Articles

AXIOM OF INFINITY AND PLATO'S THIRD MAN

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As a contribution to the critical appreciation of a central thesis in Russell's philosophical logic, I consider the Third Man objection to Platonic realism in the philosophy of mathematics, and argue that the Third Man infinite regress, for those who accept its assumptions, provides a worthy substitute for Whitehead and Russell's Axiom of Infinity in positing a denumerably infinite set or series onto which other sets, series, and formal operations in the foundations of mathematics can be mapped.

1

The Third Man objection to Plato's theory of Forms is sometimes offered as an embarrassment to Platonic realism in the philosophy of mathematics. I argue here that, far from constituting a liability to Platonic realism, the Third Man regress can be turned to realism's advantage by providing the basis for a proof justifying the existence of an infinite set, effectively replacing the need to stipulate or posit by fiat a logically unsupported Axiom of Infinity for the foundations of mathematics, as in A. N. Whitehead and Bertrand Russell's *Principia Mathematica*.

2

One way to formulate the Third Man objection to Plato's theory of Forms is to consider the full implications of the following principles:

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FORM1 For any individual x and any property F, Fx if and only if there exists an abstract archetypal Platonic Form ϕ, by virtue of which x as an instance of ϕ has property F; alternatively, we can also say that a term designating Form ϕ grammatically nominalizes the meaning or content of corresponding property F.

FORM2 Any Form ϕ is an individual thing exemplifying the same property F as all other individuals that are instances of Form ϕ, $F(\phi)$; thus, any Form ϕ applies to, in the sense of being true of, itself, just as it does to each of its instances.

3

FORM1 constitutes at most a necessary and not a sufficient condition for purposes of characterizing what is often understood historically as Plato's theory of Forms. The reason is that Plato insists that unexemplified Forms also exist. We do not follow Plato or this interpretation of the Platonic theory of Forms in this regard. We assume instead that the individuals mentioned in principle FORM1 are distinct, in which case the instantiation of a Form by a single individual can also occur. Plato, despite differences of scholarly opinion even as to whether he accepted anything resembling the theory of Forms popularly attributed to him, is plausibly interpreted as holding that there are abstract Forms, and that the Forms have an eternal changeless existence independently of whether or not they are actually instantiated by any changing spatio-temporal physical entity.

FORM1 is presented as a biconditional. It is nevertheless controversial in Plato's theory whether an individual's possession of any and every property is to be explained by reference to the existence of a corresponding Platonic Form. In Plato's dialogue, *Parmenides*, 130b1–e5, the problem is illustrated by the question of whether there must exist Forms for "disgusting" things, such as "nail, hair, and dirt". Young Socrates in the dialogue appears squeamish about acknowledging the existence of such Forms. For present purposes, however, we shall assume that there is nothing logically or metaphysically improper in admitting the existence of corresponding ideal archetypes construed as grammatical nominalizations of any and every actually exemplified or unexemplified property.

To say that Platonic Form ϕ grammatically nominalizes a given corresponding property F is just to say, for any property of being an F, corresponding Form ϕ = F, the F, the Form of F, or F-ness (F-icity, etc.),

all of which grammatical variants are standardly found in authoritative translations of Plato's discussions of the Forms. Thus, if F is the property of being a dog, then the corresponding Platonic Form ϕ is Dog, the Dog, the Form of Dog or Dogness. If F is the property of being an even number, then the corresponding Platonic Form ϕ is Even, the Even, the Form of Even or Evenness.

FORM2 expresses the Platonic thesis that an individual has or several individuals have the properties they have by virtue of imitating, participating in, or striving to become particular ideal Forms, possessing and in that sense constituting the properties of the individuals that the Form instantiates. The explanatory force of the theory is supposed to derive from the fact there is a relation between individuals and the Form of which they are instances. Many different dogs are each and all dogs, according to Plato's theory of Forms, by virtue of participating as particular instances in the ideal Form of Dog, where the ideal Dog is also a dog. If the ideal Dog were not itself a dog, then the individuals participating in the Form of Dog, being similar albeit imperfect imitations of the ideal Dog and its property of also being a dog, could not be intelligibly understood as exemplifying the property of being a dog.

