RUSSELL ON LOGICISM AND COHERENCE

Conor Mayo-Wilson
Philosophy / Carnegie Mellon U.
Pittsburgh, PA 15213, USA
conormw@andrew.cmu.edu

According to Quine, Charles Parsons, Mark Steiner, and others, Russell's logicist project is important because, if successful, it would show that mathematical theorems possess desirable epistemic properties often attributed to logical theorems, such as aprioricity, necessity, and certainty. Unfortunately, Russell never attributed such importance to logicism, and such a thesis contradicts Russell's explicitly stated views on the relationship between logic and mathematics. This raises the question: what did Russell understand to be the philosophical importance of logicism? Building on recent work by Andrew Irvine and Martin Godwyn, I argue that Russell thought a systematic reduction of mathematics increases the certainty of known mathematical theorems (even basic arithmetical facts) by showing mathematical knowledge to be coherently organized. The paper outlines Russell's theory of coherence, and discusses its relevance to logicism and the certainty attributed to mathematics.

Bertrand Russell famously claimed that mathematics is reducible to logic in the sense that (a) all mathematical terms (e.g. real number, integer, natural number, etc.) can be *defined* using only logical constants (e.g. equality, negation, etc.), and (b) all mathematical theorems can be *derived* from purely logical axioms. The conjunction of these two theses is called "logicism".

One purported consequence of logicism is that mathematical concepts and theorems *inherit* the important epistemological properties of logical concepts and theorems respectively. Why? If mathematical assertions are in fact abbreviations for logical assertions, and if all mathematical theorems can be derived from logical ones, then mathematical theorems must possess the important epistemological properties that are frequently attributed to logical theorems. In particular, one might argue that logi-

russell: the Journal of Bertrand Russell Studies
The Bertrand Russell Research Centre, McMaster U.
n.s. 31 (summer 2011): 63–79
ISSN 0036-01631; online 1913-8032

cism implies that mathematical theorems are necessary, capable of being known a priori, are certain and/or self-evident, as these are properties typically attributed to logical truths. I call this thesis "epistemic inheritance", and abbreviate it by EI.

According to many philosophers, Russell's primary motivation for defending logicism was to defend EI. For example, in describing the motivations for reducing mathematics to logic, Quine attributes EI to the logicists:

> Ideally, the obscurer [mathematical] concepts would be defined in terms of the clearer ones so as to maximize clarity, and the less obvious laws would be proved from the more obvious ones so as to maximize certainty. Ideally, the definitions would generate all the concepts from clear and distinct ideas, and the proofs would generate all theorems from self-evident truths.[1]

Similarly, Imre Lakatos, Charles Parsons, and Mark Steiner, amongst others, have all argued that EI was a central motivation for Russell's attempt to reduce mathematics to logic.[2]

As has been pointed out by Andrew Irvine and Martin Godwyn,[3] however, Russell never believed EI, and moreover, EI *contradicts* Russell's views on the relationship between mathematics and logic as they are explicitly stated in several lectures, papers, and books, including even the Preface to *Principia Mathematica*, in which he writes:

---

[1] See Quine 1969, p. 70. Quine never mentions Russell in this passage, but later parts of the essay (pp. 73–4) clearly indicate that Russell is one of the logicists who drew unacceptable epistemological conclusions from the reduction of mathematics to so-called logic. Also, Frege is not mentioned in the essay, which suggests that Quine is thinking primarily of Russell's logicist programme. Of course, Quine thinks that logicism did not meet the epistemic goals he attributes to Russell, as foundational work showed that mathematics is reducible to set theory, not logic, and furthermore, that the axioms of set theory are far from certain and self-evident.

[2] See Lakatos 1978, p. 16, Parsons 1967, pp. 193, 197, and Steiner 1975, pp. 14–17, 24. I learned of many of these passages from Irvine 1989.

[3] See Irvine 1989 and Godwyn and Irvine 2003. Douglas Lackey makes the same observation, in significantly less detail, in his editorial introduction to the 1907 lecture that provides Irvine and Godwyn with greatest support for their thesis. See Russell 1973, p. 255. Another interesting, sociological question, which I will not try to answer, is "why did philosophers ever attribute EI to Russell?" Irvine 1989 speculates that, because Frege does endorse EI, the tendency to group together Frege and Russell's programmes has led to a misunderstanding of Russell's motivations for logicism. This suggestion was also, independently, made to me by Paddy Blanchette.

