

YABLO'S PARADOX AND RUSSELLIAN PROPOSITIONS

GREGORY LANDINI
Philosophy / U. of Iowa
Iowa City, IA 52242-1408, USA
GREGORY-LANDINI@UIOWA.EDU

Is self-reference necessary for the production of Liar paradoxes? Yablo has given an argument that self-reference is not necessary. He hopes to show that the indexical apparatus of self-reference of the traditional Liar paradox can be avoided by appealing to a list, a consecutive sequence, of sentences correlated one-one with natural numbers. Yablo opens his "Paradox without Self-Reference" (*Analysis*, 1993) with the assumption that there is a sequence such that:

$$S_n: "(\forall k)(k > n \rightarrow \neg \text{True} \ulcorner S_k \urcorner)"$$

Each sentence on Yablo's list is supposed to be correlated one-one with number n . Each sentence is supposed to say that for every natural number k greater than n , the k -th sentence on the list is not true. By comparing Yablo's construction to an analogous construction with early Russellian propositions, we show that Yablo has failed to generate a paradox.

I. VICIOUS CIRCULARITY IN YABLO'S PARADOX

Self-reference has been taken to be the source of paradoxes. Poincaré and Russell are commonly associated with this thesis. In an amusing passage, the mathematician Jourdain recalls something of the dialogue between them:

Nearly all mathematicians agreed that the way to solve these paradoxes was simply not to mention them; but there was some divergence of opinion as to how they were to be unmentioned. It was clearly unsatisfactory merely not to mention them. Thus Poincaré was apparently of the opinion that the best way of avoiding such awkward subjects was to mention that they were not to be mentioned. But [as Russell put it] "one might as well, in talking to a man with

a long nose, say: ‘When I speak of noses, I except such as are inordinately long’, which would not be a very successful effort to avoid a painful topic.”¹

Poincaré maintained that one should exclude the offending cases and thereby avoid “vicious circles” of self-reference which generate paradoxes. Russell objected:

We may illustrate this by what M. Poincaré says concerning Richard’s paradox. Having first put $E =$ “all numbers definable in a finite number of words” we arrive at a paradox, due, says M. Poincaré, to our having included a number only definable in a finite number of words by means of E . This vicious circle he proposes to avoid by defining E as “all numbers definable in a finite number of words without mentioning E ”. To the uninitiated, this definition looks more circular than ever.²

Russell held that some paradoxes such as those of classes require, in order to exclude offending cases without mentioning them, a “reconstruction of logical first principles”. Others, such as Richard’s, are to be dismissed because they involve confused and viciously circular notions of “definability”.³

Is self-reference necessary for paradoxes? The question is not well crafted. There are quite different notions of “self-reference” involved in paradoxes. The self-reference involved in the Liar “This sentence is false” is provided by an apparatus of indexicals. This apparatus is quite distinct from the self-reference involved in Russell’s early ontology of propositions (as mind- and language-independent states of affairs). This is an ontological self-reference owing to the existence of *general* propositions which contain an ontological analog of a quantifier. For example, the universal quantifier in a Russellian general proposition such as

every entity being self-identical

¹ Philip Jourdain, *The Philosophy of Mr. Bertrand Russell* (London: Allen & Unwin, 1918), p. 77. Russell is quoted from “Mathematical Logic as Based on the Theory of Types” (1908), *LK*, p. 63.

² “On ‘Insolubilia’ and Their Solution By Symbolic Logic”, in B. Russell, *Essays in Analysis*, ed. Douglas Lackey (London: Allen & Unwin, 1973), p. 196.

³ I believe that this is Russell’s view in 1906. See Gregory Landini, “Russell’s Separation of the Logical and Semantic Paradoxes”, *Revue internationale de philosophie* 58 (2004): 257–94 (issue titled “Russell en héritage / Le centenaire des *Principes*”, ed. Philippe de Rouilhan).

ranges over all entities, including the proposition *every entity being self-identical*. There is yet another apparatus of “self-reference”. This is the apparatus of impredicative comprehension which is the centerpiece of the quite consistent standard second-order logic of attributes. (Attributes in this theory are mind- and language-independent intensional entities that have only a predicable nature. Russell’s paradox of attributes is not formulable in this theory.) To take an easy example from Whitehead and Russell, consider the attribute ϕ such that

$$(\forall x)(\phi x \iff (\forall \psi)(Gy \rightarrow_y \psi y \cdot \rightarrow \cdot \psi x)).^4$$

An entity x exemplifies ϕ if and only if x exemplifies every property ψ of every great general, and these properties ψ include ϕ itself. This is the “self-reference” of impredicative comprehension, and it seems central to much of ordinary mathematical reasoning. But it has nothing whatever in common with the indexical or the ontological apparatus of self-reference. In fact, these apparatus (indexicals, ontological self-reference, impredicative comprehension) do not seem to have *anything* relevant in common except a loose association produced by confused thinking.