It follows that the Platonic Forms apply to themselves, that the Form ϕ, along with all individual instances participating in Form ϕ, also has the property F that Form ϕ grammatically nominalizes. If Form ϕ grammatically nominalizes property F, therefore, then it follows that $F(\phi)$. The reflexive self-application of properties grammatically nominalized by Plato's Forms in turn logically supports the infinite regress of the Third Man.

4

The Third Man objection is now immediately forthcoming from FORM1 and FORM2.

Suppose that some (not necessarily distinct) individuals x^1 and y^1 (men existing at ontic stratum 1) are such that Fx^1 and Fy^1. It follows from FORM1 that x^1 and y^1 instantiate or participate in Platonic Form ϕ^2 (the ideal Man existing at ontic stratum 2), and from FORM2 that therefore $F(\phi^2)$. From Fx^1, Fy^1, and $F(\phi^2)$, it further follows from FORM1 that there exists another higher-order Platonic Form ϕ^3 (the Third Man, or higher-order ideal Man existing at ontic stratum 3), of which it is further true that $F(\phi^3)$. And so on. The ascending hierarchy of Forms continues infinitely in the abstract, unlimited by the restrictions of any real time

process; there is no stopping the regress and no terminus for any finite ontic stratum index n: $<x^1, \phi^2, \phi^3, \ldots, \phi^n, \ldots>$. Thus, we must conclude that if FORM1 and FORM2 are true, then, as a consequence of the Third Man regress, there exists a set with denumerably infinite cardinality, $\text{Card}\{x^1, \phi^2, \phi^3, \ldots, \phi^n, \ldots\} = \aleph_0$.

5

Platonists in the philosophy of mathematics, also known as Platonic or platonic realists, mathematical realists, or simply realists (often, Realists), are sometimes challenged by the classical problem posed for Plato's theory of Forms that was already known to the ancient Academy as the Third Man. The Third Man appears in the *Parmenides*, where it does not seem to be taken very seriously, and is discussed in Aristotle's *Metaphysics*, where it is taken very seriously indeed. It is to Aristotle, in fact, that we owe the name "Third Man" argument, since Plato in the *Parmenides* uses the more general predicative example of "large" or "largeness".[1]

The problem of the Third Man for Plato's theory of Forms, as for Platonic realism in the philosophy of mathematics, is supposed to be that adducing Platonic Forms generally and abstract Platonic mathematical entities more specifically commits the theory to violating what has come to be known as (William of) Ockham's razor, according to which we are not to multiply entities beyond explanatory necessity, *entia non sunt multiplicanda praeter necessitatem*. William Thorburn argues[2] that the Latin motto popularly cited as Ockham's razor was actually formulated, not in the thirteenth or fourteenth century, but 300 years later, in 1639, by John Ponce of Cork, Ireland, a commentator on Dun Scotus, and that the term "Ockham's razor" was first coined for the desideratum of ontic parsimony by William Hamilton in 1852. Taking note of this historical background, we nevertheless follow convention here by referring to the principle in the usual way.

The infinite regress of the Third Man can now be turned to the advantage of Platonic mathematical realism as a substitute for the Axiom of Infinity in the foundations of mathematics. For the Platonist, the prin-

[1] *Metaphysics*, 990b17–1079a13, 1039a2, 1059b8. See also Aristotle, *Sophistic Refutations*, 178b36.

[2] "The Myth of Ockham's Razor", *Mind* 27 (1918): 345–53.

ciple that there exists a denumerably infinite set or series can be supported by a solid if controversial philosophical rationale. The five Dedekind–Peano axioms of arithmetic—0 is a number, the successor of any number is a number, etc.—do not generate a denumerably infinite set of natural numbers *unless* the successor function is iteratively applied denumerably infinitely many times to 0 as basis. Recognizing the need to have an infinite set available onto which such a denumerably infinite iteration of the successor function can be mapped, Whitehead and Russell simply declare as an axiom of their logicist formalism that there exists an infinite set. This is a problematic expedient for a variety of reasons, against which we shall not rehearse any of the stock philosophical objections. Suffice it to say that in an effort to develop a logical foundation for arithmetic, positing the existence of a denumerably infinite set poses awkward questions about the project's overall circularity and its claims to ground mathematics entirely in more basic principles of logic.