> … the chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e. it must lie in the fact that the theory in question enables us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence *the early deductions, until they reach this point, give reasons rather for believing the premisses because true consequences follow from them, than for believing the consequences because they follow from the premisses.*
>
> (*PM* 1: v; my italics)

If Russell never argued for EI, and if EI contradicts other views that Russell did hold, then the following question is immediate: did Russell attribute any important epistemological consequences to logicism? More specifically, does logicism have important consequences for *mathematical* epistemology?[4]

According to Irvine and Godwyn, the answer to the above questions is "yes". Under their interpretation, Russell attached importance to logicism not because it increases the certainty that mathematicians attach to theorems of analysis, let alone basic arithmetical facts, but rather, because it improves mathematical *explanation* and facilitates *discovery*. In reducing mathematics to logic, they claim, Russell's central motivations are to (1) show how seemingly different theorems (and their proofs) can be derived from a common core of axioms, thereby *explaining* said theorems, and (2) uncover techniques for proving theorems that have been stated but are currently unproven, and (3) suggest new, fruitful concepts and theorems in mathematics.

I agree with Irvine and Godwyn's contention that Russell attached importance to the role of logic in improving mathematical explanations and in aiding discovery. In fact, later in the paper, I provide additional arguments and examples in support of this claim.

However, the central thesis of this paper is that Russell did, in fact, believe that the reduction of mathematics to logic increases the certainty of mathematical theorems (even basic arithmetical facts). Like Irvine and Godwyn, I agree that this added certainty does not result from mathematical theorems inheriting the epistemological properties of logical

[4] Note that Russell attributes many important consequences to logicism, some of which pertain to physics and philosophy broadly. See, for example, "The Philosophical Importance of Mathematical Logic" (Russell 1913, pp. 284–94, and *Papers* 6: 33–40). I will not discuss these features of Russell's views about the importance of logicism; I am only concerned with the importance of logicism to mathematical epistemology.

theorems. Rather, in reducing mathematics to logic, the certainty attached to mathematical theorems is increased because mathematical knowledge is shown to be *coherently organized*, and for Russell, coherence is an indicator of truth.[5]

This paper contains two parts. Following Irvine and Godwyn, I first argue that Russell never endorsed EI. In the second half of the paper, I discuss Russell's theory of coherence and how it bears on logicism.

Before beginning, however, two caveats are in order. First, in my analysis of Russell's work, I quote from articles and books that together span several decades of Russell's philosophical career. Because Russell's views on a number of important philosophical issues changed markedly during his life, one might wonder whether there is enough consistency in his views to support the reading I urge below. As I hope to show, however, Russell's views on coherence changed little between 1907 and the 1950s. Several passages of nearly identical timbre and wording occur in his writings over the course of 50 years.

Second, I wish to emphasize that Russell likely attributed many important philosophical consequences to logicism. My discussion of Russell's theory of coherence is meant only to deepen our understanding of the many reasons that Russell was motivated to his work in logic, and why such work might continue to be important.

1. LOGICISM, EXPLANATION, AND DISCOVERY

What is the importance of logicism, according to Russell? One standard interpretation, thoroughly debunked by Irvine and Godwyn, is that logicism implies that mathematical theorems inherit all or some of the desirable epistemological properties standardly attributed to logical theorems, such as necessity, aprioricity, certainty, and self-evidence. Yet

[5] Thagard *et al.* 2002 make a passing remark to this effect, but their interpretation of Russell is not defended in any substantial way. Russell also argues that in reducing mathematics to logic, one also acquires a *minimum* set of postulates that are *simpler* (in virtue of the number of constituents they contain). Because Russell claims, in *The Problems of Philosophy*, that simplicity is an indicator of truth of epistemology generally (PP_2, p. 23), one might infer that logicism implies mathematics is more certain because it shows how various theorems can be derived from simpler premisses. Unfortunately, this is a point on which Russell seems to have changed his mind, as he claims that, in logic, a preference for fewer and simpler premises is "merely aesthetic". See Chap. 9, "Epistemological Premisses", *An Inquiry into Meaning and Truth* (1940).

Russell repeatedly denies that axioms always possess these properties.

