

# INCOMPLETE SYMBOLS IN *PRINCIPIA MATHEMATICA* AND RUSSELL'S "DEFINITE PROOF"

RAY PERKINS, JR.  
Philosophy / Plymouth State U.  
Plymouth, NH 03264-1600, USA  
PERKRRK@EARTHLINK.NET

Early in *Principia Mathematica* Russell presents an argument that “the author of *Waverley*’ means nothing”, an argument that he calls a “definite proof”. He generalizes it to claim that definite descriptions are incomplete symbols having meaning only in sentential context. This *Principia* “proof” went largely unnoticed until Russell reaffirmed a near-identical “proof” in his philosophical autobiography nearly 50 years later. The “proof” is important, not only because it grounds our understanding of incomplete symbols in the *Principia* programme, but also because failure to understand it fully has been a source of much unjustified criticism of Russell to the effect that he was wedded to a naïve theory of meaning and prone to carelessness and confusion in his philosophy of logic and language generally. In my paper, I (1) defend Russell’s “proof” against attacks from several sources over the last half century, (2) examine the implications of the “proof” for understanding Russell’s treatment of class symbols in *Principia*, and (3) see how the *Principia* notion of incomplete symbol was carried forward into Russell’s conception of philosophical analysis as it developed in his logical atomist period after 1910.

**E**arly in *Principia Mathematica* Russell presents an informal argument that definite descriptions are incomplete symbols—that they function differently from proper names and that they have meaning only in sentential context.<sup>1</sup> A few pages later he refers to this ar-

---

<sup>1</sup> *PM* 1: 67. Throughout this paper I shall speak as though this *Principia* argument was Russell’s alone. But although the theory of descriptions may be attributed to Russell alone, we should remember, as Russell tells us in *My Philosophical Development*, p. 74, that virtually “every line” of *Principia* was “a joint product”.

gument as a “definite proof”.<sup>2</sup> This “proof” is significant not only because it is central to understanding incomplete symbols so vital to Russell’s logicism, but also because it has been taken as evidence of Russell’s alleged carelessness and confusion in philosophy of logic and language. In what follows I wish to show that a proper understanding of Russell’s “proof” not only helps absolve Russell of long-standing charges of confusion, but enables us to see more clearly how his *Principia* account of incomplete symbols fits into his idea of philosophical analysis during his atomistic period.

What many students of Russell have failed to appreciate fully is that Russell and Whitehead’s *Principia Mathematica* is more than a formal exposition of the logicist thesis that mathematics is reducible to logic. Indeed, *Principia* is infused with epistemic and ontological themes connected with Russell’s special idea of *naming*—an idea in his philosophy of logic and language that goes beyond the concerns of mathematics or formal logic as commonly understood.

### I. THE “PROOF” AND RUSSELL’S ALLEGED CONFUSION

In the Introduction, Chapter III on Incomplete Symbols, Russell is concerned to show that “ $(\exists x)(Fx)$  is always an incomplete symbol”, i.e. has no meaning in isolation but only in context. Toward the bottom of page 67 he sums up the essence of his argument using “the author of *Waverley*”. It is this summary argument (statements [1]–[3] below) which I wish to examine inasmuch as this argument has been the focus of much criticism over the last half century:

Thus all phrases (other than propositions) containing the word *the* (in the singular) are incomplete symbols: they have a meaning in use, but not in isolation. For [1] “the author of *Waverley*” cannot mean the same as “Scott”, or “Scott is the author of *Waverley*” would mean the same as “Scott is Scott”, which it plainly does not; [2] nor can “the author of *Waverley*” mean anything other than “Scott”, or “Scott is the author of *Waverley*” would be false. Hence [3] “the author of *Waverley*” means nothing.<sup>3</sup> (PM I: 67)

---

<sup>2</sup> PM I: 72. Cf. MPD, p. 85, where he calls a near-identical argument a “precise proof”.

<sup>3</sup> Statements [1] and [2] may themselves be regarded as arguments neatly translatable into the form *modus tollens*, as can be easily seen in Russell’s 1959 version: “If ‘the author of *Waverley*’ meant anything other than ‘Scott’, ‘Scott is the author of *Waverley*’ would

---

This argument, which may be found with minor alterations in several other works by Russell,<sup>4</sup> is perhaps more responsible than anything else for the widespread view that Russell confused meaning and reference, or in Fregean parlance, meaning as sense and meaning as reference. P. F. Strawson made such a criticism in his famous attack on the theory of descriptions: "the source of Russell's mistake was that he thought that referring ... if it occurred at all, must be meaning ... [and so he] confused meaning with referring."<sup>5</sup> And Strawson is only one of many who have levelled similar charges.<sup>6</sup>

Perhaps the most influential attack has been one made in 1959 by Alan White, who singles out the above *Principia* argument—or rather the nearly identical version proffered by Russell a half century later—as affording "an opportunity for giving a neat and precise proof of this confusion."<sup>7</sup> White charges Russell with committing the fallacy of equivoca-

---

be false, which it is not. If 'the author of *Waverley*' meant 'Scott', 'Scott is the author of *Waverley*' would be a tautology, which it is not. Therefore, 'the author of *Waverley*' means neither 'Scott' nor anything else—i.e. 'the author of *Waverley*' means nothing, Q.E.D." (*MPD*, p. 85).

