breve{\pi}(\breve{\nu}'0) = u \,.$

$0Px : \supset : y \in u \,.\equiv .\, x\,{}^uN_P{}'y$ (ζ)

$\zeta \,.\, 202 \,.\, \supset :. \; P \in 1\to 1 \,.\, \pi u = \breve{\nu}'0 \,.\, \breve{\pi}(\breve{\nu}'0) = u \,.\, 0Px \,.$

$\supset : y \in u \,.\equiv .\, x\,{}^uN_P{}'y$ (η)

$\eta \,.\, 203 \,.\, 164 \,.\, \supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \supset \,.\, u = {}^u\breve{\nu}_P{}'x :.$ (λ)

Frege. ⟨fol.⟩ 21

§143 continued.

$\lambda \,.\, \supset :. \; P \in 1\to 1 \,.\, \pi u = \breve{\nu}'0 \,.\, \breve{\pi}(\breve{\nu}'0) = u \,.\, \sim \exists y \ni (y \in u \,.\, u \,.\, \breve{\nu}_P{}'x) \,.$

$\supset \,.\, 0 \sim Px :.$ (ν)

$\supset :. \ldots\ldots : R \in \mathrm{Nc}\to 1 \,.\, R^N \supset 0' \,.\, u \subset \breve{\rho} \,.\, \supset_R \,.\, \sim \exists x \ni (u = \breve{\rho}'x) \supset :$

${}^uN_P \in \mathrm{Nc}\to 1 \,.\, {}^uN_P{}^N \supset 0' \,.\, u \subset {}^u\breve{\nu}_P \,.\, \supset \,.\, 0 \sim Px :.$ (ξ)

$\xi \,.\, 190 \,.\, 196 \,.\, \supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; : u \subset {}^u\breve{\nu}_P \,.\, \supset \,.\, 0 \sim Px$ (o)

$o \,.\, 199 \,. \quad \supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad : \supset \,.\, 0 \sim Px :.$ (π)

$\supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \supset \,.\, 0 \sim \in \pi$ (ρ)

$\rho \,.\, 183 \,.\, \supset :. \supset \,.\, 0 \sim \breve{N}'0 :.$ (σ)

$\sigma \,.\, 204 \,.\, \supset :. \; R \in \mathrm{Nc}\to 1 \,.\, R^N \supset 0' \,.\, u \subset \breve{\rho} \,.\, \supset_R \,.\, \sim \exists x \ni (u = \breve{\rho}'x) :$

$P \in 1\to\mathrm{Nc} \,.\, u \subset \breve{\pi} \,.\, \pi u \subset \breve{\nu}'0 :$

$\supset_P \,.\, \sim (P \in \mathrm{Nc}\to 1 \,.\, \breve{\nu}'0 \subset \pi \,.\, \breve{\pi}(\breve{\nu}'0) \subset u) :.$ (ϕ)

$\phi \,.\, 49 \,.\, \supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots : \supset \,.\, \mathrm{Nc}\ \breve{\nu}'0 \sim = \mathrm{Nc}\ u$ (χ)

$\chi \,.\, 205 \,.\, \supset :. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \supset \,.\, \alpha_o \sim = \mathrm{Nc}\ u :.$ (206)

$\supset :. \; \alpha_o = \mathrm{Nc}\ u \,.\, \supset \,.\, \exists\ \mathrm{Nc}\to 1 \wedge R \ni \{R^N \supset 0' \,.\, u \subset \breve{\rho} \,.$

$\exists x \ni (u = \breve{\rho}'x)\}$ (207)