There are class-theoretical paradoxes—such as Cantor’s paradox of the greatest cardinal—that involve impredicative comprehension of classes (or attributes). Thus there are paradoxes that don’t involve indexical “self-reference” or ontological “self-reference”. Accordingly, “self-reference” (in some forms) is not *necessary* for paradox. Is self-reference *sufficient* for paradox? This also depends on what is meant by “self-reference”. In the hope of exonerating self-reference Yablo invokes the indexically self-referential and paradox free example: “So dear Lord, to Thee we raise, this our hymn of grateful praise.” But this certainly does not exonerate the *apparatus* of indexical self-reference. One cannot exonerate a practice of detonating thermonuclear bombs on the grounds that some are duds. Whether it be the indexical, the ontological, or the impredicative form, any *apparatus* of self-reference that can generate contradictions (in the context of an otherwise quite innocuous quantification theory) is objectionable.

In his paper Yablo endeavours to show that the indexical apparatus of self-reference of the traditional Liar paradox can be avoided by appealing

⁴ This is from *Principia Mathematica to *56* (Cambridge U. P., 1964), p. 56.

to a list, a consecutive sequence, of sentences correlated one-one with natural numbers. Yablo opens with the assumption that there is a consecutive sequence, or list, of sentences such that:

$$S_n: “(\forall k)(k > n \rightarrow \neg \text{True} \ulcorner S_k \urcorner)”^5$$

Each sentence on Yablo’s list is supposed to be correlated one-one with number n . Each sentence says that for every natural number k greater than n , there is a sentence S correlated with k such that $\ulcorner S \urcorner$ is not true.

To see how the derivation of the contradiction is supposed to go, assume $\text{True} \ulcorner S_n \urcorner$. By substitution of identity this yields

$$\text{True} “(\forall k)(k > n \rightarrow \neg \text{True} \ulcorner S_k \urcorner)”$$

By Tarski’s T -schema,⁶ we have:

$$(\forall k)(k > n \rightarrow \neg \text{True} \ulcorner S_k \urcorner)$$

By universal instantiation, we have:

$$n+1 > n \rightarrow \neg \text{True} \ulcorner S_{n+1} \urcorner$$

and thus $\neg \text{True} \ulcorner S_{n+1} \urcorner$. By substitution of identity, we get:

$$\neg \text{True} “(\forall k)(k > n+1 \rightarrow \neg \text{True} \ulcorner S_k \urcorner)”.$$

By Tarski’s T -schema we have:

$$\neg (\forall k)(k > n+1 \rightarrow \neg \text{True} \ulcorner S_k \urcorner),$$

and by logic this yields:

$$(\exists k) \neg (k > n+1 \rightarrow \neg \text{True} \ulcorner S_k \urcorner).$$

Next by existential instantiation and a little logic we have:

⁵ In the original paradox Yablo used $\ulcorner S_k \urcorner$ is untrue. See S. Yablo, “Paradox without Self-Reference”, *Analysis* 53 (1993): 251–2.

⁶ Actually, Tarski has: $\ulcorner S \urcorner$ is true-in- L iff S . But this would spoil the paradox.

$$j > n+1 . \& . \text{True} \ulcorner S_j \urcorner.$$

But by universal instantiation of $(\forall k)(k > n . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner)$ we also have

$$j > n . \rightarrow . \neg \text{True} \ulcorner S_j \urcorner$$

Hence, from a little arithmetic we get: $\text{True} \ulcorner S_j \urcorner$ and $\neg \text{True} \ulcorner S_j \urcorner$. Thus $\neg \text{True} \ulcorner S_n \urcorner$. So far so good. How does the proof go from here?

