THE AXIOM OF REDUCIBILITY

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The axiom of reducibility plays an important role in the logic of *Principia Mathematica*, but has generally been condemned as an *ad hoc* non-logical axiom which was added simply because the ramified type theory without it would not yield all the required theorems. In this paper I examine the status of the axiom of reducibility. Whether the axiom can plausibly be included as a logical axiom will depend in no small part on the understanding of propositional functions. If we understand propositional functions as constructions of the mind, it is clear that the axiom is clearly not a logical axiom and in fact makes an implausible claim. I look at two other ways of understanding propositional functions, a nominalist interpretation along the lines of Landini and a realist interpretation along the lines of Linsky and Mares. I argue that while on either of these interpretations it is not easy to see the axiom as a non-logical claim about the world, there are also appear to be difficulties in accepting it as a purely logical axiom.

The most cited reason for the view that the logicism of *Principia Mathematica* was a failure is the position that the reduction of mathematics to logic fails because the theory demands non-logical principles and is in fact (in Quine's words) "set theory in sheep's clothing".[1] The non-logical principle most objected to is the axiom of reducibility, the axiom which states that for any propositional function of whatever order of a given type of argument, there is an extensionally equivalent predicative propositional function of that type. The order of a proposition involves the level of the quantifiers in the proposition. A first-order proposition will contain quantifiers ranging over individuals, a second-order proposition will contain quantifiers ranging over first-

[1] This is Quine's characterization of second-order logic in general (Quine 1970, pp. 66–7).

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level propositional functions, etc. A propositional function will be derived from a proposition by replacing either a constant or a function by a variable. A "predicative" propositional function will be one which is the lowest order compatible with its argument. The axiom is important because it gives for propositional functions otherwise defined in terms of quantification over predicative functions an equivalent predicative propositional function, from which classes can be constructed. The axiom thus serves as a comprehension principle. *Principia* has a no-class theory and defines purported predicates of classes in terms of predicates of propositional functions (*PM* ∗20.01). Russell himself says the axiom replaces the assumption of classes and in fact calls it the axiom of classes (*PM* 1: 167). In *Principia* the axiom occurs in the discussion of identity (*PM* ∗13) and in the account of mathematical induction (through ∗90 and the ancestral relation), and in the definition of real numbers. The axiom is given in two forms, one for one-place propositional functions and one for two-place propositional functions:

∗12.1 $\vdash : (\exists f) : \phi x \,.\, \equiv_x .\, f!x$ Pp

∗12.11 $\vdash : (\exists f) : \phi(x, y) \,.\, \equiv_{x,y} .\, f!(x, y)$ Pp

ϕ is a propositional function of any order which can take x as an argument, and the f with the exclamation after it is a predicative propositional function. ϕ is a real variable in Russell's terminology and f is a bound, or "apparent", variable.

The logic of *Principia Mathematica* is difficult to capture. This is because the formal system is not made explicit and incorporates the ramified theory of types. While the type theory cries out for variables with type indices, these are not supplied by Russell and Whitehead. Consequently, there have been many different formulations or reconstructions of the formal system and different accompanying interpretations, especially with regard to the ontological commitments of the theory.[2] Probably the most commonly accepted formulation is that of Church (1976). Church's formulation includes bound variables of all different types and orders and treats circumflexion as lambda abstraction. The more austere Landini formulation has all propositional function variables restricted to predicative functions, does not treat circumflexion as a term-

[2] See, for example, Hatcher 1968, Chihara 1973, Church 1976, and Landini 1998.

forming operator, and treats many instances of real variables as schematic letters, and so treats the axiom of reducibility as a schema as well as the only comprehension principle.[3]

At the heart of the issue for all the interpretations is the status of propositional functions, for it is these that are ramified and typed.[4] There are a variety of attitudes taken toward propositional functions. Russell sometimes suggests that propositional functions are merely expressions and so there are no non-linguistic correlates to them. This line would suggest a nominalist semantics for predicate variables as advocated by Landini (1998). On the other hand, Russell certainly appears to be a realist about universals during this time, and it may be natural to think that the non-linguistic correlates to propositional functions would be or ought to be universals.[5]