Frege. ⟨fol.⟩ 22

§144. ⟨53⟩ We have next to prove the converse of (207). If P, R be two such relations (*i.e.* two ω's) we correlate $\iota\pi - \breve{\pi}$ and $\iota\rho - \breve{\rho}$, and if x, y are correlated, we correlate $\iota\breve{\pi}x$ and $\iota\breve{\rho}y$. We form a series of pairs, consisting of n and x_n, where x_n is the $n+1$th term of series. We define a pair $(x\,;y)$ as $R \ni (xRy)$. Also define

$$P \simeq Q = (\alpha, \beta) \ni [\exists(x, y, z, w) \ni (\alpha = (x\,;y) \,.\, zQy \,.\, \beta = (w\,;z) \,.\, wPx\}] \quad \text{Df}$$ ⟨54⟩

or [in my notation]

$$R_{xy}(P \simeq Q)R_{wz} \,.=.\, zQy \,.\, wPx \qquad \text{Df}$$

or $R_{xy}(P \simeq Q)R_{zw} \,.=.\, zPx \,.\, wQy$ Df

Then the relation we want is $R_{ox}\,(N \simeq P)'\,R_{yz}$ [This, considered as a relation of y and z, makes z the $y+1$th term in a series beginning with x.]

We then have to prove

$$R \,\epsilon\, \text{Nc}{\rightarrow}1 \,.\, \breve{\rho}'x \subset \rho \,.\, N \simeq R = P \,.\supset.\, \breve{\nu}'0\; P'\,\breve{\rho}'x$$

[This is not an exact translation]

and $R \,\epsilon\, \text{Nc}{\rightarrow}1 \,.\, R \supset 0' \,.\supset.\, \breve{\rho}'x\breve{P}'\,\breve{\nu}'0$ [Again approximate]

Instead of (β) we prove the more general proposition

$$P, R \,\epsilon\, \text{Nc}{\rightarrow}1 : mP'y \,.\supset_y.\, y \sim P^N y : \breve{\rho}'x \subset \rho := P \simeq R = S : \supset .\, \breve{\pi}'m > S\breve{\rho}'x \quad \text{[Approximately]}$$

[The proof occupies 20 pages.]

Frege. ⟨230.030450 fol.⟩ 23 ⟨verso⟩ ⟨55⟩

§158. Here it is to be shown ~~if~~ $\exists\rho - \breve{\rho} \,.\, \exists\breve{\rho} - \rho \,.\, R \,\epsilon\, \text{Nc}{\rightarrow}1 \,.\, x \,\epsilon\, \rho \,.\supset.\, \breve{\rho}' \,\epsilon\, \text{Cls fin.}$

[This takes 23 pp.]

---

[53] This is theorem 263, the converse of 207.

[54] The notation is Frege's (§144, p. 179).

[55] This leaf completes the notes on *Gg* I. Apparently Russell misplaced or lost this leaf, which shows up as the verso of a leaf of notes on Meinong made nearly two years later. When he received Volume II Russell continued the notes with a new leaf foliated 23, repeating the last theorems of Volume I in the notation he was then using. The notation of the apparently lost folio 23 clearly matches that of the rest of the notes on Volume I.

§172. Here the converse of above is to be proved.
[This takes 14 pp.]

Frege. ⟨230.030420–F1 fol.⟩ 23_2

§§158, 172 prove
$\vdash : (\exists m, n, Q) . Q \in \mathrm{Nc}\to 1 . n \sim Q^N x . u = \overleftarrow{Q}*`m \cap \overrightarrow{Q}*`n . D$
$\equiv . u \in \cup` \overleftarrow{N}*`0$

Vol. II.
p. 1. [56] $\vdash : v \subset u . u \in \aleph_0 . \supset . v \in \aleph_0 \cup (\cup` \overleftarrow{N}*`0)$
Put $\cup` \overleftarrow{N}*`0 = \text{Cls induct}$. Then the proposition is
$\vdash : u \in \aleph_0 . \supset . \mathrm{Cls}`u \subset \aleph_0 \cup \text{Cls induct}$
p. 37. [57] $\vdash : u \in \text{Cls induct} . \supset . \mathrm{Cls}`u \subset \text{Cls induct}$
p. 44. [58] $\vdash : w \cap u = \Lambda . v \cap z = \Lambda . \mathrm{Nc}`v = \mathrm{Nc}`w . \mathrm{Nc}`z = \mathrm{Nc}`u .$
$\supset . \mathrm{Nc}`(v \cup z) = \mathrm{Nc}`(u \cup w)$
p. 58. [59] $\vdash : u \subset w . z \subset v . u \text{ sim } z . v - z \text{ sim } w - u .$
$\supset . v \text{ sim } w$