Ketland argues that the Yablo sentences do not yield contradiction since there are non-standard models of Peano Arithmetic (PA) and thus of a theory formed by adding the Yablo sentences as proper axioms to PA.⁷ The idea is that the model makes

$$“(\forall k)(k > n . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner)”$$

true for every numeral \underline{n} even though $“(\forall k)(k > n . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner)”$ is not true in the non-standard model because there are entities in the domain that are not natural numbers. This is made possible because we do not have an ω -rule. If we have a formal system where \underline{n} is the numeral for the number n , the ω -rule for the system simply tells us that if $\vdash A\underline{n}$ for every number n , then $\vdash (\forall n) A n$. Ketland agrees with Priest who maintains that a contradiction is forthcoming only if one universally quantifies $\neg \text{True} \ulcorner S_n \urcorner$ to arrive at $(\forall n)(\neg \text{True} \ulcorner S_n \urcorner)$. But this requires that Yablo replace $\neg \text{True} \ulcorner S_k \urcorner$ by $\neg(k \text{ satisfies } “S_x”)$ so that quantification is possible.

Priest holds that $\text{True} \ulcorner S_n \urcorner$ employs a free variable “ n ”. Thus he concludes that Yablo use of Tarski's T -schema, which applies only to closed formulas, is improper. Yablo must reformulate his paradox with a satisfaction relation, replacing his list of sentences (closed formulas)

$$S_n: “(\forall k)(k > n . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner)”$$

with the quite different list of open formulas of the form

$$S_x: “(\forall k)(k > x . \rightarrow . \neg(k \text{ satisfies } “S_x”))”$$

⁷ See Jeffrey Ketland, “Bueno and Colyvan on Yablo's Paradox”, *Analysis* 64 (2004): 165–72.

where “ x ” is a free object-language variable. Priest writes:

He [Yablo] asks us to imagine a certain sequence. How can one be sure that there is such a sequence? (We can imagine all sorts of things that do not exist.) As he presents things, the answer is not at all obvious. In fact, we can be sure that it exists because it can be defined in terms of Sx : the n -th member of the sequence is exactly the predicate Sx with “ x ” replaced by “ \underline{n} ”.⁸

With the truth-predicate replaced by a satisfaction relation, Priest concludes that Yablo’s paradox involves self-reference.

Bueno and Colyvan argue that Priest is mistaken. There is a free variable involved in the Yablo derivation, but it is a meta-linguistic variable for a *numeral* of the object-language. This, they claim, is innocuous and does not warrant removing the truth-predicate in favour of satisfaction of an open formula.⁹

The generalization that Priest and Ketland require is not needed. So far we have seen how to arrive at $\neg\text{True} \ulcorner S_n \urcorner$. We can continue the derivation on behalf of Yablo as follows. Since the n -th sentence in our consecutive series is

$$“(\forall k)(k > n \rightarrow \neg\text{True} \ulcorner S_k \urcorner)”$$

$\neg\text{True} \ulcorner S_n \urcorner$ yields

$$\neg\text{True} “(\forall k)(k > n \rightarrow \neg\text{True} \ulcorner S_k \urcorner)”$$

By the T -schema we have:

$$\neg (\forall k)(k > n \rightarrow \neg\text{True} \ulcorner S_k \urcorner).$$

This, in turn, yields:

$$(\exists k) \neg (k > n \rightarrow \neg\text{True} \ulcorner S_k \urcorner).$$

Existentially instantiating we have

⁸ Graham Priest, “Yablo’s Paradox”, *Analysis* 57 (1997): 236–42.

⁹ See Otávio Bueno and Mark Colyvan, “Paradox without Satisfaction”, *Analysis* 63 (2003): 152–6.

$$m > n . \& . \text{True} \ulcorner S_m \urcorner .$$

Now the m -th sentence in our consecutive sequence is

$$“(\forall k)(k > m . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner)” .$$

So we have:

$$\text{True} “(\forall k)(k > m . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner)” .$$

By Tarski's T -schema, this yields:

$$(\forall k)(k > m . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner) .$$

We need only now continue the process used in arriving at $\neg \text{True} \ulcorner S_n \urcorner$ which says that the n -th sentence in our sequence is not true. We have:

$$m+1 > m . \rightarrow . \neg \text{True} \ulcorner S_{m+1} \urcorner$$

and thus $\neg \text{True} \ulcorner S_{m+1} \urcorner$. By the T -schema and logic, we have:

$$(\exists k) \neg (k > m+1 . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner) .$$