<sup>4</sup> See "Knowledge by Acquaintance and Knowledge by Description", *Proceedings of the Aristotelian Society* 11 (1910–11), and reprinted in *Mysticism and Logic*, pp. 228–9, *Papers* 6: 159–60; "The Philosophy of Logical Atomism", *The Monist* 28–9 (1918–19), reprinted in *LK*, pp. 245–7, and *Papers* 8: 213–16; *MPD*, p. 85.

<sup>5</sup> "On Referring", *Mind* 59 (1950): 320–44 (at 328). Strawson's remark is not explicitly targeted at the *Principia* argument.

<sup>6</sup> See, for example, L. Wittgenstein, *Philosophical Investigations*, 2nd edn. (Oxford: Blackwell, 1958), p. 40; G. Ryle, "Meaning and Necessity", *Philosophy* 24 (1949): 70; A. J. Ayer, "Names and Descriptions", in *The Concept of a Person* (London: Macmillan, 1968), pp. 133, 147; J. Searle, "Russell's Objections to Frege's Theory of Sense and Reference", *Analysis* 18 (1958): 142; W. V. Quine, "Russell's Ontological Development", in *Bertrand Russell: Philosopher of the Century*, ed. R. Schoenman (London: Allen & Unwin), p. 310; L. Linsky, *Referring* (London: Routledge and Kegan Paul, 1967), pp. 53, 88. Nicholas Griffin has pointed out to me that the first person to have levelled this charge of equivocation concerning Russell's *PM* argument was E. E. Constance Jones. See her "A New Law of Thought", *Proceedings of the Aristotelian Society* 11 (1910–11): 166–86 (at 175).

<sup>7</sup> Alan R. White, "The 'Meaning' of Russell's Theory of Descriptions", *Analysis* 20 (1959): 8–9. See n.3 above. See also K. Lambert and B. van Fraassen, *Derivation and Counterexample* (Encino, Calif.: Dickerson, 1972), p. 167; and W. S. Croddy, "Russell on the Meaning of Descriptions", *Notre Dame Journal of Formal Logic* 17 (1976): 483–8. Curiously, none of these philosophers seems aware that the argument originally appeared in *Principia*. Avrum Stroll claimed that there are important differences between this version and *Principia*. See "Russell's 'Proof'", *Canadian Journal of Philosophy* 4 (1975): 653–62. But the differences are not significant regarding the alleged confusion of sense and reference. See Robert Fahrnkopf, "Stroll on Russell's 'Proof'", *ibid.*, 6 (1976): 569–78.

---

tion on “means” as between sense and reference. The essence of his argument goes: If [1] is true, “mean” must mean “has the same sense”, not “has the same reference”, because two expressions (e.g. “the morning star” and “the evening star”) may well have the same reference and be joined by the “is” of identity without being trivial like “Venus is Venus”. But if [2] is true, “mean” must mean “has the same reference”, not “has the same sense”, because two expressions may well have different senses and be joined by the “is” of identity without making a false proposition (e.g. “The morning star is the evening star”). Thus, as White’s “proof” goes, if Russell’s premisses are to be true, they must equivocate on “mean” as between sense and reference.

White’s own remark—that “anyone with a slight knowledge of . . . the English language knows that ‘the author of *Waverley*’ does mean something, both in the sense that it refers . . ., and in the sense that it has a . . . sense”—ought to have made him suspicious that his “refutation” might be too neat, that it might be overlooking something. What he and other critics of the argument have missed is the special sense of “mean” as “name” that Russell employs, a sense which was central to his philosophy of language and which, I think, vindicates his *Principia* “proof”.

To see this, one need only notice on page 67 of *Principia*, in the paragraph before the summary argument, that Russell insists that “Scott” and “the author of *Waverley*” are not “two names for the same object”, which, he says, “illustrates the sense in which ‘the author of *Waverley*’ differs from a true proper name.” And I believe that naming in Russell’s special sense is the key to understanding his argument correctly. Thus, when he concludes “‘the author of *Waverley*’ means nothing” he means that “the author of *Waverley*” names nothing, because it is not a true proper name.<sup>8</sup>

Russell’s special sense of “name” with one of its most distinguishing features is clearly set forth in *Principia* (I: 66):

Whenever the grammatical subject of a proposition can be supposed not to exist without rendering the proposition meaningless, it is plain that the grammatical subject is not a proper name, i.e. not a name directly representing some object.

On this view of names, most ordinary proper names—indeed, all those

---

<sup>8</sup> I have defended Russell’s 1959 version of his “proof” in “On Russell’s Alleged Confusion of Sense and Reference”, *Analysis* 32 (1971): 45–51.