In *Principia*, Russell and Whitehead motivated the ramified theory of types by the vicious-circle principle, which states that "whatever involves *all* of a collection must not be one of the collection" (*PM* 1: 37). Their first explication of the principle is in terms of sets and in particular of the way a set is defined. Given the "no class theory" and the role of propositional functions, Russell and Whitehead rephrase the principle in terms of propositional functions as the position that "the values of a function cannot contain terms only definable in terms of the function" (*PM* 1: 40). Much work has been done on the vicious-circle principle,[6] and it is easy to see that the principle suggests a conceptualism of the kind associated with predicative logics as studied by Feferman (1964) and Hazen (1983). The conceptualist view, simply put, is that propositional functions are constructions of the mind. As Gödel pointed out in 1944, this

[3] Landini does not want to talk about propositional functions or function variables at all, given Russell's view that they, like propositions, are "incomplete symbols." He would phrase his position as the view that all predicate variables are predicative.

[4] In the 1908 article, "Mathematical Logic as Based on the Theory of Types", Russell treated propositions as single entities and as entities which could be substituted for variables. In that presentation it was propositions which came in different orders and types. Russell rejects propositions as entities in the Introduction to *Principia*. Because of his account of propositions and propositional functions as certain kinds of constructions, B. Linsky would say that even though in one sense there are no propositions and propositional functions which are entities in *Principia*, they are both constructions which come in (ramified) types.

[5] The identification of universals and propositional functions is rejected, though for different reasons, by both Landini 1998 and Linsky 1999.

[6] See Gödel 1944, Chihara 1973, Hazen 1983 and Goldfarb 1989, for example.

conception does not fit well with Russell's realist view of universals.[7]

Some more recent realist interpretations treated propositional functions as higher-order entities which are not constructions, but have a structure which lends support to the rationale of the ramified theory.[8] If we think of propositional functions as entities which come in different logical types, and these types are ramified, so that non-predicative functions are entities distinct from predicative functions, then the axiom of reducibility will certainly seem to be a substantial non-logical claim. If we think that propositional functions come into being as a result of some mental activity, then it appears clear that the axiom of reducibility is not only not a logical principle, but, as Hazen rightly puts it (1988, p. 374), something in which the conceptualist notion of property provides no grounds for belief. Without being explicit on how they are taking "propositional functions" and "predicative propositional functions", both Wittgenstein and Ramsey argued that there are logically possible worlds in which the axiom of reducibility is false.[9]

I. TWO INTERPRETATIONS OF "PRINCIPIA"

I want to look at two interpretations of *Principia* and see how the axiom of reducibility fares within them. I pick these interpretations because they more naturally lead to the view that the axiom of reducibility can plausibly be seen as a logical principle. The justification for the axiom on both of these views will involve understanding infinite disjunctions and conjunctions of predicative propositional functions as themselves being predicative propositional functions. One of these views is a nominalist view, the other a realist.

The nominalist position is articulated in Landini (1998). This position takes very seriously the origins of *Principia*'s logic in the substitution theory and the manuscripts in which Russell moved from this theory to the type theory of *Principia*. It also takes very seriously Russell's remarks in Chapter II of the Introduction to *Principia*. What Landini sees as the biggest shift from the substitution theories of 1905–06 is Russell's abandonment of propositions as entities, which is evident in Russell and

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[7] See Gödel 1944; also Hazen 1983.

[8] See, for example Goldfarb 1989 and Linsky 1998 and 2006 for two different such accounts.

[9] See Ramsey 1925, p. 57, and Wittgenstein 1922, 6.1232, 6.1233.