First, Russell emphasizes that the primitive axioms, from which mathematical theorems are proven, are less obvious than their consequences. "The most obvious and easy things in mathematics", he writes, "are not those that come logically at the beginning; they are things that, from the point of view of logical deduction, come somewhere in the middle."[6] It follows that, for Russell, mathematical axioms are not always self-evident.[7]

Second, since the search for mathematical axioms lies on the frontier of mathematical knowledge, and concepts, such as that of a class, had led to contradictions in the recent past, Russell denied that logical principles are certain. He writes, "In mathematical logic it is the conclusions that have the greatest degree of certainty: the closer we get to the ultimate premisses the more uncertainty and difficulty do we find."[8]

Third, the axioms of infinity and multiplicativity, which assert the existence of particular classes, are not necessary, as one can imagine worlds, for example, in which only finitely many objects exist. In fact, Russell claims that no existential postulate is a logical truth:

> Among "possible" worlds, in the Leibnizian sense, there will be worlds having one, two, three, ... individuals. There does not even seem any logical necessity why there should be even one individual—why, in fact, there should be any world at all.[9]

Finally, Russell argues that mathematical axioms are justified by a pseudo-inductive strategy, namely, whether they can be used to derive

---

[6] *IMP*, p. 2. Nearly identical passages occur in *PM* 1: 1 and Russell 1907, pp. 273, 294. In the 1907 lecture, Russell also explicitly claims that "Some of [Frege's] logical premises are not very obvious" (p. 279).

[7] According to Russell, some logical truths are self-evident, such as in the case of the law of non-contradiction. Russell claims that the law of excluded middle is also obvious, but this is a point on which he changed his mind. See Russell 1907, p. 279.

[8] See Russell 1913, p. 285; *Papers* 6: 33. Similarly, he claims that by modifying Frege's axioms to avoid the paradoxes, one cannot be assured "with certainty" the resulting axioms do not imply a contradiction (1907, p. 280). Here Russell's claim does not anticipate Gödel's second incompleteness theorem. Rather, he seems to indicate one can only see whether a set of axioms implies a contradiction by repeatedly trying to prove one.

[9] *IMP*, p. 203. This is another point on which Russell changed his opinion. One axiom of *Principia Mathematica* postulates the existence of at least one object, but in a note to the above passage Russell claims that this axiom is a "defect of logical purity".

accepted theorems of arithmetic and analysis. Hence, although the arithmetical theorems that they justify might be a priori, it's not clear that Russell's axioms are capable of being known a priori. Moreover, concerning the existential axioms that form a foundation for most of mathematics, Russell claims, "Existence-theorems, where individuals are concerned, are now theorems as to existence in the *philosophical* sense; hence it is natural that they should not be demonstrable a priori."[10] Additionally, because Russell often seems to identify a priori and necessary truths, his arguments against the necessity of certain logical axioms also likely indicate that he denied their status as a priori truths.

Even if Russell had claimed that logical axioms always possessed the desirable epistemic properties discussed above, there is yet another reason to think that Russell never endorsed EI. Recall that logicism is the conjunction of the two theses: (a) all mathematical terms can be *defined* using only logical constants, and (b) all mathematical theorems can be *derived* from purely logical axioms. Call these the "definitional" and "derivational" theses, respectively. Despite Russell's repeated insistence that mathematics and logic are identical, he admits that the truth of the derivational thesis is not settled by the work in *Principia Mathematica*. At the conclusion of *Introduction to Mathematical Philosophy*, he writes:

> We have sufficiently defined the character of the primitive *ideas* in terms of which all the ideas of mathematics can be *defined*, but not of the primitive *propositions* from which all the propositions of mathematics can be *deduced*. **This is a more difficult matter, as to which it is not yet known what the full answer is.**
>
> We may take the axiom of infinity as an example of a proposition which, though it can be enunciated in logical terms, **cannot be asserted by logic to be true.** (*IMP*, pp. 202–3; his italics, my bolding)

Hence, although mathematical terms, like "real number", might merely be abbreviations for logical terms, Russell is unsure whether mathematical theorems, about real numbers for example, can be proven from purely logical axioms. Following the above passage, Russell argues that the derivational thesis of logicism might still be true, but proving it requires either (i) widening the concept of "tautology" so as to include existential assertions like the axiom of infinity, or (ii) proving standard mathemati-

[10] "The Paradox of the Liar", Russell 1906, fol. 65.

cal theorems without the aid of existential assertions like the axiom of infinity. In retrospect, it is clear that (ii) is impossible: one can construct finite models of ZF minus the axiom of infinity (thus proving the axiom's independence from the remaining axioms), or models of ZF in which the axiom of choice fails. Furthermore, without such existential assertions, many standard theorems of mathematics are unprovable. If the derivational thesis of logicism were true, then such assertions must shown to be logical, which is what Russell explicitly denies in some of the passages cited above.