If the Third Man regress as we have presented it is accepted as a positive feature of Platonism in the philosophy of mathematics, and if the philosophical underpinning of the two principles FORM1 and FORM2 on which the regress rests are accepted as the Platonist understands them, then a Platonist can substitute the denumerably infinite regress of the Third Man for the Axiom of Infinity. Whitehead and Russell formulate the Axiom of Infinity in these terms:

$*120.03$ Infin ax $. = : \alpha \in \text{NC induct} . \supset_{\alpha} . \exists ! \alpha$ Df

"Infin ax"… is an arithmetical hypothesis which some will consider self-evident, but which we prefer to keep as a hypothesis, and to adduce in that form whenever it is relevant. [The axiom] states an existence-theorem. In the above form, it states that, if α is any inductive cardinal, there is at least one class (of the type in question) which has α terms…. Hence by induction, every inductive-cardinal must exist. (*PM* 2: 203)

The Axiom of Infinity *qua* axiom, even if intuitive to many, lacks philosophical rationale, and the Third Man regress based on FORM1 and FORM2 has a kind of built-in justification and even necessity or inevitability for the Platonist. The Platonist in mathematical ontology can claim a distinct advantage over Whitehead and Russell's logic, which simply lays it down that there exists an infinite set. If, as mathematical Platonic realists, we accept, as we should, the underlying principles of Platonism by adopting (some equivalent version of) FORM1 and FORM2 as essential

to Plato's theory of Forms, then we can derive the existence of a denumerably infinite set of Forms onto which we can one-one map iterations of the Dedekind–Peano successor function to 0 and its series of products under the induction to generate the denumerably infinite set of all natural numbers. Alternatively, though, in an obvious sense, equivalently, if we begin with 0, then we can also appeal to Wittgenstein's method of defining the positive integers in *Tractatus Logico-Philosophicus* 6.021, when he writes: "A number is the exponent of an operation." Here we let the operation be the unstoppable successive application of FORM2 to the results of FORM1 directed toward any choice of individuals generating the Third Man regress. We do not need to assume that there exists a denumerably infinite set of any sort of entities; if we are Platonists who accept FORM1 and FORM2, then we can prove that there exists such a set, guaranteed by the denumerably infinite regress of the Third Man, which we can then exploit for all the usual purposes in the foundations of mathematics.

Whitehead and Russell are well known as sufficiently Platonistic and friendly to certain formulations of Platonism in their philosophies of mathematics, as elsewhere in their metaphysics, to find it congenial to their general way of thinking to accept a version of the proposed replacement for the Axiom of Infinity or philosophical rationale for a corresponding Third Man Theorem of Infinity. If we need an infinite set for mapping and modelling purposes in the foundations of mathematics, we need not wave the magic wand of axiom postulation, in the manner of Whitehead and Russell, and simply declare that such a set exists; we can *prove* instead, if we are Platonic realists, as Whitehead and Russell are in the most fundamental aspects of their mathematical ontology, that there must exist as many such denumerably infinitely regressive self-generating sets as there are Platonic Forms or universals.

6

It might be objected that FORM2, whatever its imaginable appeal to Plato or Aristotle, appears to violate Type Theory and also the Vicious Circularity Principle in *Principia Mathematica*. If true, such a concern would cast doubt on whether Whitehead and Russell could welcome the Third Man argument as an acceptable replacement for the Axiom of Infinity.

The question here is not really whether or not Whitehead and Russell could incorporate this substitute for the Axiom of Infinity without rip-

ples throughout the rest of the *Principia Mathematica* logical system. There is nevertheless a reasonable way of construing FORM2, so that it does not run counter to simple Type Theory or the Vicious Circularity Principle prohibiting impredicative definitions. If we think of a Platonic Form as an individual thing, such as the Dog, then the property of being a dog, or dogness, is not identical with the Form, and we can have, for example, within Type Theory constraints: Dogness[Type 1](the Dog [Type 0]). Alternatively, if the Dog or Platonic Forms more generally are construed as themselves universal properties, which Plato himself does not obviously encourage in the dialogues, then we have the basis for an interesting criticism of post-1912 Russell. In Chapter 9, "The World of Universals", in *The Problems of Philosophy*, Russell explicitly accepts universals, beginning minimally with similarity relations among particulars, but soon expanding the domain to include qualities by virtue of their explanatory usefulness, once we have opened the floodgates to universal relations and given up strict nominalism. If these universal predicates are typed, then Russell's argument for insisting that similarities are universal is rendered deductively invalid. We can then type similarities (Type 1 or 2, etc.) of similarities (Type 0 or 1, etc., respectively), and thereby avoid the conclusion that appeal to similarities commits us ontically to the existence of similarities as universal. The construction, Similarity[Type N+1](Similarity[Type N]), does not represent a universal similarity, but rather similarities of distinctly different ordered types.