---

that putatively name fictitious objects, as well as all those whose objects are “known to the speaker only by report and not by personal acquaintance” — would not be true proper names, but would in fact be disguised descriptions, as Russell had already explained in Chapter I (*PM* 1: 31). A genuine name picks out its referent *directly* without the help of any properties the object may possess; the object is known by acquaintance, and the name’s meaning, in the only sense in which it has meaning, is its reference. It would be, as Russell so often put it, a constituent of the judgment or proposition which we understand.<sup>9</sup> And it would constitute an integral part of the meaning (sense/intelligibility) of the sentence containing the name so that if the name were supposed meaningless, i.e. referentless, the expressed proposition/judgment would be rendered meaningless (nonsense). A principle underlying Russell’s position here — let’s call it  $R_1$  — can be expressed as follows:

$R_1$  If “ $N$ ” and “ $M$ ” are two genuine proper names for the same object, then, in the only sense in which such symbols have meaning, “ $N$ ” and “ $M$ ” will have the same meaning, and their connection by an “is” of identity ( $N = M$ ) will express a trivial truth, the same one that “ $N = N$ ” expresses.<sup>10</sup>

We can see how this special sense of “mean” enables Russell’s “proof” to succeed — provided, of course, that we recognize that Russell is there treating “Scott” as a proper name in this strict and special sense.<sup>11</sup> Let’s recast the argument making the appropriate changes for “mean”. The crucial replacements are for “mean(s)” in [3] and [2], and for the first “mean” in [1]. The argument, we must keep in mind, is concerned with

---

<sup>9</sup> See *PM* 1: 43. On p. 66 he treats “Socrates” as a genuine name in “Socrates is mortal” which “expresses a fact of which Socrates himself is a constituent.” But strictly speaking, “Socrates” is not actually a name (see n.12 below). See also “Knowledge by Acquaintance and Knowledge by Description”, *ML*, pp. 219–20 (*Papers* 6: 154–5); Russell gave up propositions as non-linguistic entities by 1907.

<sup>10</sup> A. N. Prior has made a similar claim about this sort of symbol in “Is the Concept of Referential Opacity Really Necessary?”, *Acta Philosophica Fennica* 16 (1963): 194–5. Cf. Saul Kripke, “Naming and Necessity”, in *Semantics of Natural Language*, ed. D. Davidson and G. Harmon (Dordrecht: Reidel, 1972), pp. 253–5. Russell’s genuine names would be “rigid designators” in Kripke’s sense, although the converse would not hold.

<sup>11</sup> Notwithstanding Russell’s *treatment* of “Scott” as a name in his *Principia* argument on p. 67, “Scott” isn’t actually a name in Russell’s technical sense, not only because he lacked personal acquaintance with Scott, but also because on Russell’s view of the nature of acquaintance at the time, “Scott” could only be a name when used by Scott himself.

---

descriptions and names and with showing that the former don't "mean" in the same sense as the latter. Russell also held that sentences (propositions) have "meaning" in the perfectly familiar sense that if two sentences "mean" the same they make the same assertion, or, as we might also say, are synonymous. We needn't worry whether Russell thought that sentences named objects in the same way that true proper names did.<sup>12</sup> With the appropriate substitutions for "mean", Russell's argument becomes:

- [1'] "the author of *Waverley*" cannot name the same object that "Scott" names, or "Scott is the author of *Waverley*" would make the same trivial assertion as "Scott is Scott", which it plainly does not;
- [2'] nor can "the author of *Waverley*" name anything other than what "Scott" does, or "Scott is the author of *Waverley*" would be false.
- [3'] Hence "the author of *Waverley*" names nothing, i.e. is not a name.

It might be thought that there are obvious counterexamples to show that Russell's premiss [1']—and R<sub>1</sub>—are not true. For example, "Phosphorus" (morning star) and "Hesperus" (evening star) are apparently two names for the same object (Venus), yet "Phosphorus is (=) Hesperus" hardly seems the same trivial truth as "Phosphorus is (=) Phosphorus." Indeed, one might well doubt the truth of the former, but not of the latter. Yet surely Russell would insist, as he did a few years later,<sup>13</sup> that names in such a case are not being used and understood as genuine names, but rather as truncated descriptions, i.e. they pick out their referents indirectly via certain properties, e.g. as "the object called 'Phosphorus'." Thus the "counterexample" is really not a counterexample at all.<sup>14</sup>

---

<sup>12</sup> In *Principia*, Russell says that sentences are incomplete symbols having meaning only in the context of judgment. That sentences are not names for facts or anything else is clearly articulated several years later under the influence of Wittgenstein. See "The Philosophy of Logical Atomism", *LK*, p. 187 (*Papers* 8: 167).