This is an important point. Although Russell repeatedly endorses logicism, thus underscoring its importance, his later writings reveal a hesitation concerning the derivational thesis. This hesitation suggests that there are two conclusions we ought to draw about how to understand Russell's logicism: (1) Russell attached great importance to the definitional thesis of logicism alone, and (2) the search for axioms for all of mathematics is *independently valuable whether or not the axioms necessary for deriving mathematical theorems are purely "logical"*.

What, then, is the epistemic value of logicism (if any)? According to Irvine and Godwyn, there are at least two philosophically important consequences of logicism for mathematical epistemology that simultaneously explain Russell's explicitly stated goals for his work in logic and his admiration for nineteenth-century geometry and set theory. First, in reducing mathematics to logic, one obtains better *explanations* of many mathematical theorems by unifying their proofs under a single set of axioms. Second, one discovers new and fruitful *concepts*, such as that of equinumerosity and cardinal number, and one discovers new techniques for proving theorems. However, Russell never claimed that logicism implies that mathematical theorems inherit properties like necessity, aprioricity, self-evidence, and so on from logical theorems. So the oft-repeated interpretation of Russell, in which he is depicted as defending EI, is not supported by textual evidence.

A central question, however, remains: is there any sense in which the reduction of mathematics to logic increases the certainty of mathematical theorems? In other words, even if mathematical theorems do not inherit self-evidence from the logical axioms from which they are eventually derived, is there any way in which the logical reconstruction of mathematics increases the evidence one has for accepting a given theorem as true? Russell repeatedly suggests that the answer to this question is "yes", and the balance of the paper is dedicated to explaining why.

2. COHERENCE AND THE EPISTEMIC SIGNIFICANCE OF LOGICISM

For Russell, the law of non-contradiction is an a priori, necessary, self-evident truth. If a set of propositions implies a contradiction, then at least one of its members must be false. Therefore, for Russell, the existence of a potential contradiction or inconsistency in one's beliefs counts as evidence against such beliefs. Because the logical analysis of mathematics led to the discovery, and (one hopes) eradication of paradoxes concerning set construction principles, it follows that logicism, if true, would reduce the chance of error and evidence against the theorems of mathematics. This much seems to be agreed upon by Russell scholars.

What is perhaps less well recognized is that Russell also endorses the converse of the above thesis: the absence of a contradiction in a set of assertions counts as evidence in *support* of such assertions. In discussing Frege's premisses for arithmetic, and their modification due to the discovered contradiction, Russell writes:

> ... if we have seemed to discover precisely *why* our previous premises led to contradictions, so that what (apart from consequences) seemed reasonably true, now seemed obviously false, and if the whole kind of reasoning from which the contradictions sprang is ruled out by our new premises, we may have a reasonable confidence that we have at least made the right kind of modification, and that if more modification is required, it will be more of the same.... Thus, Frege's premises undoubtedly give a first approximation, and the exact **truth** must be very much like them. (Russell 1907, p. 280; his italics, my bolding)

A similar assertion is made at the outset of *Principia Mathematica*:

> The proof of a logical system is its adequacy and its *coherence*. That is: (1) the system must embrace among its deductions all those propositions which we believe to be true and capable of deduction from logical premises alone ... and (2) the system must lead to no contradictions. (*PM* 1: 12; my italics)

If by "proof of a logical system" we understand Russell to mean "truth of the axioms", then the two passages are very similar. Yet Russell's conclusion is odd. There are many consistent axiomatizations of mathematics, and in the absence of an argument that logical consistency of a set of propositions is, modulo some additional assumptions, evidence for their truth, Russell's conclusion that either Frege's axioms or those in *Prin-*

cipia must be near to the "truth" seems unjustified.