Whitehead's account of truth and falsehood (*PM* 1: 41–7), where they endorse the no-propositions theory and the multiple relation theory of judgment. They begin with a basic ontology of such complexes as "*a*-in-the-relation *R*-to-*b*" and define elementary judgments as those which assert such complexes (*PM* 1: 44). These judgments have "first-order truth". They then define a general judgment as one which asserts a propositional function of every individual, but state that these judgments do not correspond to one complex, but to many. These judgments will have "second-order truth" (*PM* 1: 45). The discussion of the higher-order functions (*PM* 1: 48–55) makes reference to making assertions about all functions of a given order (*PM* 1: 51–2) but does not use ontological language such as "property", "quality", etc. So while Russell and Whitehead no longer attempt to ground the account of propositional function variables in the manner of the substitution theory which Russell held prior to September 1906, they still seem to subscribe to the same cut-down ontology of that time.[10]

On this nominalist view, there are no different types of entities; the theory of types and orders is a theory of the symbolism, not the symbolized. Quantification over predicates should be understood substitutionally; the theory of types is a theory of structured variables.

The difference between this position and realist ones can be illustrated by considering Russell's own treatment of non-predicative functions. The example (now often repeated) Russell gives of a non-predicative function of individuals is the function "x has all the predicates of a great general" (*PM* 1: 56). Russell uses the example to explain the force of the axiom of reducibility, as that axiom will state that there is a predicative function equivalent to this function. Russell here defines a "predicate" of an object as a predicative function which is true of that object. Since "x has all the predicates of a great general" refers to a totality of predicates, it is not itself a predicate. Russell puts $f(\phi ! \hat{z})$ for $\phi ! \hat{z}$ is a predicate required in a great general, and then expresses the claim that this second-order pred-

[10] For a while after his adoption of the theory of descriptions Russell sought to treat symbols for propositional functions as incomplete symbols explained in terms of entity variables and substituting one entity into another. See Landini 1998 for a thorough discussion of this theory. He rejected the theory in the "Paradox of the Liar" (1906) and initially adopted a richer ontology which involves propositions as entities which come in distinct orders. This is the theory present in "Mathematical Logic as Based on the Theory of Types" (1908). Propositional function variables were still treated as with the substitution theory, though (see *LK*, p. 77).

icate applies to Napoleon as

(1) $(\phi) : f(\phi ! \hat{z})$ *implies* $\phi !$(Napoleon).

Now how should we understand $f(\phi ! \hat{z})$?

On this nominalist semantics, f is a free variable that should be assigned a well-formed formula which contains the function $(\phi ! \hat{z})$—for example, $(x)(Gx \supset \phi ! x)$, where "Gx" just is "x is a great general". (1), then, just is

(2) $(\phi) : (x)(Gx \supset \phi ! x) \supset \phi !$(Napoleon).

In this case we should note the $\phi ! \hat{z}$ does not appear in a subject position, but only in a predicate position. It appears in (1) as if in subject position, but a further analysis reveals that it is not. What is important for the nominalist semantics is that the function variables be understood substitutionally, not objectually.

On a realist interpretation, the semantics would assign to f a property that other properties have, a property of being one of the properties common to great generals, such that

$$(\phi) : f(\phi ! \hat{z}) \equiv (x)(Gx \supset \phi ! x).$$

However, $f(\phi ! \hat{z})$ would not be identical to the property $(x)(Gx \supset \phi ! x)$.

What sort of world would be one where the axiom of reducibility is false? It would be a world where a higher-order propositional function held of a group of individuals, but no predicative function held of exactly that group. But, one could ask, in virtue of what does the higher-order propositional function hold?

We can get clear on this by taking an example mentioned by Hylton in the context of his questioning Russell's claim that variables in *Principia* range only over predicative propositional functions.[11] Suppose, to modify Russell's example slightly, that *having a property common to great generals* is true of Fred and also true of Mary, although Fred has this in

[11] Hylton 1990, p. 308n. The reference is to the claim in *Principia* ∗12 that "it is possible … without loss of generality, to use no apparent variables except such as are predicative" (1: 165). Landini's interpretation requires that there be no variables for non-predicative functions (Landini 1998, p. 264).

virtue of having property F (say being arrogant) and Mary has this property in virtue of having property G (say being deceitful), but otherwise Fred and Mary do not have any genuine properties in common.[12] In that case the second-level function, $(\exists\phi) : (x)(Gx \supset \phi!x) \,.\, \phi!\hat{z}$, would be true of Fred and Mary, but no predicative function would be.