Suppose now that we insist on the same Type Theory syntax stratifications permitting only predicates of increasingly higher order to be applied to predicates of correspondingly decreasing lower orders in the case of FORM2. Then we have, at the first steps of the Third Man regress, … F[Type 2](F[Type 1])(ϕ[Type 0])) … . In that case, contrary to Plato's simplified assumptions, there is no single unified universal property F, but rather only distinct properties of ascending types. These properties would be different anyway, independently of Type Theory restrictions, for a Platonist who is also an extensionalist, because the predicates in question have manifestly different extensions. The point is that we still thereby generate an infinite regress, in this case, of different properties, each of a higher type, the denumerably infinite set and series of whose members can then be used as a replacement for the Axiom of Infinity to provide the formal model we need for mappings of other sets, series and operations in the foundations of mathematics.

As to the Vicious Circularity Principle that stands guard against im-

predicative definitions in *Principia Mathematica*, FORM2 is also blameless, partly because it is a thesis rather than a definition. Nor is the infinite regress generated by the Third Man a circularity, which is another thing altogether, to be precluded by the Vicious Circularity Principle. We have suggested in any case that the denumerably infinite regress of the Third Man is virtuous rather than vicious, because it accomplishes something useful and productive for the foundations of mathematics without simply laying it down as an axiom that a denumerably infinite set exists.

7

A problem that remains concerns Ockham's razor and the inadvisability of multiplying entities beyond explanatory necessity. This is a cautionary quasi-aesthetic principle to be taken to heart in efforts at theory construction and in deliberating choices among alternative rival theories that are in other ways explanatorily adequate. If a Platonic realist in the philosophy of mathematics hopes to make available the denumerably infinite regress of Forms as a substitute for the Axiom of Infinity, then it will be necessary to reconcile the acceptance of FORM1 and FORM2 with the ontic constraints of Ockham's razor.

To bite the bullet on the denumerably infinite regress of Forms without cracking Platonism's teeth, the following solution might be found acceptable—at least to Platonists who are already ideologically committed to the metaphysics of Forms. We consider answering the objection by observing that all of the Forms in the hierarchy associated with a particular selection of individuals, x^1, y^1, etc.—those Forms, in other words, beginning with ϕ^2, ϕ^3, … , and extending indefinitely—are numerically distinct but qualitatively indistinguishable. The Forms in each of these categories are numerically distinct because each comprehends (in the sense of being instantiated by) a different extension of individuals: ϕ^2 comprehends x^1, y^1, etc.; ϕ^3 comprehends x^1, y^1, etc., plus ϕ^2; ϕ^4 comprehends x^1, y^1, etc., plus ϕ^2 and ϕ^3; and so on. In general, ϕ^n at any fixed position in the hierarchy will comprehend x^1, y^1, plus ϕ^2, …, ϕ^{n-1}. The Forms in each category are nevertheless qualitatively indistinguishable from each other because they do not represent a different ideal of Man (the ideal Human Being), or ideal Couch, ideal City-State, or Unity, Duality, Evenness, Primeness, Triangularity, etc. Thus, it should serve the Platonist to choose arbitrarily any of the Forms in the appropriate hierarchy category to invoke in the kinds of explanations Platonists want

to give of the properties of individuals, for example, of why it is that all triangles are rightly so called, what it is that they all have in common, or what makes them all triangles.

As a further palliative to soothe concerns about violating Ockham's razor, the Platonist might argue that there is after all a definite explanatory need to admit each and all of the denumerably infinitely many Forms implied by the Third Man regress entailed by a philosophical commitment to FORM1 and FORM2. If the Forms are adduced to explain the properties of individuals, and if the Form at the lower reaches of each of the hierarchies of Forms is an individual, as the Platonist believes, then a higher-order Form must be further adduced to explain its properties. That situation might reasonably be understood and construed as explanatory necessity, by and for the Platonist, rather than an objectionable explanatory vacuity and ontic largesse.