---

[60] p. 166. $*T = \ddot{P}\dot{Q}(P|T = Q)$ Df [assuming $T \llcorner P$ means $P|T$]
$\vdash . *T \in \mathrm{Nc}\to 1$
The above Df is useful for powers of T: for we have
$\vdash n \in \text{No fin} . \supset . T^n *T\, T^{n+1}$ and $\vdash . \overleftarrow{*T}`T$ = finite powers of T
$\vdash : P(*T*)T . Q(*T*)T . \supset . P|Q = Q|P$
(This is $\vdash : m, n \in \text{No fin} . \supset . T^m | T^n = T^n | T^m$)

---

[56] Russell here begins to identify theorems by page number. This is theorem 428 on p. 37. Russell now uses $v \subset u$ in the contemporary manner for inclusion of v in u, reversing the direction from the earlier notes. He now represents Frege's $0 \cap (\mathfrak{N}v \cap \cup`f)$ by $(\cup``0)$ which he in turn replaces by *Class induct*, or "inductive class".

[57] Part *N*, beginning at p. 37, is devoted to proving theorem 443 on p. 43.

[58] Part *Ξ* begins at p. 44 and ends at p. 58 with the proof of theorem 469.

[59] Part *O*, pp. 58–68, proves various "*Folgesätze*" (corollaries) including this, theorem 472 on p. 61.

[60] Pp. 69 to 162 are Part *I* "Kritik der Lehren von den Irrationalzahlen" [Critique of theories of irrational numbers] of the volume's division III "Die reellen Zahlen" [The real numbers]. It is a discussion of the theories of Cantor, Heine, Thomae, Dedekind, Weierstrass, and others. Russell begins the notes where the formal development resumes; p. 163 starts subdivision 2, "Die Grössenlehre" [The theory of measurement]. See Dummett, Chaps. 19–21, for a discussion of Frege's criticisms.

p. 169. $\text{ð}‘s = \grave{R}\{(\exists Q) : Q \in S : R = Q \,.\, \mathbf{v} \,.\, R = \breve{Q} \,.\, \mathbf{v} \,.\, R = Q|\breve{Q}\}$ Df

Frege calls $\overleftarrow{*T}‘T$ a ~~Positivklasse~~ *Positivalklasse*, ⟨61⟩ and $\text{ð}‘\overleftarrow{*T}‘T$ the corresponding *Grössengebiet*. (The case of $R = \breve{Q}$ gives negative terms, and $R = Q|\breve{Q}$ gives zero.)

p. 171. $\text{ɟ}‘s = :.\, s \subset 1 \rightarrow 1 : P, Q \in s \,.\, \supset_{P,\,Q} .\, P|\breve{P} \sim\in s \,.$
$$D‘P = \breve{D}‘Q \,.\, P|\breve{Q}, \breve{P}|Q \in \text{ð}‘s \,.\, P|Q \in s \quad \text{Df}$$

[Here "ɟ" is an *f* upside down]. $\grave{s}(\text{ɟ}‘s)$ is the class of *Positivalklassen*.

p. 176. $\vdash : \text{ɟ}‘s \,.\, P, Q \in s \,.\, \supset \,.\, P|\breve{P} = Q|\breve{Q}$

p. 174. $\vdash : \text{ɟ}‘s \,.\, P, Q \in s \,.\, \supset \,.\, P|\breve{Q} \in \text{ð}‘s$ [Proposition 526] ⟨62⟩

Frege. Gg. Vol. II. ⟨fol.⟩ 24

p. 180 $\vdash : P \in \text{ð}‘s \,.\, \breve{P} \sim\in s \,.\, \text{ɟ}‘s \,.\, R \in s \,.\, P \sim\in s \,.\, \supset \,.\, P = R|\breve{R}$ ⟨63⟩

p. 181 $\vdash : \text{ɟ}‘s \,.\, P, Q \in s \,.\, \supset \,.\, Q|P|\breve{P} = Q$

$\vdash : \text{ɟ}‘s \,.\, P, Q \in s \,.\, \supset \,.\, P|\breve{P}|Q = Q$ ⟨64⟩

p. 185 $\vdash : \text{ɟ}‘s \,.\, Q \in s \,.\, Q|P = Q \,.\, \supset \,.\, P \sim\in s$