If one held that whenever there are the predicates F and G then there is always the assertion that one or the others of these hold, i.e. that Boolean operations on predicative functions yield further predicative functions, then the disjunction of all the first-level functions asserting the various properties would be itself a predicative function equivalent to the second-level function *having a property of a great general.* Similarly the conjunction of those functions would yield a predicative function equivalent to *having all the properties of a great general.* If as realists we identified predicative propositional functions with genuine universals and thought each of these logically independent of each other, then we might wonder whether this step is legitimate. Similarly, if as conceptualists we thought that some mind had to think the disjunction for it to exist, we might also wonder whether this step is legitimate. Russell, though, clearly thought it was legitimate, for in a footnote (*PM* 1: 56) he says, "When a (finite) set of predicates is given by actual enumeration, their disjunction is a predicate, because no predicate occurs as apparent variable in the disjunction." So Russell himself allows that any finite disjunction or conjunction of propositional functions of a given order is itself a function of that order. It is easy to see that at the finite level, the axiom of reducibility would be justified: for every higher-order function that quantified over functions where what was involved was only a finite collection of such functions, there would always be a lower-level function equivalent to it. The difficulty occurs in the case where we are dealing with an infinity of such disjunctions or conjunctions.

If we follow the nominalist semantics and we understand function quantification substitutionally, then it appears that to come up with a falsification of the axiom of reducibility, we need a case where a non-predicative function is present but not captured by any finite disjunction or conjunction of propositional functions, as there are no sentences or (linguistic) propositional functions of infinite length. It isn't clear wheth-

[12] Hylton attributes the example to Thomas Ricketts. Hylton's concern is that from $(x)(Gx \supset Ax)$ and $(x)(Gx \supset Dx)$ and from Af and Dm we can deduce $\exists\psi(\psi \mathrm{f} \,.\, \psi \mathrm{m})$, where the ψ must be a non-predicative function.

er Russell's "assertions" in the semantics he develops might leave open the possibility that there is a possible infinite assertion of the properties and relations which hold of an individual. This "infinite assertion" would give the predicative function required by the axiom of reducibility. As we shall see, Ramsey proposed a system which allows that all generalizations, whether over individuals or functions, are infinite truth-functions, and he asserted that our inability to write the propositions of infinite length is "logically a mere accident" (Ramsey 1925, p. 41). If we accept these infinite conjunctions and disjunctions of predicative functions as themselves predicative functions, then every intended model of *Principia* will be a model of the axiom of reducibility.[13] The axiom is necessary, just as universal generalization is necessary, because we do not have a language which is infinite.

So far we have been discussing the nominalist line which preserves (as Nino Cocchiarella has pointed out) what drove much of Russell's early philosophy of logic: the doctrine of the unrestricted variable and the view that all entities, therefore, are of the same type.[14] Without Ramsey's notion of an infinitely long sentence, the axiom will be dubious.

Peter Hylton and Warren Goldfarb have suggested that despite the ramified hierarchy, Russell was a realist about propositional functions. Goldfarb, in particular, defends this interpretation against those who see the constructivism involved in the ramified theory as being alien to Russell's realism.[15] His position is that Russell was committed to various different ontological types of abstract entities: propositional functions in particular, but also propositions. These intentional entities are the values of the various bound propositional function variables. On this kind of position it is easy to imagine that the axiom of reducibility is false in

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[13] In fact, while Ramsey suggests that he rejects the axiom of reducibility, this is what he does by "widening the concept of a predicative function" to include these infinite conjunctions and disjunctions (Ramsey 1925, pp. 38–9). Ramsey's criticism of the axiom of reducibility should therefore be understood that once we understand a predicative function in this way, the axiom is true, and is shown to be such by the concept of a truth function and so is not needed as a separate axiom.

[14] Cocchiarella, though, does not think Russell maintained this position in *Principia*, while Landini does (Cocchiarella 1980, pp. 105–8, Landini 1998, Chap. 10).