<sup>13</sup> See his "Philosophy of Logical Atomism", p. 246 (*Papers* 8: 216), where he insists that "Scott is Sir Walter" is a trivial truth (he says "a pure tautology, exactly on the same level as 'Scott is Scott'") when the names are used as genuine names. But it is not trivial, he says, when the names are actually used as truncated descriptions, e.g. as "the person called 'Scott'" and "the person called 'Sir Walter'." This distinction is really implicit in Russell's remarks about "Apollo" at *PM* I: 31. Indeed, the distinction is implicit in his 1905 account of descriptions. See *OD* in *LK*, p. 54 (*Papers* 4: 425–6).

<sup>14</sup> See J. D. Carney and G. W. Fitch, "Can Russell Avoid Frege's Sense?" *Mind* 88 (1979): 384–93, where the Phosphorus/Hesperus example is used with Russell's notion of naming as a way of escaping Frege's need to postulate senses to explain his (Frege's) puzzle concerning identity.

---

Indeed, from R<sub>1</sub>, another principle concerning names in belief (and other non-extensional) contexts seems to follow. Let's call this R<sub>2</sub>:

R<sub>2</sub> If "*N*" and "*M*" are (used by *S* as) genuine names for the same object, then *S* believes that  $N = M$  iff *S* believes that  $N = N$ .

This principle is plausible: by R<sub>1</sub>, " $N = N$ " would be the very same trivial truth as " $N = M$ ", and thus the truth-values of "*S* believes ..." would be the same in both cases.

Of course, this is not to say that Russell's two principles and his notion of naming are ultimately acceptable. They may not be. But equivocating fallaciously on "means" as between sense and reference is not one of Russell's shortcomings in the "proof" in question.<sup>15</sup>

Avrum Stroll, in an original criticism, has argued that Russell's argument is flawed, quite apart from any alleged equivocation between sense and reference, on the grounds that, if accepted, it leads logically to the obliteration of the distinction between names and descriptions.<sup>16</sup> His tactic is to show that the "mirror-image" argument of the original—which results from substituting "the author of *Waverley*" for "Scott" and vice versa, and which, he says, should be sound if the original is—will afford a proof that "*Scott*" means nothing. Thus:

"Scott" cannot mean the same as "the author of *Waverley*" or "The author of *Waverley* is Scott" would mean the same as "The author of *Waverley* is the author of *Waverley*", which it plainly does not; nor can "Scott" mean anything other than "the author of *Waverley*", or "The author of *Waverley* is Scott" would

---

<sup>15</sup> Some philosophers have failed to take account of these principles in the course of their criticism of Russell's *Principia* argument. Thus Karel Lambert, in an otherwise astute essay, thinks that Russell's premiss [1'] is dubious because it likely violates "the principle of the substitutivity of identity" in a non-extensional context like "... is trivial" (*Free Logic: Selected Essays* [Cambridge: Cambridge U. P., 2003], p. 8). And Mark Sainsbury, without explicitly mentioning the *Principia* argument, implies that it would be unsound by virtue of its first premiss, because "that names name the same does not guarantee that they mean the same", and he attributes to Russell a failure to realize that "names cannot everywhere be interchanged *salva veritate* even if they name the same: 'John believes that Tully was bald' may differ in truth-value from 'John believes that Cicero was bald'" (*Russell* [London: Routledge, 1985], pp. 79, 107). Sainsbury is here directing his criticism at Russell's "law of identity" in "On Denoting". But Russell's doctrine of names in 1905 was not significantly different from what it was in 1910.

<sup>16</sup> Stroll, "Russell's 'Proof'", pp. 658–9. Cf. his *Twentieth-Century Analytic Philosophy* (New York: Columbia U. P., 2000), p. 24.

---

be false. Hence “Scott” means nothing.

And so, by Stroll’s analysis, it would seem that Russell’s original argument proves too much—neither names nor descriptions mean (name) anything, and, at least as far as Russell’s argument shows, there is no difference between names and descriptions. Thus, Russell’s argument is flawed.

But apart from the fact that the *Principia* argument proceeds on the explicit assumption that “Scott” is being treated as a genuine name, there is at least one serious problem with Stroll’s *reductio* given our reading of “mean” in the first, fourth and fifth lines of the mirror-image argument above. The first premiss is unwarranted.<sup>17</sup> On our reading it becomes:

“Scott” cannot name the same object that “the author of *Waverley*” names, or “the author of *Waverley* is Scott” would mean the same as “the author of *Waverley* is the author of *Waverley*.”

To see how this could be false, recall that Stroll’s claim is that the mirror-image argument is sound if the original is. So let’s suppose Russell’s argument sound. Then, as its conclusion asserts, “the author of *Waverley*” means (names) nothing, i.e. it’s an incomplete symbol. But then the first premiss of the “mirror-image” argument might be false. This is because “Scott”, in naming what “the author of *Waverley*” names, *viz.* nothing, would be an incomplete symbol, i.e. a truncated description. But which description? Presumably any one which had the same meaning (i.e. named nothing) as “the author of *Waverley*”. But this could be any description, e.g. “the author of *Marmion*”. But the fact that symbols may have the same meaning in this sense does not guarantee that “the author of *Waverley* is Scott” would mean the same as (make the same assertion as) “the author of *Waverley* is the author of *Waverley*.” Thus, I think Stroll’s *reductio* can’t succeed, at least not on our reading of “means as ‘names’”.