We may put
$$P\, s\rangle\, Q \,.=.\, \text{ɟ}‘s \,.\, P, Q, P|\breve{Q} \in s \qquad \text{Df} \ ⟨65⟩$$

p. 187 $s\text{Ⅎ}u \,.=\grave{R}\{P \in s \,.\, R|\breve{P} \in s \,.\, \supset_P .\, P \in u\}$ Df

$s\text{ł}u \,.=\grave{R}\{P \in s \,.\, P|\breve{R} \in s \,.\, \supset_P .\, P \sim\in (s\text{Ⅎ}u) :$
$$R \in (s\text{Ⅎ}u) \,.\, R \in s \,.\, \text{ɟ}‘s\} \quad \text{Df}$$

i.e.

$s\text{ł}u \,.=\grave{R}\{\text{ɟ}‘s\, R \in s \,.\, R \in (s\text{Ⅎ}u) : P \in s \,.\, P|\breve{R} \in s \,.$
$$\supset_P .\, P \sim\in (s\text{Ⅎ}u)\} \quad \text{Df} \ ⟨66⟩$$

i.e. all relations less than R belong to u, but no relations greater than R do so.

p. 189. $\vdash : (s\text{ł}u) \in 0 \cup 1.$

p. 190. $\grave{p}‘s \,.=::\, s\text{ł}u = \Lambda \,.\, \supset_u : x \in (s\text{Ⅎ}u) \cap s \,.\, \supset_x .\, s \subset u :.$

---

[61] Frege coins the term "*Positivalklasse*", first used in the title on p. 168 and defined at p. 171 to distinguish the notion from "*Positivklasse*". See Dummett, pp. 277–8.

[62] Theorems 558 and 559 on p. 179 establish this result.

[63] Theorem 561.

[64] Theorems 562 and 565.

[65] Theorem 585. "$P\, s\rangle\, Q$" is Russell's notation and it is not used below.

[66] Frege's definitions Ω and AA.

$$R \in s \,.\, \supset_R .\, P|\breve{R} \sim\in s :\supset_P .\, P \sim\in s : \jmath`s \quad \text{Df}$$

i.e. $'p`s \,.=::\, \jmath`s : P \in s \,.\supset_P .\, (\exists R) \,.\, R \in s \,.\, P|\breve{R} \in s : s\ell u = \Lambda \,.$

$$\exists`s \cap (s\dashv u) \,.\supset_u .\, s \subset u \quad \text{Df}$$

i.e. $'p`s \,.=:\, \jmath`s : P \in s \,.\supset_P .\, (\exists R) \,.\, R \in s \,.\, P|\breve{R} \in s : \exists`s - u \,.$

$$\exists`s \cap (s\dashv u) \,.\supset .\, \exists`(s\ell u) \quad \text{Df} \ \langle 67\rangle$$

p. 191ff. Axiom of Archimedes.

~~$sGP \,.=\, \hat{\hat{x}}\hat{y}[(\exists T) \,.\, x(*P^*)T \,.\, y|\breve{T} \sim\in s]$ Df~~

$G_s`P = \hat{\hat{R}}\hat{S}[(\exists T) \,.\, R(*P^*)T \,.\, S|\breve{T} \sim\in s]$ Df

$\vdash :\, 'p`s \,.\, P, Q \in s \,.\supset .\, Q(G_s`P)P$

[This is axiom of Archimedes.] ⟨68⟩

Frege, *Gg. Vol. II.* ⟨fol.⟩ 25

p. 195 ⟨69⟩ $\vdash : \jmath`s \,.\, P \in s \,.\, Q \in s\ell\ \overrightarrow{G_s`P`}P \,.\supset .\, Q|P \in (s\dashv\ \overrightarrow{P|\breve{P}`}G_s`P)$

p. 204 $\vdash :\, 'p`s \,.\, P, Q \in s \,.\supset .\, Q|P = P|Q$

p. 207 $\vdash :\, 'p`s \,.\, P, Q, Q|\breve{P} \in s \,.\supset .\, \breve{P}|Q \in s$