Russell does provide such an argument, and the first step in understanding the argument is recognizing that, despite his suggestion in the second passage above, "coherent" means more than logically consistent. In several essays Russell argues that, when many propositions in a set bear some relation *R* to one another, then the set becomes more "obvious", "probable", or "credible", than each proposition individually. Russell later collects all such relations *R* together under a single relation that he calls "coherence". In the *Principia* passage immediately above, the relation *R* is consistency. In "The Regressive Method of Discovering the Premises of Mathematics", the relation *R* is derivability:

> Assuming the usual laws of deduction, two obvious propositions of which one can be deduced from the other both become more nearly certain than either would be in isolation; and thus in a complicated deductive system, many parts of which are obvious, the total probability may become all but absolute certainty. (1907, p. 279)

By 1912 Russell realized that derivability is one of several ways in which one proposition might provide evidence for or against another. Thus, in a passage of nearly identical timbre to the previous one, Russell replaces "derivability" with "mutual coheren[ce]" and "obviousness" with "probability":

> In regard to probable opinion, we can derive great assistance from *coherence*, which we rejected as the *definition* of truth, but may often use as a *criterion*. A body of individually probable opinions, if they are mutually coherent, become more probable than any one of them would be individually.[11]

In his 1959 philosophical autobiography, Russell gives this thesis the name "the coherence theory of probability":

[11] PP_2, p. 140. In an earlier passage (pp. 25–6), Russell makes a similar claim: "… by organizing our instinctive beliefs and their consequences, by considering which among them is most possible, if necessary, to modify or abandon, we can arrive, on the basis of accepting as our sole data what we instinctively believe, at an orderly systematic organization of our knowledge, in which, though the *possibility* of error remains, its likelihood is diminished by the interrelation of the parts and by the critical scrutiny which has preceded acquiescence."

I do not accept the coherence theory of *truth*, but there is a coherence theory of *probability* which is important and I think valid. Suppose you have two facts and a causal principle which connects them, the probability of all three may be greater than the probability of any one, and the more numerous and complex the inter-connected facts and principles become, the greater is the increase of probability derived from their mutual coherence. (*MPD*, p. 204)

When Russell wishes to be careful in distinguishing the mathematical concept of probability from the concept of evidential support, he speaks of a coherence theory of "credibility" rather than probability:

Given a number of propositions, each having a fairly high degree of intrinsic credibility, and given a system of inferences by virtue of which these various propositions increase each other's credibility ... [we] arrive at a body of interconnected propositions having, as a whole, a very high degree of credibility. (*HK*, p. 395)

The above passages, similar in wording and spirit, appear in works spanning over 50 years of Russell's life. Moreover, the contexts in which the passages occur vary widely. The passages from *Principia* and the 1907 essay occur within the context of discussion of what might constitute evidence for mathematical axioms and a formal system more generally. In contrast, the passages from *The Problems of Philosophy* occur within discussions of scepticism concerning the existence of physical objects, and they explain how Russell's coherence theory is relevant to philosophical theses in general. Finally, the passages from *Human Knowledge* and *My Philosophical Development* occur within discussions of the use of probability in inductive inference and discovery of causal relationships, respectively.

The frequency with which Russell discusses his coherence theory, the period of time over which the discussions occur, and the variety of subjects to which he found it applicable, together suggest that Russell attached importance to the theory. My central claim is that, in reducing mathematics to logic, Russell was attempting, in part, to show that mathematical knowledge can be coherently organized, thereby increasing one's evidence for mathematical assertions. Before defending this claim, it is necessary to explain what Russell's coherence theory asserts. There are three aspects of the theory that deserve clarification: (1) what does Russell mean by coherence? (2) what relations do coherent propositions bear to one another? and (3) of what epistemic properties (e.g. obviousness, high probability, high credibility) of a set of propositions does coherence provide evidence?

First, it is important to ascertain what Russell means by coherence. In the *Problems*, Russell explicitly defines coherence as follows: "Two propositions are coherent when both may be true, and are incoherent when one at least must be false" (PP_2, p. 123). In the above passage, and others, however, Russell discusses the coherence of more than two propositions. How is the coherence of a set of many propositions defined? Although Russell does not say so explicitly, the above passage suggests that a set of propositions ("body of individually probable opinions") is coherent just in case any two such propositions are pairwise coherent ("mutually coherent").

Unfortunately, this definition does not seem to be the one employed when Russell discusses examples of incoherent beliefs. Consider Russell's discussion of dreams in the *Problems*, where he claims, "If our dreams, night after night, were as coherent one with another as our days, we should hardly know whether to believe the dreams or the waking life" (p. 140). First, the phrase "were as coherent ... as" suggests that coherence occurs in various gradations, whereas Russell's earlier definition renders coherence a binary variable: propositions cohere if and only if they are pairwise non-contradictory. Thus, Russell likely uses the word "coherence" to refer to a quantitative measure of how probable one proposition is, on supposition that another proposition (or set of other propositions) is true.