We shall see that Russell’s “proof” has important implications for understanding *Principia*’s account of class symbols. But before we examine that connection a final point concerning equivocation should be addressed. In the introductory sentence just before premiss [1], Russell does use

---

<sup>17</sup> James Carney has made a similar criticism although for different reasons. See his “Russell’s ‘Proof’, Again”, *Canadian Journal of Philosophy* 10 (1980): 587–92.

---



“meaning” in a way that seems equivocal, at least ambiguous (see above p. 30).<sup>18</sup> As we have argued, Russell means that these incomplete symbols don’t name in isolation, and presumably they don’t name in use (context) either. So their meaning in use must be meaning in some other sense. But in what sense? I think the answer is that descriptions not only don’t name, but in *Principia* they are without meaning in the sense of being, in isolation, *undefined* symbols. And to say they have meaning *in use* is simply to say that when they (symbols of the form “ $\iota xFx$ ”) occur in the context of a sentence of the form “ $G(\iota xFx)$ ” they do contribute to *the meaning of a whole sentence*, meaning which is assigned through explicit definition as on page 68 as:

$$(D) \quad G(\iota xFx) = \exists x [(y) (Fy \leftrightarrow y = x) \ \& \ Gx] \quad \text{Df.}$$

(Here the scope marker is omitted for convenience; a primary scope is assumed.) “The object which is *F* is also *G*” is to mean “There is exactly one object which is *F* and that object is also *G*.” Notice that the *definiens* does not contain “ $\iota xFx$ ”, so there is no question of that symbol naming anything. As Russell says: “Thus ‘ $\iota xFx$ ’ is merely symbolic and does not directly represent an object ...” (*PM* I: 68).

## II. DESCRIPTIONS, CLASS SYMBOLS AND ONTIC IMPLICATIONS

It’s important to realize that the incompleteness of symbols does not mean that there are no objects corresponding to them (although, such objects, if existent, will not be constituents of the expressed fact or judgment). Obviously it’s true in some cases that  $\iota xFx$  exists, e.g. it is true that the author of *Waverley* exists, and, indeed, it is certainly true that the referents of genuine proper names exist. Yet Russell’s remarks in *Principia* have sometimes led to misunderstanding on this point. For example J. O. Urmson in his classic history of analytic philosophy between the wars writes that Russell seems to think that “... to show that *X* is an incomplete symbol is tantamount to showing that there are no *Xs*.”<sup>19</sup>

---

<sup>18</sup> A. P. Martinich has made such a claim. See his “Russell’s Theory of Meaning and Descriptions (1905–1920)”, *Journal of the History of Philosophy* 14 (1976): 198–9. See also A. Stroll, “Descriptions Again”, *Analysis* 34 (1973): 27–8.

<sup>19</sup> Urmson, *Philosophical Analysis: Its Development between the Wars* (London: Oxford U. P., 1967), p. 30.

---

Urmson takes as his evidence Russell's passage in *Principia*, Chapter III, where he is discussing classes symbols and their connection with descriptions. Urmson quotes the passage as follows:

In the case of descriptions, it was possible to *prove* that they are incomplete symbols. In the case of classes, we do not know of any equally definite proof.... It is not necessary for our purposes, however, to assert dogmatically that there are no such things as classes. It is only necessary for us to show that the incomplete symbols we introduce as representative of classes yield all the propositions for the sake of which classes might be thought essential. (PM I: 72)

This does make it look like Russell thought that to show that “*X*” is an incomplete symbol is to show that there are no *X*s. For he could be understood to mean in the above passage that if one could prove that class symbols are incomplete that would be tantamount to proving that there are no classes. Yet surely Russell didn't think that there was no author of *Waverley* just because he had proved “the author of *Waverley*” to be an incomplete symbol. So what's going on here?<sup>20</sup>

Urmson's editing of the above passage obscures the fact that Russell believed that there was more than one way to prove “*X*” an incomplete symbol, and that he was actually thinking of a proof for the incompleteness of class symbols along an alternative route. Urmson's ellipsis at the end of the second sentence omits a sentence and a footnote. Russell actually says, “In the case of classes we do not know of any equally definite proof, *though arguments of more or less cogency can be elicited from the ancient problem of the One and the Many*” (my italics). And he adds the following footnote:

Briefly, these arguments reduce to the following: If there is such an object as a class, it must be in some sense *one* object. Yet it is only of classes that *many* can be predicated. Hence, if we admit classes as objects, we must suppose that the same object can be both one and many, which seems impossible.<sup>21</sup> (PM I: 72n.)