p. 209 $\vdash :.\, \jmath`s : Q, R \in s : P \in s \,.\supset_P .\, P|Q|\breve{R}, P|R|\breve{Q} \in s :\supset .\, Q = R$

p. 211 $\vdash : \jmath`s \,.\, 'p`s \,.\, P, Q, R \in s \,.\, P|\breve{R}, Q|\breve{R} \in s \,.$

$$\supset .\, \dot{R}^2|Q|P|\breve{Q}|\breve{P} \in s$$

p. 230 $\vdash : \jmath`s \,.\, 'p`s \,.\, P, Q \in s : R \in s \,.\, P|\breve{R}, Q|\breve{R} \in s \,.$

$$\supset_R .\, S|\breve{R}|\breve{R} \sim\in s :\supset .\, S \sim\in s$$

or $\vdash : \jmath`s \,.\, 'p`s \,.\, P, Q, S \in s \,.\supset : (\exists R) \,.\, P|\breve{R}, Q|\breve{R}, S|\breve{R}|\breve{R} \in s$

p. 239. $\vdash :\, 'p`s \,.\, P, Q \in \eth`s \,.\supset .\, P|Q = Q|P$

---

[67] Theorem 602 says that *su* has one member, as does this.
[68] Russell approximates Frege's symbol.
[69] The theorems on each page are:
p. 195: Theorem 619.
p. 204: Theorem 674 (the main result of Part E).
p. 207: Theorem 641, lemma b to theorem 674.
p. 209: Theorem 644, lemma c.
p. 211: Theorem 666, lemma d.
p. 230: Theorem 673, lemma e.
p. 239: Theorem 689 of Part Z, the last result in *Gg* 2.

Frege Gg. I. p. 61. ⟨230.030420–F1 unfoliated⟩

Grundgesetze.

$\vdash : p . \supset . q \supset p$ Pp
$\vdash : p \supset p$ Pp
$\vdash : (x) . f!x . \supset . f!y$ Pp
$\vdash : (\phi) . F!(\phi!\hat{x}) . \supset . F!(f!\hat{x})$ Pp
$\vdash : g!(a = b) . \supset : g!\{\phi!b . \supset_{\phi} . \phi!a\}$ Pp
$\vdash : p \equiv q . \vee . p \equiv \sim q$ Pp
$\vdash : \grave{x}(f!x) = \grave{y}(g!y) . \equiv : (z) : f!z . \equiv . g!z$ Pp
$\vdash . x = \text{ɿ}`\grave{y}(x = y)$ Pp

Regeln.

$\vdash . -(-\xi) \equiv -\xi$ Pp
$\vdash :. p . \supset . q \supset \text{ɿ} r : \supset : q . \supset . p \supset r$
$\vdash : p \supset q . \supset . \sim q \supset \sim p$
$\vdash : p . \supset . p \supset q : \supset . p \supset q$
$\vdash . (C\mathfrak{j}y) . \supset \vdash . (x)(C\mathfrak{j}x)$
$\vdash$. Therefore
$\vdash : p \supset q . q \supset r . \supset . p \supset r$
$\vdash : p \supset q . \sim p \supset q . \supset . q$

VII. OTHER NOTES ON "GRUNDGESETZE", VOL. II

Frege Vol. II. ⟨220.148001c⟩ ⟨p.⟩ 1

$\sim (n, m)$: $Q \epsilon \mathrm{Nc}{\to}1 . A = m \to n . n \sim Q^N n . mQ^* \epsilon .$
$\epsilon \sim Q^* n .$
$\epsilon \ni \{Q \epsilon \mathrm{Nc}{\to}1 . \exists(m, n) \ni [A = m \to n - n \sim Q^N n . mQ^* \epsilon .$
$\epsilon\, Q^* n]\} = A\ell Q$
Thus $A\ell Q$ consists of all terms of the Q-series from m to n both inclusive.