Next, by existential instantiation we have:

$$t > m+1 . \& . \text{True} \ulcorner S_t \urcorner .$$

But by universal instantiation of $(\forall k)(k > m . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner)$, we also have

$$t > m+1 . \rightarrow . \neg \text{True} \ulcorner S_t \urcorner .$$

Hence, we get the contradiction: $\text{True} \ulcorner S_t \urcorner$ and $\neg \text{True} \ulcorner S_t \urcorner$.

We have avoided using an ω -rule, and we have avoided Priest's use of a satisfaction relation. Priest's concern that the existence of Yablo's *list* might be questioned was on the right track, however. Conspicuous by its absence in the formulas on Yablo's list is any explicit mention of the alleged function which generates the list. But Yablo has to specify a syntactic form for sentences on his list and correlate them with natural numbers. Recall that he has:

$$S_n: “(\forall k)(k > n . \rightarrow . \neg \text{True} \ulcorner S_k \urcorner)”.$$

The expression $\neg \text{True} \ulcorner S_k \urcorner$ is supposed to say that there is a sentence S numbered k in the list that is not true. It is in virtue of this that Yablo avoids having to follow Priest in employing the relation of satisfaction. Suppose, however, that we make this explicit. Let us put

$$\#n = “(\forall k)(k > n . \rightarrow . \neg \text{True} \# k)”$$

where $\#$ is the function generating the list of formulas. Now observe that a symbol “ $\#$ ” occurs in each of the sentences on the list. But how are we to be assured that the expression “ $\#$ ” occurring in each of the sentences *must* be semantically interpreted so that it picks out the function $\#$ which generates the list? To specify a function, one must give its domain and range. In the present case, however, the range of the alleged Yablo function $\#$ is stated not simply as a set of sentences of a syntactic form, but together with the specific *semantic* instruction that the sign “ $\#$ ” in each of the sentences is to be read as referring to the very function in question. This is not only self-reference; it is self-referential incoherence.

2. RUSSELLIAN PROPOSITIONS

Yablo cannot assure that there is a function establishing a list if he characterizes the sentences on the list by appeal to that very function’s sign occurring in each of the sentences. The semantic interpretation of the signs of each sentence on Yablo’s list is necessarily outside of the syntactic specification of the list of sentences. To see the problem of assuring the existence of the function generating the list, it is instructive to investigate whether a Yablo paradox could be formed with an ontology of early Russellian propositions. Russellian propositions contain entities as constituents, so there is no concern about the semantic interpretation of signs occurring in sentences.

Russellian propositions are mind- and language-independent intensional entities. They are akin to states of affairs, intensionally construed so that even logical equivalence is not sufficient for identity. In the ontology of early Russellian propositions, the logical connectives are relation signs. They flank terms to form formulas. Modern statement connectives such as the arrow flank formulas to form formulas. For the early Russellian language of propositions, let us adopt the horseshoe sign (“ \supset ”) as

a relation sign. Thus, where x and y are individual variables, $x \supset y$ is a well-formed formula. The expression $x \rightarrow y$ is quite ungrammatical. Conversely, where A and B are formulas, the expression $A \supset B$ is ungrammatical in the early Russellian language of propositions. To express the modern $A \rightarrow B$ one must transform (nominalize) a formula A into a term $\{A\}$ and write $\{A\} \supset \{B\}$. Similarly, to express the modern $A \rightarrow (B \rightarrow A)$, one must put:

$$\{A\} \supset \{\{B\} \supset \{A\}\}.$$

We can avoid the tedious proliferation of braces by simply taking subject positions of a relation sign to perform the nominalization. Thus, using dots for punctuation, we have: $A . \supset . B . \supset A$ as well as $x . \supset . y \supset x$. Where α , β , and δ are any terms, we can then adopt the following axioms and definitions for the propositional logic:

$$\begin{aligned} A_1 & \quad \alpha . \supset . \beta \supset \alpha \\ A_2 & \quad \alpha . \supset . \beta \supset \delta : \supset : \alpha \supset \beta . \supset . \alpha \supset \delta \\ A_3 & \quad \sim \sim \alpha \supset \alpha \end{aligned}$$