[15] See Goldfarb 1979, especially pp. 30–4. Goldfarb sees the constructivism Russell is accused of advocating with the vicious-circle principle as appropriate to one who would advocate typing sets according to their presentation. Goldfarb argues that a specification of a propositional function determines the function presented in a way that justifies the type restrictions on propositional functions without constructivism (p. 31).

some possible world.[16] But on the two variants of realism I will discuss, the axiom of reducibility does appear to be true in all possible worlds. These related views have been advocated by Bernard Linsky (1999) and Edwin Mares (2007).

Linsky makes a distinction in type between entities and universals, but he does not identify universals with propositional functions. Universals for Linsky come in different types, depending on which type of entity falls under them, but these types are not identical to the ramified types of *Principia*, which apply to propositions and propositional functions. On Linsky's view, both propositions and propositional functions should not be thought of as independent entities as they are constructions, but they are not "logical fictions" as some constructions are, but rather structured entities. They are not fictions in the sense that "the average man" is a fiction in the proposition "the average man is five foot nine", presumably because there is no method of eliminating them in the same way. Linsky sees the construction as a metaphysical relation among entities and does not see these as "creations of the human mind" (Linsky 1999, p. 29). They are structured entities, and so "not single entities". Linsky is thus able to address the passages in *Principia* which advocate the no-propositions and the no-propositional functions position while not adopting a nominalist attitude toward these.[17] Linsky was anticipated on this view by Philippe de Rouilhan, who distinguished three kinds of constructivism. Rouilhan's constructions$_1$ are the logical constructions which can be eliminated in analysis (such as "the average man" above); his constructions$_2$ are the structured entities that are constructed in the sense that the totality of elementary propositions is constructed out of individuals and first-level concepts which apply to them, and first-order propositions are those constructed of these and such operations as generalization over entities, etc. His constructions$_3$ are mental constructions favoured by Feferman and Hazen (Rouilhan 1996, p. 217). Like Goldfarb and Linsky, Rouilhan thinks constructions$_3$ are not pertinent to Russell's view. Rouilhan sees propositions and propositional functions alike as constructions$_2$, as does Linsky. On Linsky's view there is a sharp distinction between universals, which are not constructed entities, and proposi-

[16] Goldfarb sees the axiom as showing the failure of Russell's logicism (p. 38).

[17] So Linsky, even with his realist view which is akin to that of Goldfarb, can accept these passages in *Principia* despite his realism. Church (1976, p. 748) and Goldfarb (1989, p. 34) reject these passages as not consistent with the logic of *Principia*.

tional functions, which are.[18]

Edwin Mares (2007) has developed a semantics somewhat based on Linsky's ontology for *Principia*. Mares, though, identifies predicative functions with universals. Mares and Linsky both accept a formal system for *Principia* which includes free and bound variables for non-predicative functions, and in this way they sharply disagree with Landini. Mares' hereditary predicative function symbols denote universals (Mares 2007, p. 237). On his reconstruction, universals come in a hierarchy of simple types. The semantics for quantification using variables of hereditary predicative types is objectual, and the semantics for the non-hereditary function variables (i.e. the non-predicative functions of *Principia*) is substitutional. There are no ontological counterparts to the non-predicative functions and no ontological "variables". The models for the semantics involve frames which are sets of facts which include, for any finite sequence of entities in the model, a relation in which all and only those entities stand. Mares accepts the "Boolean combination thesis", that any conjunction or disjunction of properties of a given type yields a property of that same type. Using this thesis, Mares first displays Leibnizian concepts of individuals and then forms disjunctive properties of these that would apply to any set of entities of the same type (Mares 2007, p. 239). He then continues these constructions for universals of different levels. Mares strengthens the Boolean combination thesis to include infinite conjunctive and disjunctive properties for any set of properties of the same type. This then allows for the validation of the Axiom of Reducibility on all models. As Russell himself said, "The axiom of reducibility is equivalent to the assumption that 'any combination or disjunction of predicates is equivalent to a single predicate'" (*PM* 1: 58–9). Mares builds this assumption into his models and then demonstrates Russell's claim.