---

<sup>20</sup> David Pears, in his important work on Russell's atomism, denies Urmson's general claim, but says (wrongly, I think) that in the *Principia* passage, Russell makes a “slip”. See *Bertrand Russell and the British Tradition in Philosophy* (London: Collins/Fontana Library, 1967, 1972), pp. 24–5.

<sup>21</sup> In *MPD*, p. 80, Russell mentions another important source of his scepticism about the existence of classes, viz. Cantor's proof that  $2^n$  is always greater than  $n$ , even when  $n$  is infinite. If all the things in the world number  $n$ , then the class of all things has  $n$  members and  $2^n$  sub-classes. Thus there are more classes than things, which seems to show

---

These arguments purport to show that the notion of class (*qua* object) is inconsistent. And clearly, if one had a "definite proof" along these lines, class symbols would have to be incomplete symbols, and not genuine names, since their apparent nominata would be non-existent. To the extent that Russell had doubts about the cogency of such arguments his "proof" would be less than "definite". But if there *were* such a definite proof of the incompleteness of class symbols, or of any "X", along these lines, then we *could* "assert dogmatically" that there are no Xs.

Russell's footnote also suggests why he thought he couldn't get a proof along the familiar route, i.e. the route that he had used on page 67 for descriptions. Proofs along that line require in the premisses a true sentence of the form  $a = \iota xFx$ , where "a" occupies the place of a genuine name. But owing to arguments like the one in the footnote on page 72, Russell had serious doubts about whether there were such objects as classes, and so, whether there were any true identity sentences of the required form.<sup>22</sup>

In *Principia* Russell is officially agnostic regarding the existence of classes. He treats class symbols as incomplete symbols on the model of descriptions—they are not genuine proper names, and they are defined in use only. The general strategy is to preserve the idea of classes as extensions of propositional functions, and as identical if and only if they have the same members or are determined by formally equivalent propositional functions.<sup>23</sup> Thus, for example, we can say that the class of humans is identical with the class of featherless bipeds, just in case all and only things which have the property of being human have the property of being featherless and bipedal. If we symbolize "the class of things that are F" using the class abstract " $\{x: Fx\}$ ", we can follow *Principia's* treatment of class symbols in use and render "The class of things that are F is G" (which may be symbolized as " $G\{x: Fx\}$ ") by the following definition:

---

that classes are not things.

<sup>22</sup> Cf. Russell, *The Principles of Mathematics*, pp. xv–xvi: "In the case of classes, I must confess, I have failed to perceive any concept fulfilling the conditions requisite for the notion of class ... in order that the mind may have that kind of acquaintance with them which it has with redness or the taste of pineapple."

<sup>23</sup> *Principia* explicitly gives five requisites that a satisfactory theory of classes must fulfil. See 1: 76–7.

---

(C)  $G\{x: Fx\} = \exists H[(y) (H!y \leftrightarrow Fy) \ \& \ G(H!\hat{x})]$ <sup>24</sup> Df.

I.e. (loosely) “The class of things that are  $F$  is  $G$ ” is to mean “There is a propositional function (or property)<sup>25</sup>  $H$  such that  $H$  is formally equivalent to  $F$ , and  $H$  is  $G$ .”

The sentence thus derived will always be extensional, i.e. true if and only if  $Hy$  is formally equivalent to  $Fy$ , and this sentence may be regarded as being what one means when one formulates a sentence using a class symbol in grammatical subject position purporting to be about a class. But, like the case of descriptions (where “ $\iota xFx$ ” is eliminated), the class symbol disappears from the *definiens* and there is no symbol, or complex of symbols, purporting to name a class. Such sentences, as the definition shows, are really about propositional functions or properties.

Nevertheless, there is an important difference between Russell’s treatment of descriptions and class symbols. Although in both cases we have symbols that do not name entities which are constituents of the facts/judgments involved, in the case of true description-sentences we are (sometimes) committed to the existence of  $\iota xFx$ , as, for example, in “The author of *Waverley* was Scotch.” That is because such sentences assert, in part, that the author of *Waverley* exists, i.e. that *there is* exactly one object which authored *Waverley*, as we can see in (D) above (p. 37).<sup>26</sup> In the case of true sentences containing class symbols we are never committed to the existence of  $\{x: Fx\}$ —an extension—but rather only to the sorts of things that can be values of the apparent variable “ $H$ ” in (C) above, *viz.* intensions such as propositional functions or properties.<sup>27</sup> This explains, I think, what Russell means later in *Principia* when, notwith-

<sup>24</sup> See *PM* I: 76 and \*20.01, p. 190.

<sup>25</sup> This function or property is said to be “predicative” in *Principia*’s technical sense of determining a legitimate totality in conformity with the theory of types. The issue of whether Russell’s propositional functions are, in this context, properties or linguistic objects is controversial. See Scott Soames, “No Class: Russell on Contextual Definition and the Elimination of Sets”, *Philosophical Studies* 139 (2008): 213–18; Michael Kremer, “Soames on Russell’s Logic: a Reply”, *ibid.*, pp. 209–12.