Modus Ponens: From A and $\{A\} \supset \{B\}$, infer B .

$$\begin{aligned} \text{df } (\sim) & \quad \sim \alpha =_{\text{df}} \alpha \supset f \\ \text{df } (\&) & \quad \alpha \& \beta =_{\text{df}} \sim (\alpha \supset \sim \beta) \\ \text{df } (\equiv) & \quad \alpha \equiv \beta =_{\text{df}} (\alpha \supset \beta) \& (\beta \supset \alpha) \end{aligned}$$

Russell allows general propositions as well. Thus, for example, we can nominalize the formula $(x)(x \supset x)$ to arrive at the term $\{(x)(x \supset x)\}$. As expected, the following is an axiomatization of the quantificational logic of propositions:

$$\begin{aligned} A_4 & \quad (x)Ax \supset A [\alpha/x], \\ & \quad \text{where the term } \alpha \text{ is free for free } x \text{ in } A. \\ A_5 & \quad (x)(\alpha \supset Ax) . \supset . \alpha \supset (x)Ax, \\ & \quad \text{where } x \text{ is not free in } \alpha. \\ A_6 & \quad \alpha = \alpha \\ A_7 & \quad \alpha = \beta . \supset . A \supset A^*, \\ & \quad \text{where } A^* \text{ results from replacing one or more free occurrences} \\ & \quad \text{of } \beta \text{ for free occurrences of } \alpha \text{ in } A. \end{aligned}$$

Universal Generalization: From Ax infer $(x)Ax$.

df $(\exists) (\exists x)Ax =_{df} \sim (x) \sim Ax$

df $(x) A \supset_x B =_{df} (x)(A \supset B)$

df $(\neq) \alpha \neq \beta =_{df} \sim (\alpha = \beta)$

In the context of the quantificational theory, the constant f can be defined thus:

$$\text{df } (f) f =_{df} \{(x)(y)(x \supset y)\}.$$

This quantification theory is semantically complete. To establish fine-grained identity conditions for propositions, one needs to add new axioms which set out the structure propositions. Let us follow Church¹⁰ and add the following axiom schemata for the identity of propositions:

$$A_8 \quad \{(x)Ax\} = \{(x)Bx\} \supset (x)(\{Ax\} = \{Bx\})$$

$$A_9 \quad \{\alpha \supset \beta\} = \{\delta \supset \sigma\} : \supset : \alpha = \delta . \& . \beta = \sigma$$

$$A_{10} \quad \{\alpha \supset \beta\} \neq \{(x)Ax\}.$$

This theory of propositions is consistent.

In this theory of propositions, there is quantificational self-reference. According to the theory, there are general propositions, and these general propositions quantify over a realm which includes themselves. For example, the following is an instance of A_4 :

$$(x)(x \supset x) . \supset . \{(x)(x \supset x)\} \supset \{(x)(x \supset x)\}.$$

This form of ontological self-reference is the result of Russell's early thesis that there are general propositions which contain ontological counterparts of quantifiers. It is not a source of paradox for one can readily apply the usual consistency proof for quantification theory to Russell's theory. That is, for any axiom remove all quantifier symbols and their variables. What results are quantifier-free propositional variables or nominalized quantifier-free propositional formulas. Thus, any contradiction in the quantificational theory would have a transformation into the quantifier-free propositional system. Since the propositional system is consistent, so

¹⁰ See Alonzo Church, "Russell's Theory of the Identity of Propositions", *Philosophia Naturalis* 21 (1984): 513–22.

is the quantificational system.

The consistency of the quantificational theory of propositions has eluded many interpretations of Russell's philosophy. The reason is that such interpretations freely import into the *formal* theory notions such as "assertion", "belief", or indexicals such as "this", "I" and "now" which do not properly belong to it. The importation comes because of a confusion about the notion of a Russellian proposition and cavalier attitudes about the distinction between a formal system and a system that might hold of natural language. In Russell's formal system of propositions, only well-formed formulas of his formal language may be nominalized to form terms of the language.¹¹ Goldfarb misses this when he writes that in Russell's system of propositions the Epimenides paradox can be generated. All that is required, he says, is that the value of the propositional function A in

$$(p)(Ap \supset \sim p)$$

is uniquely satisfied by this proposition itself.¹² But there is no formula A of Russell's formal language of propositions to fit the bill.¹³ Consider an attempt to formulate a Liar paradox of Russellian propositions by invoking:

$$s \text{ believes } \{(x)(s \text{ believes } x \supset \sim x)\} \\ (y)(s \text{ believes } y \cdot \supset \cdot y = \{(x)(s \text{ believes } x \supset \sim x)\}).$$

Such a paradox involves a contingent psychological theory of belief which is quite outside the logic of propositions.