$xR^* d . a\, 0'e - dRa - e \,\epsilon\, \breve{\rho}\, x \cap \rho a . \supset . e \,\epsilon\, \breve{\rho} x \cap \rho d$

Mem. If documentary view of history right, ought to do all possible sums in Arithmetic. ⟨70⟩

---

p. 166 $*T = ℩\,\mathrm{Rel} \cap S \ni [P(S)Q \,.\!\equiv\!.\, TQ = P]$ Df

$P(*T)Q \,.\!\equiv\!.\, TQ = P$ [or $TP = Q$?]

$*T \in 1\rightarrow\mathrm{Nc}$

$T(*T^*)Q \,.\, T(*T^*)P \,.\supset.\, PQ = QP$

p. 169 $.\, \eth s = R \ni [Q \in s \,.\supset_Q.\, R \sim= \breve{Q} \,.\, R \sim= Q\breve{Q} :\supset.\, R \in s\,]$

$= R \ni [R \in s \,.\vee.\, \exists\, s \cap Q \ni \{R = \breve{Q} \,.\vee.\, R = Q\breve{Q}\}]$ Df

⟨p.⟩ 2

p. 171.

$\cancel{\sim(P \in s) \,.\vee \sim(Q \in s) \,.\vee.\, \sim P = (\sim \breve{Q})\,\breve{P} \,.\, Q \in \eth s\,.}$

$\text{ɟ}s = :.\, s \subset 1\rightarrow 1 : P, P' \in s \,.\supset_{P,\,P'}.\, P\breve{P} \sim\in s \,.\, \pi = \breve{\pi}' \,.\, \breve{P}P',$

$\breve{P}P' \in \eth\, s \,.\, PP' \in s$ Df

$\text{ɟ}s$ means that s is a *Positivalklasse*.

p. 176. $\text{ɟ}s \,.\, P, P' \in s \,.\supset.\, P\breve{P} = P'\breve{P}'$

p. 180. $P \in \eth\, s \,.\, \breve{P} \sim\in s \,.\, \text{ɟ}s \,.\, R \in s \,.\, P \sim\in s \,.\supset.\, P = R\breve{R}$

p. 181. $\text{ɟ}s \,.\, P, Q \in s \,.\supset.\, Q = QP\breve{P} = P\breve{P}Q$

p. 185. $P\breve{Q} \in s \,.\!\equiv\!.\, P > Q$ in s

p. 187. $s\text{Ⅎ}u \,.= R \ni \{P \in s \,.\, R\breve{P} \in s \,.\supset_P.\, P \in u\}$ Df

[This is segment defined by u. B.R.]

[i.e. it is the class of R's such that all lesser relations belong to u.]

$s\text{ł}u = P \ni \{E \in s \,.\, E\breve{P} \in s \,.\supset_E.\, \exists Q \ni (Q \in S \,.\, E\breve{Q} \in s \,.\, Q \sim\in u) :$

$Q \in s \,.\, P\breve{Q} \in s \,.\supset_Q.\, Q \in u : P \in s \,.\, \text{ɟ}s\}$ Df

This defines class of limits.

p. 189. $s\text{ł}u \in 0 \cup 1$

⟨p.⟩ 3

---

[70] This appears to be a note to himself referring to "On History" (*Papers* 12: 73–82), which Russell reports working on in March 1903, when Vol. II of *Gg* would have recently arrived. Russell had discussed the documentary view of history with George Trevelyan the previous month (Journal, *Papers* 12: 19) and included a discussion of it in the essay.

p. 190.

$$\text{'}ps \,.\, = \,:.\, \exists s - u \,.\, \exists s \cap s\text{Ⅎ}u \,.\, \supset_u .\, \exists s\text{ł}u : A \in s \,.$$
$$\supset_A .\, \exists s \cap E \ni (A\breve{E} \in s) : \text{ɟ}s \quad \text{Df}$$

$\text{'}ps$ means: s is a *Positivklasse*. In this kind of class there is no minimum, and every segment has a limit.

p. 192. $Q(sgP)R \,.\, = \,.\, \exists T \ni \{R(*P*)T \,.\, Q\breve{T} \sim\in s\}$ Df

Thus $Q(sgP)P$ asserts "There is a finite multiple of P which is greater than Q."