<sup>26</sup> This is merely another way of saying that some descriptions have denotations. See his “Knowledge by Acquaintance”, *ML*, p. 229; *Papers* 6: 160. Michael Kremer has made similar observations about ontic commit. See Kremer, pp. 211–12. See also Kevin Klement’s defence of Russell *vis à vis* Soames in “The Functions of Russell’s Having No Class”, *Review of Symbolic Logic* 3 (2010): 633–64.

<sup>27</sup> See Russell’s remark in *PM* I: 72, that, while a class is an extension and its symbol is incomplete, its use “always acquires its meaning through a reference to intension.”

standing his official agnosticism towards classes, he calls them “fictitious objects” (I: 188). Ordinarily to say that something is fictitious is to imply that it is non-existent. And, for Russell, classes are non-existent in the following sense: in *Principia*, classes (*qua* individual objects, if any), are not among—or need not be assumed to be among—the objects which may be values of the apparent (bound) variables ranging over objects in *Principia*'s universe of discourse. And this, of course, is really what is meant by Russell's official agnosticism regarding the existence of classes.

To be sure, in *Principia* one finds true propositions of the form “ $(\exists\beta)$  (...  $\beta$  ...)” where the position of “ $\beta$ ” is occupied by a class symbol.<sup>28</sup> But these propositions are not in expanded (primitive) notation. When they are expanded, they contain no symbol (or complex of symbols) purporting to name a class, nor do they require any apparent (bound) variables taking classes (as opposed to propositional functions or properties) as their values.

Nevertheless, Russell himself was not always completely unambiguous about this issue of ontic commitment regarding incomplete symbols. For example, in his more popular account of the logicist project a few years after *Principia*, after stating that he wants a definition of class symbols “on the same lines as the definition of descriptions”, he writes:

We shall then be able to say that the symbols for classes are ... not representing objects called “classes”, and that classes are in fact, like descriptions, logical fictions, or (as we say) “incomplete symbols”.

(*IMP*, p. 182; cf. *LK*, p. 253, *Papers* 8: 221)

This seems ambiguous owing to use-mention carelessness. Russell could mean:

1. Classes are ... like objects corresponding to descriptions, logical fictions, or (as we say) their apparent names are “incomplete symbols”.

Or

2. Class symbols are ... like descriptions, logical fictions, or (as we say) “incomplete symbols”.

---

<sup>28</sup> E.g. *PM* \*20.54, I: 195.

---

On (1), such symbols (for classes and descripta) would be logical fictions in the sense that these symbols drop out in the *analysans*—the expanded notation—of what sentences containing them mean; they would not be among the primitive symbols needed in a *Principia*-like language for discoursing about the world. While (1) may be regarded as true, it ignores the important difference between the ontic implications of Russell's analyses of these two kinds of incomplete symbols. Yet (2) seems odd in applying "fiction" to symbols. After all, Russell had called the putative *objects* "fictitious" in *Principia*.<sup>29</sup> But in his lectures on logical atomism he uses the term for *both* class symbols and classes.<sup>30</sup> However, (2) seems untrue by virtue of implying that the analysis of descriptions eliminates the putative objects corresponding to descriptions in the same way that the analysis of class symbols eliminates the putative objects corresponding to class symbols.

We can fix these apparent shortcomings by distinguishing two senses of "logical fiction" corresponding to Russell's two kinds of incomplete symbols. Let's say that putative objects,  $X$ 's (or their symbols), are **logical fictions<sub>1</sub>** if and only if their symbols are eliminated (on the model of descriptions) through a contextual definition. In this wide sense, both descripta and classes would be logical fictions. But in a narrower sense, we may say that  $X$ 's (or their symbols) are **logical fictions<sub>2</sub>** if and only if their symbols are eliminated through a contextual definition which does not require that  $X$ 's be among the values of the apparent (bound) variables in the *definiens*. In this sense, only classes (or their symbols) would be logical fictions. Thus, all *logical fictions<sub>2</sub>* are *logical fictions<sub>1</sub>*, but not conversely.

### III. SOME RELATED THOUGHTS ON RUSSELLIAN ANALYSES

Russell's use of the term "logical fiction" in his atomist period seems usually to intend it in our second sense. And, as David Pears observed in his important work on Russell's atomism, his use usually conveyed a

---

<sup>29</sup> *PM* 1: 188. And see his *Our Knowledge of the External World*, p. 206 (*OKEW*<sub>4</sub>, p. 160), where he refers to his doctrine that "classes are fictions".

<sup>30</sup> See *LK*, pp. 253, 265 (*Papers* 8: 221, 230–1). However, in those lectures he may mean to use the phrase "logical fiction" to apply only to class symbols or classes and not to descriptions or their descripta, although his intent is not completely clear.