The same holds for attempts to generate an Epimenides Liar paradox for Russell's formal theory of propositions by appeal to relations of "assertion" instead of "belief". This is often missed. In *The Principles of Mathematics* Russell entertained the view that there is, in addition to the psychological notion, a purely logical notion of "assertion" which accounts for the difference between a proposition p occurring in the prop-

¹¹ See Gregory Landini, *Russell's Hidden Substitutional Theory* (New York: Oxford U. P., 1998), p. 201.

¹² Warren Goldfarb, "Russell's Reasons for Ramification", in C. Wade Savage and C. Anthony Anderson, eds., *Rereading Russell* (Minneapolis: U. of Minnesota P., 1989), p. 29.

¹³ The same argument is given in Graham Stevens, *The Russellian Origins of Analytic Philosophy* (London: Routledge, 2005).

osition $\{p \supset q\}$ and that very same proposition occurring assertorically. But in Appendix A of the *Principles* Russell retreated from this position, and it never again returned in his philosophy. Indeed, Russell himself formulated a propositional Liar paradox in a 1905 manuscript entitled “On Fundamentals” (*Papers* 4). This didn’t deter him in the least from proceeding with the development of a logic of propositions replete with general propositions and ontological self-reference.

Let us now consider Yablo’s construction formulated in an ontology of Russellian propositions. To formulate Yablo’s construction, add to the Russellian logic of propositions the axioms governing elementary arithmetic. For convenience, let us use special letters m, n, j, k for numbers. Thus,

$$(k)Ak =_{\text{df}} (x)(Nx \supset Ax)$$

$$(\exists k)Ak =_{\text{df}} (\exists x)(Nx \ \& \ Ax),$$

where “ Nx ” is for “ x is a natural number.” A propositional Yablo paradox would have us assume that there is a function $\#$ that is one-one from natural numbers n , to propositions of the form

$$\{k > n \cdot \supset_k \cdot \sim \# k\}.$$

Let’s call this assumption “(P_{Yablo})”. Notice that (P_{Yablo}) assumes that the function $\#$ is itself a constituent of each of the propositions in its own range. Then we get the following derivation of a contradiction:

- Suppose 1. $k > n \cdot \supset_k \cdot \sim \# k$
- | | |
|--|----------------------------|
| 2. $\#(n+1) = \{k > n+1 \cdot \supset_k \cdot \sim \# k\}$ | (P _{Yablo}) |
| 3. $n+1 > n \cdot \supset \cdot \sim \#(n+1)$ | 1, <i>ui</i> . |
| 4. $\sim \#(n+1)$ | 3, <i>arithmetic, mp</i> . |
| 5. $\sim \{k > n+1 \cdot \supset_k \cdot \sim \# k\}$ | 2, 4, <i>sub =</i> . |
| 6. $j > n+1 \cdot \& \cdot \# j$ | 5, <i>ei, logic</i> |
| 7. $\# j = \{k > j \cdot \supset_k \cdot \sim \# k\}$ | (P _{Yablo}) |
| 8. $j > n+1 \ \& \ \{k > j \cdot \supset_k \cdot \sim \# k\}$ | 6, 7, <i>sub =</i> . |
| 9. $k > j \cdot \supset_k \cdot \sim \# k$ | 8, <i>simp</i> . |
| 10. $j > n$ | 8, <i>simp, arithmetic</i> |
| 11. $j > n \cdot \supset \cdot \sim \# j$ | 1, <i>ui</i> . |
| 12. $j > n \cdot \supset \cdot \sim \{k > j \cdot \supset_k \cdot \sim \# k\}$ | 7, 11, <i>sub =</i> . |
| 13. $\sim \{k > j \cdot \supset_k \cdot \sim \# k\}$ | 10, 12, <i>mp</i> . |
| 14. $\sim \{k > n \cdot \supset_k \cdot \sim \# k\}$ | 1–13. <i>reductio</i> |

- | | |
|--|-----------------------|
| 15. $m > n . \& . \# m$ | 14, <i>ei, logic</i> |
| 16. $\# m = \{k > m . \supset_k . \sim \# k\}$ | (^P Yablo) |
| 17. $m > n \& \{k > m . \supset_k . \sim \# k\}$ | 15, 16, sub =. |
| 18. $k > m . \supset_k . \sim \# k$ | 17, <i>simp.</i> |

Next continue as from the beginning to get a contradiction.