Moreover, Russell's dream example makes little sense if one interprets coherence as pairwise non-contradiction. If one were to form one's beliefs on the basis of dreams from successive evenings, the resulting body of beliefs need not be pairwise contradictory. In fact, such beliefs taken *together* might be satisfiable and/or fail to imply a contradiction. Why? Different dreams may have entirely different content matter, and so, despite their fantastical character, they may not contradict each other in the slightest. Therefore, it is likely that Russell used the word "coherence" to refer to a quantitative measure of the evidential support that one proposition lends to another.

This leads us to the second question: what relation might a proposition p bear to proposition q such that p lends (or decreases) evidential support to q? In other words, what properties of a set of propositions Γ determine its coherence? A cursory glance at the above passages (in the context) suggests the following list:

- Is Γ satisfiable?
- Is a contradiction derivable from Γ?
- Are some propositions in Γ derivable from others?
- How many propositions does Γ contain?
- Are the propositions in Γ complex or simple?[12]

Furthermore, the passages in which Russell discusses inductive inference suggest that he thinks there might be probabilistic analogues of some of the logical notions in the list above.[13] For example, when Russell speaks of the "inter-connectedness" of propositions in a set Γ, he might be discussing whether, for any proposition $\gamma \in \Gamma$, the probability/credibility of γ conditional on $\Gamma \setminus \{\gamma\}$ is greater than, equal to, or less than the probability/credibility of γ *simpliciter*.[14] Russell is, however, not explicit on this point, and although conditional probability affords one way of making the notion of coherence precise, any number of measures might also be reasonable and lead to different assessments.

Unfortunately, even among the relations that Russell explicitly claims constitute measures of coherence, a number are still vague. Consider, for example, Russell's claim that the number of propositions in a set is one criterion for judging whether the set is coherent. When represented in a formal language, any finite number of assertions can be written as a single assertion by forming a long conjunction. So, one might ask Russell, "can propositions be delineated so that there is a distinction between finitely many propositions and only one?" Similarly, one might wonder whether cardinality is an appropriate measure of "number" of propositions in an infinite set, if one's goal is to quantify the complexity of the set.

The last aspect of Russell's coherence theory in need of clarification is the following: of what epistemic properties of a set of propositions (e.g.

[12] Russell defines one proposition to be simpler than another if the former's constituents are a proper subset of the latter.

[13] In his later writings Russell claims that even demonstrative (i.e. deductive) inference is probabilistic because long, intricate mathematical arguments require significant care in reading and constructing to avoid error (see Russell 1948, p. 383). For the late Russell, then, logical inferences are the endpoint on a continuum of non-demonstrative inferences.

[14] Russell implicitly employs the concept of conditional probability when explaining how *credibility* of a proposition ought to be determined by probabilities (i.e. *frequencies*), when such probabilities are known (1948, p. 385).

obviousness, high probability, high credibility) does coherence provide evidence? I claim that, although Russell repeatedly speaks of a coherence theory of *probability*, he likely uses the word "probability" to refer to a measure of evidential support that does not satisfy the axioms standardly used in probability theory today. Why? Recall Russell's claim that "A body of individually probable opinions, if they are mutually coherent, become more probable than any one of them would be individually." If taken literally, the claim contradicts a simple fact about probability, namely that the conjunction of two or more propositions is less probable than either conjunct, i.e., $p(A \& B) \leq p(A), p(B)$ for all A and B. However, Russell recognizes a far more general fact about probability, namely that if ϕ entails ψ, then ϕ is less probable than ψ. In the *Problems*, he writes:

> For the probability that Socrates is mortal is greater, on our data, than the probability that all men are mortal. (This is obvious, because if all men are moral, so is Socrates; but if Socrates is mortal, it does not follow that all men are mortal.)
>
> (PP_2, p. 80)

This passage suggests that Russell uses the word "probability" in two different ways in his earlier writings, and perhaps this is why he felt it necessary to distinguish carefully between probability and credibility in *Human Knowledge*, which contains Russell's most sustained discussion of probability and its interpretations (e.g. frequentist and subjective Bayesian).[15]

Thus, Russell's theory of coherence is, unfortunately, incomplete. But the above sketch is sufficient for discussing its importance with respect to logicism. As the reduction of mathematics to logic eradicates contradictions in one's premisses, finds *logically* simple premisses, increases the number of propositions that one accepts by producing new theorems, and uncovers derivability relations between propositions, it follows that the logical analysis of mathematical theorems renders mathematical knowledge more coherent, and therefore, according to Russell, more

[15] There is one snarl for this interpretation. Notice that Russell again uses the word "probability" in the passage from *My Philosophical Development* quoted above, and that passage was written *after* he wrote *Human Knowledge*. I expect that the use of the word "probability" was a mere lapse of thought, as Russell carefully distinguishes between probability and credibility several pages before that passage.

probable.