(Axiom of Archimedes).

p. 204. $\text{'}ps \,.\, P, Q \in s \,.\supset.\, PQ = QP$

p. 209. $C, D \in s\ \text{ɟ}s \,.\, A \in s \,.\supset_A.\, AC\breve{D}, AD\breve{C} \in s : \supset .\, C = D$

p. 211. $B, P, Q, P\breve{B}, Q\breve{B} \in s \,.\, \text{'}ps \,.\supset.\, B^2QP\breve{Q}\breve{P} \in s$

p. 230. $\text{'}ps \,.\, P, Q, E \in s \,.\, P\breve{E}, Q\breve{E} \in s \,.\supset_E.\, A\breve{E}^2 \sim\in s : \supset .\, A \sim\in s$

p. 239. $\text{'}ps \,.\, P, Q \in \eth\text{‘}s \,.\supset.\, PQ = QP$

VIII. NOTES ON "GRUNDLAGEN"

⟨220.148001b⟩ ⟨fol.⟩ 1

119 pp Breslau 1884

Frege

Empiricism (Mill)—won't explain 0 and 1; [won't explain generality]. Won't explain big numbers. "If definition of each particular number affirmed a separate physical fact, one couldn't enough admire a man who can reckon with 9 figures." ⟨71⟩

Induction itself depends on arithmetic, through probability.

Synthetic à priori Note, "synthetic" has vague meaning. Most useful: "Not deducible from logic alone". In this sense, detailed proof that arithmetic *analytic*.

NC not a property of *things*: 1000 green leaves are each green, not each 1000. 2 books are 1 pair of books. Thus physical objects are not subjects of NC.

NC not *subjective* or object of *psychology* anymore than the North Sea. NC and colour equally objective, but not equally properties of sensible *objects*. Objective ≠ palpable. If NC were subjective, there would be

---

[71] *Grundlagen*, p. 10.

many 2's.
NC not a presentation.
NC not same as collection of objects which *has* a number; how about 0 and 1?

⟨fol.⟩ 2

Opinions on 1.
Is 1 a property of objects?
"1 man" seems like "wise man".
Taken this way, *everything* is one.
Yet *one* is opposed to *many.*
Numbers not obtained by *abstraction*: what shall we abstract from the moon to get 1? Or how get 0 this way?
NC is asserted of a *concept.*
"Venus has 0 moons" means "Venus's moons are a 0".
"Kaiser's carriage is drawn by 4 horses" means "Horses drawing etc. are a 4."
This removes an *ambiguity* of NC to be ascribed. E.g. book and pair of books.
Existence also is to be asserted of a concept: same as denial of 0.

⟨fol.⟩ 3

Numbers are *objects*, though not in space. [Wrong]
Definition of NC
Take e.g. set of parallel lines. What is meant by saying they all have the same *direction*?
Can *define* "direction of line *a*" as "all lines parallel to *a*". Similiarly "shape of triangle *ABC*" is "all triangles similar to *ABC*". *Principle of abstraction.* Two concepts "equinumerous" [similar] when 1→1 between terms under them.
Nc'*F* = extension of concept "equinumerous with *F*". Df
0 = Nc'(not ~~equal to~~ identical with itself) Df
1 = Nc'(identical with 0) Df
Then 1 follows 0 immediately.
Infinite NC'*s*

Nc‘(Nc fin) is infinite. ⟨72⟩
What Cantor calls “powers” are NC‘s.
[Observe with above definition of NC, no need of *counting*.]

⟨fol.⟩ 4

Hope to have made probable that arithmetical laws are analytic and therefore à priori, and arithmetic mere prolongation of logic.

Classes and Concepts. Classes must be defined by *intension*—even enumeration, which is only possible with *finite* classes, is really giving intension, i.e.
identical with *a* or with *b* or etc.

Finite and Infinite
NC refl
Nc induct

⟨verso of fol.⟩ 4 ⟨in ink⟩

$dUa \,.\, \supset_a . \, a \sim\epsilon\, \delta : \supset_d . \, d \sim\epsilon\, \gamma$
$d \,\epsilon\, \gamma \,.\, \supset_d \,: (\exists a) \,.\, dUa \,.\, a \,\epsilon\, \delta$

---

[72] The NC of the concept “finite cardinal number” is infinite.