- | | |
|--|-----------------------------|
| 19. $\# (m+1) = \{k > m+1 . \supset_k . \sim \# k\}$ | (^P Yablo) |
| 20. $m+1 > m . \supset . \sim \# (m+1)$ | 18, <i>ui.</i> |
| 21. $\sim \# (m+1)$ | 20, <i>arithmetic, mp.</i> |
| 22. $\sim \{k > m+1 . \supset_k . \sim \# k\}$ | 19, 21, sub =. |
| 23. $t > m+1 . \& . \# t$ | 22, <i>ei, logic</i> |
| 24. $\# t = \{k > t . \supset_k . \sim \# k\}$ | (^P Yablo) |
| 25. $t > m+1 \& \{k > t . \supset_k . \sim \# k\}$ | 23, 24, sub =. |
| 26. $k > t . \supset_k . \sim \# k$ | 25, <i>simp.</i> |
| 27. $t > m$ | 26, <i>simp, arithmetic</i> |
| 28. $t > m . \supset . \sim \# t$ | 18, <i>ui.</i> |
| 29. $\sim \# t$ | 27, 28, <i>mp.</i> |
| 30. $\# t \& \sim \# t$ | 23, 29 contradiction |

But there is no paradox. The derivation simply shows that (^PYablo) is false. That is, it shows that there is no such function as #.

In a theory of Russellian propositions paired with the axioms of elementary arithmetic, the principles for propositional identity will assure that for any function g , there is a one-one function f from natural numbers n to propositions of the form $\{k > n . \supset_k . \sim g(k)\}$. But this is quite different from the assumption (^PYablo) and does not yield a contradiction. The assumption (^PYablo) is that there is a one-one function # from natural numbers n to propositions of the form $\{k > n . \supset_k . \sim \# k\}$. This proposition contains the very function # as its own constituent. There is no reason whatever to think such a function exists. Not only has (^PYablo) failed to avoid self-reference, it has failed to produce a contradiction.

It might at first be thought that the reason there is no function # is that it has been impredicatively comprehended. This is mistaken. Call the following thesis “(F)”:

A function is a mapping from a class A to a class B . The members of B *must* have an existence that is independent of the existence of the function doing the mapping.

It must be understood that (F) is perfectly compatible with the *impredicative* comprehension of attributes (functions are many-one attributes) and classes. To illustrate, consider the use of impredicative comprehension of a class in Cantor's power-class theorem. One supposes for *reductio ad absurdum* that there is a function from a class A onto its power-class (the class of all its subclasses). Impredicative comprehension then assures the existence of a class C of all and only those entities x that are members of A but not members of fx . This class is a subset of A and thus in the range of f . Hence there is some a which is a member of A and such that $fa = C$. But then a is a member of C if and only if a is not a member of C . Thus Cantor concluded that there can be no such function f . Now the impredicative comprehension of the class C came under heavy fire from those who thought it involved an impredicative "definition". But those who maintain a realism about classes justifiedly don't find impredicative "definition" involved in the impredicative comprehension of classes. The impredicative comprehension of C is quite compatible with thesis (F) since the existence of all classes that are subsets of A is determinate quite independently of the *reductio* assumption of the existence of the function f . The situation is quite otherwise with (p Yablo). No realist intuition pushes us to believe there is a function $\#$ assumed by (p Yablo). The existence of entities (propositions) of its range is not given by any realist assumption of propositions. The function $\#$ is in violation of thesis (F). Quite clearly, the assumption of the existence of the function $\#$ made by (p Yablo) is false. Thus there is no (p Yablo) paradox of Russellian propositions.