The major difference between Mares and Linsky is that Mares has no ontological counterparts for non-predicative functions. For Mares, quantification over predicative propositional functions should be treated objectually, and quantification over non-predicative functions substitutionally. Linsky thinks of all propositional functions, including the non-predicative ones, as constructions of universal and individuals. It might be natural to think that, on this view, the axiom of reducibility would

[18] There may be a concern that the type differences between individuals and the various types of universals are not reflected in the language, which only shows the type differences between individuals and propositional functions.

certainly not look like a logical principle, but as a grand metaphysical claim. Linsky does in fact call it a "metaphysical principle … of great generality" (Linsky 1999, p. 108), and he suggests that "predicative propositional functions inherit some of the character of universals" (p. 107). He remarks: "Higher type propositional functions do not really introduce new properties of things. They may characterize new ways of thinking of or classifying them, but they do not introduce any new real universals" (p. 106). Mares' position makes this explicit, by identifying the predicative functions with universals, and so gives us a better understanding of how the axiom of reducibility could be viewed as a logical principle. Part of Linsky's position is that for Russell, at least at this stage, the mark of a logical truth is its extreme generality.

There are, however, some consequences of the assumption involved in Mares' strengthened Boolean thesis which have to be faced. Rouilhan also discusses a realist constructivism of the kind Linsky suggests, but rather than seeing this interpretation as validating the axiom of reducibility, he sees the axiom of reducibility as "hardly worth more … than a contradiction" (Rouilhan 1996, p. 273). Rouilhan is picking up on some remarks Gödel made in his contribution to the Schilpp volume on Russell, in particular Gödel's criticism of the axiom as making the constructivism an illusion (Gödel 1944, p. 143) and bringing in an infinity of "occult qualities" (p. 152). In fact, Max Black had already pointed out that given that at least at the type level of classes of ratios, there need to be distinct proxies for each real number, and thus there must be at least a continuum of predicative functions at that level (Black 1933, pp. 115–16). Russell and Whitehead themselves mention this point in the "Prefatory Statement" to Volume II (*PM* 2: vii). What Rouilhan means when he suggests that at least with an infinity of individuals the axiom of reducibility is not worth more than a contradiction is not that the system is contradictory, but rather that the axiom of reducibility contradicts the spirit of the interpretation, since constructing the propositional functions by the methods suggested in *Principia* would not lead to this many functions. This plethora of predicative functions at each type level is what Rouilhan attributes to Gödel's remark about "occult qualities".

But is it really correct to think that these infinite disjunctions and conjunctions of perfectly respectable predicative functions are somehow "occult qualities"? Gödel blended Rouilhan's constructions_1, constructions_2, and constructions_3 or Linsky's structured entities with his logical fictions (Gödel 1944, p. 143), and called into question the claim that these functions could be treated as incomplete symbols. But if we think of predicative functions as somewhat familiar structured entities, and conjunctions and disjunctions of these also as constructed entities, then we are not jumping to a whole new kind of entity by admitting these new predicative functions. They seem, in other words, less "occult" than some new universal which applies to all and only the members of whatever class we have.[19]

II. THE IDENTITY OF INDISCERNIBLES, RAMSEY, AND THE COUNTEREXAMPLES TO THE AXIOM OF REDUCIBILITY

There is a close connection between the axiom of reducibility and the definition of identity in *Principia Mathematica*, and this connection is important for the purported counterexamples to the axiom. Linsky has also recognized this close link (1999, pp. 104–9).

Russell defined identity in *Principia* as follows:

$$*13.01 \qquad x = y \,.\!=\!: (\phi) : \phi ! x \,.\supset .\, \phi ! y \qquad \text{Df.}$$

From this, given the axiom of reducibility, the general theorem follows:

$$*13.101 \qquad \vdash : x = y \,.\supset .\, \psi x \supset \psi y \,.$$

And from this we get:

$$*13.12 \qquad \vdash : x = y \,.\supset .\, \psi x \equiv \psi y \,,$$

which is the principle which justifies the substitution of identicals. Russell requires reducibility to prove ∗13.101 and therefore ∗13.12. He remarks that without the axiom we would be led to identities of different degrees, so that strict identity would have to be taken as a primitive idea. Perhaps a better way of putting it is that, without the axiom of reducibil-

[19] Linsky himself suggests that if the axiom is taken as saying that any arbitrary class of objects has a "nature", then it is clearly a highly contentious metaphysical principle (Linsky 1999, p. 106n.). On the other hand, he defends the axiom of reducibility by pointing out that "it does not assert the existence of some new, non-logical category of entity, but rather just of a predicative propositional function coextensive with an arbitrary propositional function" (p. 101).

ity, the definition ∗13.01 would be defective as a definition of identity since there might be distinct individuals which agreed on all their predicative properties, but differed with respect to some non-predicative properties.

