DID *PRINCIPIA MATHEMATICA* PRECIPITATE A "FREGEAN REVOLUTION"?

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I begin by asking whether there was a Fregean revolution in logic, and, if so, in what did it consist. I then ask whether, and if so, to what extent, Russell played a decisive role in carrying through the Fregean revolution, and, if so, how. A subsidiary question is whether it was primarily the influence of *The Principles of Mathematics* or *Principia Mathematica*, or perhaps both, that stimulated and helped consummate the Fregean revolution. Finally, I examine cases in which logicians sought, in the years immediately following publication of the *Principles* and *Principia*, to integrate traditional logic into the Fregean paradigm, focusing on the case of Henry Bradford Smith. My proposed conclusion is that there were different means adopted for rewriting the syllogism, in terms of the logic of relations, in terms of the propositional calculus, or as formulas of the monadic predicate calculus. This suggests that the changes implemented as a result of the adoption of the Russello-Fregean conception of logic could more accurately be called by Grattan-Guinness's term *convolution*, rather than *revolution*.

My question—did *Principia Mathematica* precipitate a "Fregean revolution"?—can be analyzed to reveal at least two related but separate questions. (1) Did *Principia* precipitate a revolution in the history of mathematics? (2) Assuming that the answer to the first question is affirmative or negative, but still attributable to Russell, was that revolution "Fregean" and, if so, "Fregean" in what sense?

For much of the twentieth century, historiography of logic has responded without always clarifying the distinction between the two questions.

So far as (1) is concerned, the answer has been an equivocal "No". Rather, the generally accepted credit for Russell's role in spreading the legacy of Frege gives the lion's share to *The Principles of Mathematics*, for the two appendices which gave Russell's exposition and analysis of Frege's logical theory and the discussion and proposed solution, in the theory of types, to the Russell paradox. The argument runs that, except for a brief, and entirely negative, reaction in a handful of reviews of Frege's 1879 *Begriffsschrift*, logicians ignored Frege's work until it was recalled to their attention by Russell in the *Principles*. The more recent consensus, however, is that Frege's work was not quite as ignored in the period between 1879 and 1903 as had been thought.[1] It would be more accurate, I suggest, to argue that, whereas the *Principles* (re)introduced Frege's work to the community, it was *Principia* that disseminated the logicist viewpoint and thereby indirectly cemented Frege's reputation as the founder of the logicist programme and mathematical logic.

In this sense, William C. Kneale's assertion that *Principia* played the pivotal role in (re)directing attention to Frege and his work is somewhat idiosyncratic. In Kneale's words, the "Fregean" revolution owed much to Russell. He explained by saying that, so far as he could recall, no notice had been taken of Frege's death in 1925 and that he believed that at that time,

> … few philosophers had any inkling of the fact that as far back as 1879, in a pamphlet called *Begriffsschrift* …, Frege had produced the first complete system of formal logic. This is not to say that his ideas had been entirely neglected in his lifetime. On the contrary, they had won great triumphs among mathematicians and philosophers through *Principia Mathematica*, the famous work of Whitehead and Russell, which appeared first in 1910 and reached a second edition in the year of Frege's death. But Frege's own works were not read. On all suitable occasions Russell made generous acknowledgment of Frege's priority in the attempt to reduce arithmetic to logic; but there was a widespread impression that it was not worth while to read Frege's own writings, and so the greatness of his achievement was not realized in his lifetime.[2]

Kneale does not make mention of the *Principles of Mathematics*,

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[1] See, e.g., Risto Vilkko, "The Reception of Frege's *Begriffschrift*", *Historia Mathematica* 25 (1998): 412–22, and Jan Woleński, "The Reception of Frege in Poland", *History and Philosophy of Logic* 25 (2003): 37–51.

[2] William C. Kneale, "Gottlob Frege and Mathematical Logic", in A. J. Ayer *et al.*, *The Revolution in Philosophy* (London: Macmillan; New York: St. Martin's Press, 1957), pp. 26–40 (at 26–7). *Cf.* Avrum Stroll, "On the First Flowering of Frege's Reputation", *Journal of the History of Philosophy* 4 (1966): 72–81.

which, as we know, devoted considerable space not only to Frege's system, but to the theory of types as a means of circumventing the Russell paradox to which Frege's system, as presented in the *Grundgesetze*, gave rise.

Kneale's assertion therefore seems to ignore either the existence of the *Principles* or, at the minimum, suggests that it is subsidiary to the role of *Principia* in bringing Frege, and mathematical logic, to the fore. Rather, considering that Kneale says merely that Russell "made generous acknowledgment of Frege's priority in the attempt to reduce arithmetic to logic; but there was a widespread impression that it was not worth while to read Frege's own writings ...,"[3] we might conclude that, if Kneale had it aright, then Russell, rather than Frege directly, was the instigator, but not the progenitor, of the "Fregean" revolution.