3. AN INDEXICAL YABLO PARADOX?

Once again Yablo has failed to establish the existence of the appropriate function which generates his list. The assumption (p Yablo) is false. This may not be the end of the matter, however. Perhaps there is still a way to revive a Yablo paradox. Our discussion of Russellian propositions was appealing because it avoided some thorny matters concerning indexicals and use and mention. But if we are to investigate new forms of the Yablo paradox it is into the thorns that we must go.

The self-reference involved in the traditional Liar is not the result of impredicative comprehension or the ontological self-reference of Russellian general propositions. It is the result of the apparatus of indexicals. Yablo's paradox was presumably intended to show that indexical self-

reference is not a necessary condition for a Liar paradox. Perhaps we can save a Yablo paradox by relying on indexicals that are not self-referential.

Indexical reference is a very complicated practice of human communication that involves the adoption of pragmatic rules (conventions) governing the use of indexical words such as “this”, “here”, “now”, and “I”. Our practice may well be inconsistent, but a theory rendering an apparatus of indexicals cannot be. A theory of the apparatus of indexical reference is an all or nothing affair. A viable account of the *apparatus* of indexical reference must, if it permits indexical self-reference, show that indexical self-reference never yields contradictions. It is very doubtful that there is such an account. Hence most who eschew para-consistency typically reject accounts of indexical reference which embrace indexical self-reference.

There are many controversies as to how the referential apparatus of indexicals works, and what a competent user of a language must know in order to successfully communicate using indexical words. Consider coming upon a truck on which there is a sign which reads:

This truck makes sudden stops.

What does the word “this” refer to? Common sense might suggest that that it refers to the truck. But which truck? There might be several in one’s visual field. The salient truck, the one used to track the referent of “this”, is the truck on which *this* sign is placed. Perhaps then “this” refers to the sign, not the truck! The point here is that it is a very complicated matter to set out what a competent speaker of a language must know in successfully communicating with indexicals. There are special problems that arise with “pure” indexicals such as “I” as opposed to demonstratives such as “this” and “that”. One thing seems clear: sentences involving indexical words do not have truth-conditions *simpliciter*. They have truth-conditions relative to the contexts of their utterance. Quite obviously, different contexts of utterance can change the referents of the indexicals uttered. If one utters “This sentence is false”, pointing to a blackboard on which the concatenation of signs “This sentence is false” occurs, then the utterance “this” (given the context and the pragmatic rules) refers to the sentence token “This sentence is false” on the blackboard. But it certainly doesn’t follow that “This sentence is false” is false. As we have just noted, sentences involving indexical words don’t have truth-conditions *simpliciter*.

Concerns about pragmatics and the use of indexicals infest attempts to revive the Yablo paradox. We have observed that in Yablo's original formulation, the sentences on his list do not contain a name of the function generating the list. But we found this to be illicit and, once we made the function name explicit in the sentences in question, Yablo's argument collapsed. One might try to revive the Yablo by appeal to a list each member of which is of the form

“*n*Every sentence below is untrue.”

(To make the tokens clear we have included a numeral in the expression.) Thus there is a one-one function from each natural number n to each token of the above form.

This attempt to use pragmatics to salvage Yablo's technique fails. Each token sentence contains the indexical “below”. This indexical gets a meaning only when the sentence token is uttered in a context. But no sentence on this list has been uttered. Hence “below” has no referent. When we then ask as to the truth-condition for a given sentence on the list such as “*n*Every sentence below is untrue”, we are stuck. The sentences don't have truth-conditions *simpliciter*. A context of utterance for each is required, and there are none. One cannot at one and the same time set forth the list of sentence tokens of a syntactic type and fix a pragmatic interpretation for the tokens so that the word “below” occurring in each sentence token on the list refers to the sentence below it on the list. That is an attempt to generate a list of sentences and at one and the same time to demand that each sentence be given a specific pragmatic interpretation. The latter presupposes the list has already been generated and tokens on it uttered. Once again, that is not only self-reference, it is impossible self-reference.

The lesson seems clear. In reviving the Yablo paradox one cannot both render a consecutive sequence and demand that it be semantically or pragmatically interpreted in a particular way. Far from showing that a Liar paradox can be produced with self-reference, Yablo has failed even to have produced a paradox.¹⁴

¹⁴ Special thanks to Valia Alori and Francesco Orilia for comments on an early version of this paper.