The link between the axiom of reducibility and definition ∗13.01 is helpful in understanding Ramsey's criticism of the axiom of reducibility, and also Russell's own qualms about the axiom as he expressed them in *Introduction to Mathematical Philosophy.*

In his 1925 "Foundations of Mathematics", Ramsey criticized the axiom of reducibility in strong language as illegitimate and a defect of the whole system. His major concern is that the axiom is not a tautology (Ramsey 1925, p. 28).

Ramsey is famous for dividing the contradictions into two groups (1925, p. 20). One group involves the mathematical and logical contradictions and the other the semantical ones, or those which "contain some reference to thought, language, or symbolism, which are not formal but empirical terms." Now it is easy to read Ramsey as therefore rejecting the whole system of orders and just adopting a simple theory of types. This is not quite what Ramsey does. Instead, what he proposes is not a rejection of Russell's theory of orders, but a different understanding of what a predicative function is.[20] In fact, Ramsey explicitly includes as predicative functions, functions constructed by disjunction or conjunction which can include an infinite number of arguments (pp. 38–9). He distinguishes his account of predicate functions from what he takes to be Russell's by just this permission of including propositional functions formed from infinite truth functions (p. 39).

As Mares' reconstruction includes as predicative functions those functions formed by infinite Boolean operations, Mares' predicative functions will be co-extensive with Ramsey's, but not with what Ramsey takes to be Russell's. The axiom of reducibility will therefore be true, with Ramsey's understanding of "predicative function".

Ramsey says that on his view, therefore, all functions are predicative functions (1925, p. 41). What about the non-predicative function we ascribed to Napoleon above? Ramsey would say that while the expression contains the quantification over functions, this should not be understood

[20] Ramsey points out that, on his view, order is a characteristic of a symbol and not a "real" characteristic (Ramsey 1925, p. 47).

as giving us a new (ontological) function which is in some way distinct from the predicative functions from which it was developed. As Ramsey put it, "the fact it asserts to be the case does not involve the totality of functions; it is merely our symbol which involves it" (p. 41). This position conflicts neither with the position advocated by Landini nor with that advocated by Mares and Linsky, for on the former view there are no ontological correlates to propositional functions at all, while on the latter there are correlates only to the predicative functions.

Ramsey's insistence that Russell's predicative functions (which at the first level, at least, he calls "elementary functions" to distinguish them from his "predicative functions") helps us understand his counterexample to the axiom of reducibility:

> For it is clearly possible that there should be an infinity of atomic functions, and an individual *a* such that whichever atomic function we take there is another individual agreeing with *a* in respect of all the other functions, but not in respect of the function taken. Then $(\phi) \,.\, \phi!x \equiv \phi!a$ could not be equivalent to any elementary function of *x*. (1925, p. 57)

There will be no other individual but *a* to which this function will apply, but in Ramsey's world there is no predicative function (in what he takes to be Russell's sense) that individuates, so to speak, *a*. Ramsey has built his counterexample from a rejection of Leibniz's law with respect to atomic functions.