We are now ready to ask the questions, "According to Russell, why does coherence increase the credibility of a set of propositions? Why is coherence a mark of or truth?" In particular, one must answer the obvious objection that coherence is not a mark of truth because fictional novels, plays, and so on, can be coherent.

Three of the passages cited above (from Russell 1907, p. 279; PP_2, p. 140; and HK_2, p. 395) suggest that coherence of a set of propositions is a mark of truth only when at least some of the propositions are *independently supported by available evidence.* In the passage from "The Regressive Method of Discovering the Premises of Mathematics", Russell indicates coherence is an indicator of truth for a set of propositions only when the propositions themselves are "obvious". In the passage from the *Problems*, coherence lends greater support to beliefs that are "individually probable", and in *Human Knowledge* coherence is relevant only to propositions "each having a fairly high degree of intrinsic credibility". A passage from *Human Knowledge* confirms this point:

> The edifice of knowledge may be compared to a bridge resting on many piers, each of which not only supports the roadway but helps the other piers to stand firm owing to the interconnecting girders. The piers are the analogues of the propositions having some intrinsic credibility, while the upper portions of the bridge are the analogues of what is only inferred. (HK_2, pp. 395–6)

In other words, Russell's epistemology mixes foundationalist and coherentist themes. Some justified beliefs are "probable", "obvious" or gain the status of "facts" because they are self-evident or directly inferred from observation. Others gain support from adding coherence to a set of independently obvious or probable opinions. Thus, a piece of fiction, for Russell, would lack coherence, as most of its claims would be ill supported by evidence like direct observation or obviousness, and so there is no reason to believe the worlds constructed in novels, plays, and movies are real. In contrast, the laws of physics cohere with "facts" learned by direct observation of the world, and the axioms of mathematics cohere with the basic "facts" of mathematics described in elementary arithmetic and geometry.

Thus, Russell anticipates the objection that coherence is not always an indicator of truth. Only when some propositions in a set Γ are probable, self-evident, or obvious, Russell claims, do the propositions in Γ gain support from the coherence of the entire set. Still a question lingers: if coherence is not an indicator of truth generally, why does it ever provide *additional* evidence for a set of propositions?

Although Russell does not answer this question specifically, his discussion of criteria for "good" premisses for mathematics and causal inference, I think, can be used to provide a Russellian response. Clearly, when one proves a complex mathematical theorem from simple premisses, one gains greater evidence for the truth of the theorem. By exactly symmetric reasoning, proving that particular premisses are *necessary* for deducing an obvious arithmetical fact provides greater evidence for the premisses; one has, in essence, obtained a proof of the premisses from an obvious arithmetical fact. That the assertions being proven are called "premisses" and that the assumptions in such proofs are "theorems" makes no difference. This is what Russell means when he claims that a formal system ought to be judged by its "adequacy".

One might object that accepted mathematical theorems might be deduced from infinitely many different sets of axioms, and so there are rarely axioms that can be proven to be "necessary" for deriving ordinary mathematics. Of course, certain axioms are necessary for proving certain facts *modulo other axioms.* I say more about this below. So I hypothesize that, for Russell, coherence is a measure of "how necessary" the premisses in an axiomatic system are for deriving the consequences. This is why Russell emphasizes that the "interconnectedness" of propositions in a "complicated deductive system" is an indication of their coherence.