In so far as (2) is concerned, historiography of much of the twentieth century has held that there was a revolution in logic, one which overthrew the Aristotelian paradigm, based upon the grammatical subject-predicate structure of ordinary language, in which the syllogism was the quintessential logical form of deductive inference. This same historiography also claims that the Fregean paradigm employed the function-theoretic syntactic structure of analysis in place of the old subject-predicate structure and that modus ponens was its preferred form of inference. Unlike the algebraic logic of the middle and late nineteenth century that offered merely an algebraic reformulation of the Aristotelian syllogism, the new paradigm also exhibited a quantification theory that was made possible by the introduction of the function-theoretic syntax. Whether Russell is given credit for consummating this seminal change in paradigm or not, and whether the greater part of Russell's credit is attributed to the *Principles* or to *Principia*, this ostensibly seismic shift is inevitably traced back to Frege, and to Russell's influence in helping to establish Frege's conception of logic. Paul Lorenzen (1915–1994), for example, has called Frege's *Begriffsschrift* "a logical masterpiece comparable in originality and import only to Aristotle's *Analytics*";[4] and William and Martha Kneale, in their exposition of Frege's doctrines, have come to the opinion that "it is not unfair either to his predecessors or to

[3] Kneale, p. 26.

[4] Paul Lorenzen, *Die Entstehung der exakten Wissenschaften* (Berlin/Göttingen/Heidelberg: Springer Verlag, 1960), p. 156.

his successors to say that 1879 is the most important date in the history of the subject." If I do not completely misjudge these and many similar statements, it is not only the singular magnitude of Frege's logical achievements that they refer to; it is also the bewildering impression that Frege created his logic, as it were, *ex nihilo*.

The ink was barely dry on the second edition of *Principia* when Paul Ferdinand Linke (1876–1955), Frege's friend and colleague at Jena, helped formulate the concept of a "Fregean revolution" in logic, when he wrote:

> … the great reformation in logic … originated in Germany at the beginning of the present century … was very closely connected, at least at the outset, with mathematical logic. For at bottom it was but a continuation of ideas first expressed by the Jena mathematician, Gottlob Frege. This prominent investigator has been acclaimed by Bertrand Russell to be the first thinker who correctly understood the nature of numbers. And thus Frege played an important role in … mathematical logic, among whose founders he must be counted.[5]

The concept of a Fregean revolution was explicitly expressed by Donald Angus Gillies; he argued the history of logic was marked by a "Fregean Revolution" in which the old Aristotelian paradigm of syllogistic logic was overturned and replaced by a mathematical, or "Fregean", paradigm.[6]

Heinrich Scholz (1884–1958) expressed succinctly the principal points of the canonical historiographic interpretation and thus confirmed and aptly summarized the conception of the "Fregean revolution", while accounting at the same time for Russell's role in carrying through that "revolution"; he wrote that:[7] "Wir sprechen von einem Sonnenaufgang, wenn wir den großen Namen Leibnizens nennen"—that "mentioning the name of Leibniz is like referring to a sun rising."[8] But he added that:

Between Leibniz and Russell there lies a tremendous amount of labor of which only the *most important phases* can be touched upon. In the 18th century

[5] Paul Ferdinand Linke, "The Present State of Logic and Epistemology in Germany", trans. Edward L. Schaub, *The Monist* 36 (1926): 222–55 (at 226–7).

[6] Donald A. Gillies, "The Fregean Revolution in Logic", in D. A. Gillies, ed., *Revolutions in Mathematics* (Oxford: Clarendon Press.; paperback edn., 1995), pp. 265–305.

[7] Heinrich Scholz, *Geschichte der Logik* (Berlin: Junker und Dünnhaupt Verlag, 1931; reprinted as *Abriss der Geschichte der Logik* Freiburg/Munich: Verlag Karl Alber, 1959), p. 48.

[8] Scholz, *Concise History of Logic*, trans. Kurt F. Leidecker (New York: Philosophical Library, 1961), p. 50.

and still under the influence of Leibnizian ideas, Lambert and Gottfried Ploucquet (1716–1790) … worked on the construction of the logical calculus. Then, for a time, leadership passed to the English. Quite independently of Leibniz and the German research work of the 18th century two English mathematicians Augustus de Morgan (1806–1878 [*sic*]) and George Boole (1815–1864) invented around the middle of the 19th century a new logical calculus which was later expanded by the German mathematician Ernst Schröder (1841–1902) into a grandiosely planned *Algebra der Logik*. Since 1889 we meet the new type of logic with basic improvements in the work of the Italian mathematician G. Peano who did a great deal for the axiomatization of arithmetic. For the first time the most important propositions were presented by him in symbolic notations for larger and larger areas of mathematics.