Russell's own objections to the axiom of reducibility are interesting in this regard. When he mentions qualms about the axiom in the Introduction to *Principia*, his concern does not seem to be with the truth of the axiom (or its logical status), but with whether it should be included as an axiom because it might follow from a "more fundamental and more evident axiom" (*PM* 1: 60). He is more critical in his *Introduction to Mathematical Philosophy*. Here he says that the admission of the axiom is a "defect" (*IMP*, p. 193). His reason for holding this is that it is possible that the axiom may be false in some possible world. But he does not directly come up with a possible world in which the axiom is false. Instead, he takes the axiom together with definition ∗13.01 as a "generalized form of Leibniz's law" (*IMP*, p. 192). It is the necessary truth of the identity of Indiscernibles which he then calls into question. "There might well, as a matter of abstract logical possibility, be two things which had exactly the same predicates, in the narrow sense in which we have been using the word 'predicate'" (*ibid.*, p. 192). This by itself would not call the axiom of reducibility into question unless these items were in fact distinguished by a non-predicative propositional function, and Russell does not supply us with that possibility.[21] As we have seen, Mares, in giving a reconstruction that validates the axiom of reducibility, builds into his models a very strong thesis of the identity of indiscernibles. But if Leibniz's law is rejected as a definition of identity, this by itself would not show that the axiom of reducibility is flawed.[22]

The counterexample given by Wittgenstein in a 1913 letter to Russell does not mention the identity of indiscernibles, but makes very strong assumptions about the existence of predicative functions:

> … imagine we lived in a world in which nothing existed except $\aleph_0$ *things* and, over and above them, ONLY a *single* relation holding between infinitely many of the things and in such a way that it did not hold between each thing and every other and further never held between a finite number of things. It is clear that the axiom of reducibility would certainly *not* hold good in such a world. But it is also clear to me that whether or not the world in which we live is really of this kind is not a matter for logic to decide.[23]

The principle that whenever there is a predicate propositional function there are also the predicative propositional functions which are formed by the Boolean operators (including negation) is not respected in this example.

III. CONCLUDING REMARKS

[21] Hazen (1983, p. 364) has supplied a possibility which involves a rejection of materialism and propositional attitudes as being individuated by the propositions involved: "It would seem possible for there to be two spirits sharing all their non-psychological properties and believing … exactly the same propositions at every level lower than some specified level, but differing in their attitudes toward some proposition of that level." This particular counterexample requires that the property of believing *p* has to be non-predicative if *p* is a proposition which contains a non-predicative function. Russell had rejected understanding belief as a relation between persons and propositions, but while he gave an account of the truth of propositions involving non-predicative functions, he did not given an account of beliefs involving these.

[22] Ramsey rejects definition ∗13.01, but sees the rejection of Leibniz's law as being independent from the question of the validity of the axiom of reducibility (1925, p. 30).

[23] See Wittgenstein 1974, p. 39. The letter in question is R.23.

The axiom of reducibility appeared false to Ramsey because he took Russell's constructions of predicate propositional functions to be limited to those which can be formed by finite applications of Boolean operations to atomic functions. The axiom appeared suspect to Gödel (and later Rouilhan) because it posits the existence of so many predicative propositional functions. Gödel thought that if ramification has its roots in constructivism, this posit is unjustified as is the axiom of reducibility. His own view was that we should be Platonists about sets, and so we need neither ramification nor the axiom. While Rouilhan rejected Gödel's understanding of constructivism, he still thought that the operations of construction won't result in all the propositional functions required for the truth of the axiom of reducibility even if we don't think of constructions as mental. If we think of propositional functions as real entities, the axiom appears to make non-logical existence claims. If we give a nominalist interpretation, then the axiom seems false unless we see it as somehow capturing what we would intend to say by an infinite disjunction or conjunction of predicative propositional functions, were we able to formulate these. Finally, on Linsky's view, especially as developed by Mares, the axiom is true because of a truth about the constructions which are predicative propositional functions which are themselves abstracted from facts.

On this view, the axiom plays a role similar to the role that quantificational axioms play. If we could express infinite conjunctions of propositions, as Ramsey suggested, we wouldn't need to have separate quantificational axioms. Similarly, if we could express infinite conjunctions and disjunctions of predicative propositional functions, we wouldn't need the axiom of reducibility to give us the predicative propositional function which is equivalent to these. This last view is attractive as it lends support to the understanding of the axiom as something that is at least more of a logical principle than a purely metaphysical one. Russell's own qualms about the axiom may perhaps be seen as a reason for wondering whether this last view in fact does capture the logic of *Principia*, but it is not clear whether his difficulty is solely with the axiom or with Leibniz's law.[24]

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