To understand this point more fully, it is helpful to consider Russell's analogy between mathematics and the sciences. In inductive inquiry, one can never show that a law or set of laws (e.g. Newton's laws) are necessary in order to explain particular phenomena (e.g. projectile motion). One can, however, gain confidence that if the true laws were to differ from hypothesized ones in specific ways, then one should not have observed the phenomena that he or she did. Hence, hypothesized laws become "approximately necessary" for explaining observable facts. Russell explains:

> In induction, if p is our logical premise and q our empirical premise, we know that p implies q, and in a text-book, we apt to begin with p and deduce q. But p is only believed on account of q. Thus we require a greater or less probability that q implies p, or, what comes to the same thing, that not-p implies not-q. If we can *prove* that not-p implies not-q, i.e. that p is the only hypothesis con-

sistent with the facts, that settles the question. But usually what we do is to test as many alternative hypotheses as we can think of. (1907, p. 274)

Similarly, in mathematics, one can show that a particular axiom is necessary for a proof *modulo other axioms.* For example, modulo ZF and classical logic, the axiom of choice is known to be necessary to prove any number of elementary mathematical results. The theorem that a countable union of countable sets is countable, for instance, is known to require the axiom of choice.[16] Continuing with this example, although the axiom of choice might not be necessary for such a proof modulo some other set of axioms, if one thinks that the axioms one has used in showing the necessity of choice are few in number and simple, then one has gained greater evidence that choice is absolutely essential in explaining the mathematical "phenomena". This last hypothesis also suggests a relationship between Russell's views on simplicity and coherence. In short, coherence of a set of propositions can be an indicator of truth because it measures to what degree one's premisses are "approximately necessary" for proving those facts one does take as self-evident.

Russell's theory of coherence, therefore, provides a partial answer to a desire for certainty in mathematics: even if the reduction of mathematics to logic is not completely successful (because certain axioms may not be logical), the increased organization of mathematical knowledge yields a coherent body of theorems that is of greater security than the disconnected collection of mathematical subjects with which one began.[17]

REFERENCES

Godwyn, Martin, and Andrew D. Irvine. 2003. "Bertrand Russell's Logicism". In Nicholas Griffin, ed. *The Cambridge Companion to Bertrand Russell*. Cambridge, UK: Cambridge U. P.

Irvine, Andrew D. 1989. "Epistemic Logicism and Russell's Regressive Method".

---

[16] I doubt that Russell would countenance this theorem as a basic fact, but it is one that is essential in modern mathematics.

[17] Acknowledgements: I am first and foremost indebted to Teddy Seidenfeld for helpful comments on the earliest drafts of this essay. The arguments in the paper also benefited greatly from discussions with participants at the *PM*@100 conference. Special thanks are due to Marjorie Senechal for prompting me to consider Russell's views on probability more carefully and to revisit sections of *Human Knowledge.*

Philosophical Studies 55: 303–27.
Lakatos, Imre. 1978. “Infinite Regress and Foundations of Mathematics”. In his *Mathematics, Science, and Epistemology*. Ed. John Worral and Gregory Currie. New York: Routledge.
Parsons, Charles. 1967. “Foundations of Mathematics”. In *The Encyclopedia of Philosophy*. Vol. 5. Ed. Paul Edwards. New York: Macmillan.
Quine, Willard Van Orman. 1969. “Epistemology Naturalized”. In his *Ontological Relativity and Other Essays*. New York: Columbia U. P.
Russell, Bertrand. 1906. “The Paradox of the Liar”. Unpublished manuscript. RAI 220.010930. Bertrand Russell Archives. McMaster University.
—. 1907. “The Regressive Method of Discovering the Premises of Mathematics”. In Russell 1973.
—. 1913. “The Philosophical Importance of Mathematical Logic”. In Russell 1973 (titled erroneously “The Philosophical Implications …”). Reprinted in *Papers* 6.
—. 1919. *Introduction to Mathematical Philosophy*. London: Allen & Unwin. Online: http://people.umass.edu/klement/russell-imp.html
—. 1940. *An Inquiry into Meaning and Truth.* New York: Norton.
—. 1948. *Human Knowledge: Its Scope and Limits*. New York: Simon and Schuster.
—. 1959. *My Philosophical Development*. London: Allen and Unwin.
—. 1973. *Essays in Analysis*. Ed. Douglas Lackey. New York: George Braziller.
—. 1997. *The Problems of Philosophy.* Introduced by John Perry. New York: Oxford U. P.
Steiner, Mark. 1975. *Mathematical Knowledge*. Ithaca, NY: Cornell U. P.
Thagard, Paul, Chris Eliasmith, Paul Rusnock, and Cameron Shelley. 2002. “Knowledge and Coherence”. In *Common Sense, Reasoning, and Rationality*. Ed. Renée Elio. New York: Oxford U. P.
Whitehead, A. N., and Bertrand Russell. 1925. *Principia Mathematica*. 2nd edn. Vol. 1. Cambridge, UK: Cambridge U. P. (1st edn., 1910.)