---

point about the *kind* of analysis employed<sup>31</sup>—a kind sometimes called reductive, or new-level analysis (in contrast to same-level analysis), whereby talk about one kind of “thing” is replaced with talk about another kind of “thing”.<sup>32</sup> In Russell’s best-known reductive analyses, talk about the “things” to be analyzed was ultimately reducible to talk about propositional functions. Numbers (or numerals) in *Principia* and material objects (or their symbols) in *Our Knowledge of the External World* would be logical fictions in our narrower sense, i.e. logical fictions.<sup>2,33</sup>

Another feature of Russell’s analyses closely related to reductivity is the fact that they were almost always *revisionary*, sometimes radically so, i.e. they were designed to replace problematic pre-analytic notions by more “legitimate” ones. What Russell did in effect was to doubt—on grounds independent of, and antecedent to, a new analysis—the legitimacy of our belief in *Xs* as thought of in some pre-analytic way. We saw this in the case of classes. This is also the case with numbers and material objects, to take two other well-known examples. Numbers, thought of pre-analytically, had generated a host of muddles (*MPD*, pp. 53–5); material objects before 1914 (e.g. in *The Problems of Philosophy*) had involved problematic assumptions, especially that of a *ding-an-sich*-like cause of sense-data; and classes, as we have noted, had engendered several puzzles.<sup>34</sup> Russell’s analyses generally had the effect of purging “*X*” of its ordinary but “illegitimate” meaning by treating “*X*”—or rather sentences in which “*X*” occurs—in terms of more “legitimate” notions. Thus, discourse putatively about numbers was to be regarded as discourse about certain kinds of classes; discourse about material objects, as about certain series of classes of sensibilia; and discourse about classes, as about certain propositional functions or properties formally equivalent to functions

---

<sup>31</sup> See *Bertrand Russell and the British Tradition in Philosophy*, pp. 17f. and 110. Pears takes Russell’s period of logical atomism to be roughly 1905 to 1919. This seems reasonable for reasons we need not elaborate here.

<sup>32</sup> See Urmson, p. 39.

<sup>33</sup> Russell also uses the term “logical construction” as interchangeable with “logical fiction” (i.e. *logical fiction*<sub>2</sub>) or “symbolic fiction” (“The Ultimate Constituents of Matter” [1915], in *Mysticism and Logic*, p. 129; *Papers* 8: 77) or “symbolically constructed fictions” (“The Relation of Sense-Data to Physics”, *ML*, pp. 156–7; *Papers* 8: 12). The term “logical construction” seems to appear at the time (1914) that Russell developed his reductive analysis of material objects.

<sup>34</sup> See n.21 above.

---

determining those classes.<sup>35</sup> Given this revisionary motivation, it would be unreasonable to complain that the definitions associated with Russell's analyses fail to capture accurately "what we ordinarily mean" by " $X$ ". Yet some philosophers associated with the so-called Oxford school have made these very sorts of complaints.<sup>36</sup>

In so far as these analyses are revisionary and reductive, they are also applications of the principle that Russell called "Occam's razor", in the sense that they intend to shave away the "illegitimate", unnecessary portion of the pre-analytic notion being analyzed. Russell's celebrated analysis of descriptions has usually been taken as an example of non-reductive, same-level analysis. But although we have seen that it has important differences with Russell's more overtly reductive analyses, e.g. of classes, it has some reductive similarities as well. And although Russell often presented his analysis as capturing and preserving what people ordinarily mean by "The so and so is  $F$ ", his analysis is, in certain respects, revisionary—most notably as regards grammatical form—but also as regards ordinary linguistic meaning.<sup>37</sup> But that is another story for another time.

Our revisitation to Russell's *Principia* "proof" has shown, I hope, that it did not trade on equivocation, and that his analyses of incomplete symbols in *Principia* involved important differences between descriptions and class symbols regarding ontic commitment. We have seen that his special notion of naming, with its semantic and epistemic features, is central to his analysis of incomplete symbols, which, in turn, was itself vital, not only to *Principia*'s logicist programme, but also to his wider conception of philosophical analysis during his logical atomist period.<sup>38</sup>

---

<sup>35</sup> In his "Philosophy of Logical Atomism", Russell's language often makes clear the revisionary character of his analyses, e.g. of a chair as a series of classes of sense-data. He says of the analysis "That is what you mean by saying—or what you ought to mean by saying..." (*LK*, p. 275; *Papers* 8: 238). (My emphasis.)

<sup>36</sup> See "Some Replies to Criticism", *MPD*, pp. 214–54.

<sup>37</sup> See, for example, Strawson, pp. 320–44, especially concerning existential presupposition and truth value; see Russell's reply in *MPD*, pp. 178–9. For an excellent overview of the critical literature and a defence of Russell's theory and his related doctrine of ordinary proper names as truncated descriptions, see Peter Hylton's "The Theory of Descriptions", in *The Cambridge Companion to Bertrand Russell*, ed. Nicholas Griffin (Cambridge: Cambridge U. P., 2003), pp. 228–40.

<sup>38</sup> I wish to express my appreciation to Gregory Landini, Kevin Klement and Nicholas Griffin for several valuable suggestions for improvements on an earlier draft of this paper.

---