Unquestionably the greatest genius of modern logic of the 19th century was, however, the German mathematician Gottlob Frege (1848–1925). More than anyone else he contributed to the interpretation of basic mathematical concepts in terms of the fundamental concepts of logic which operate with exact determinations right from the start. The first one to do so, he raised the logical calculus to a level at which it turns into the "interlude" [*sic*][9] of which Leibniz had spoken. Nevertheless, he did not exert a direct and definitive influence, but in a roundabout way he did so by way of Russell's masterwork. The reason for this was that in spite of his thorough reflections he was not able to find the type of plastic symbolism which we need for a "conceptual script." In this great task only the authors of the *Principia Mathematica* succeeded. With the appearance of this opus the new logic was called into being.[10]

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[9] "Interlude" ["*Zwischenspiel*"] appears plainly to be a mistranslation by Kurt F. Leidecker: Scholz certainly intended "*Zeichenspiel*", as found in the original German (p. 57), by which he clearly meant manipulation of signs or calculus of signs, i.e. what Schröder and Peano called "pasigraphy".

[10] Scholz, pp. 58–9; my emphasis. In the original German (pp. 56–7), we read: "Zwischen Leibniz und Russell liegt eine gewaltige Arbeit, von der hier nur *Allerwichtigste* mitgeteilt werden kann. Noch unter dem Einfluß Leibnizischer Ideen haben im 18. Jahrhundert Lambert und Gottfried Ploucquet (1716–1790), … an dem Aufbau eines Logikkalküls gearbeitet. Dann geht die Führung zunächst an die Engländer über. Ganz unabhängig von Leibniz und der deutschen Forschung des 18. Jahrhunderts haben die beiden englischen Mathematiker Augustus de Morgan (1806–1878 [*sic*]) und George Boole (1815–1864) um die Mitte des 19. Jahrhunderts einen neuen Logikkalkül geschaffen, der dann von dem deutschen Mathematiker Ernst Schröder (1841–1902) zu einer groß angelegten "Algebra der Logik" erweitert worden ist. Mit wesentlichen Verbesserungen tritt uns die neue Logik in den Werken des um die Axiomatisierung der Arithmetik hochverdienten italienischen Mathematikers G. Peano seit 1889 entgegen, hier zum erstenmal so, daß für immer größere Teile der Mathematik die wichtigsten Sätze in symbolischer Anschreibung vorgelegt werden. Das größte Genie der neuen Logik im 19. Jahrhundert ist aber unstreitig der deutsche Mathematiker Gottlob Frege (1848–1925) gewesen; denn er hat mehr als irgend ein anderer für die Interpretation der mathematischen Grund-

This view of the merits of Frege's work permeated much of the historiography of logic. Thus, for example, Kurt Friedrich Gödel (1906–1978) also gave short shrift to the significance of the work of the Booleans, dealing with it with far greater disdain and brevity than did Scholz:

> … it was almost two centuries after his [Leibniz's] death that his idea of a logical calculus really sufficient for the kind of reasoning occurring in the exact sciences was put into effect … by Frege and Peano.…
>
> It was in this line of thought of Frege and Peano that Russell's work set in.… It was only in *Principia Mathematica* that full use was made of the new method for actually deriving large parts of mathematics from a very few logical concepts and axioms.[11]

Likewise, Willard Van Orman Quine (1908–2000), while acknowledging the accomplishments of the algebraic logicians, chief among them Charles Peirce, and even Peano, Richard Dedekind, and Alonzo Church, credited Frege with priority for most of the achievements that contributed to making mathematical logic the logic of the twentieth century, writing that:

> Frege scooped Peirce in quantification, he scooped Dedekind and Peirce in the theory of chains, he scooped Peano in class abstraction and Church in function abstraction. But the *Begriffsschrift* was scarcely noticed except for an unappreciative review by Schröder. Frege's important *Grundlagen der Arithmetik* of 1884 and *Grundgesetze der Arithmetik* of 1893 fared little better.[12]

Perhaps the starkest expression of the origin of mathematical logic

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begriffe durch die Grundbegriffe einer mit einem genau bestimmten Ausgangsmaterial operierenden Logik getan und den Logikkalkül selbst erst eigentlich auf die Stufe gehoben, auf der er zu dem Leibnizischen "Zeichenspiel" wird. Und dennoch ist er selber nicht direkt, sondern erst auf dem Umwege über das Russellsche Meisterwerk durchgedrungen. Warum? Weil er, trotz alles Nachdenkens, die plastische Symbolik nicht hat finden können, die von einer "Begriffsschrift" gefordert werden muß. Dieses große Werk ist erst den Verfassern der Principia Mathematica gelungen. Mit diesem Werk ist die neue Logik … endgültig geschaffen worden.…"

[11] Kurt Gödel, "Russell's Mathematical Logic", in Schilpp, pp. 125–6; reprinted: Solomon Feferman, John W. Dawson, Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, Jean van Heijenoort, eds., Kurt Gödel, *Collected Works*, Vol. 2: *Publications 1938–1974* (New York: Oxford U. P., 1990), pp. 119–41.

[12] Willard Van Orman Quine, "Peirce's Logic", in Kenneth Laine Ketner, ed., *Peirce and Contemporary Thought: Philosophical Inquiries* (New York: Fordham U. P., 1995), pp. 23–31 (at 23).

with Frege and his *Begriffsschrift* emanated from Jean van Heijenoort, who bluntly wrote, in the posthumously published "Historical Development of Modern Logic", that:[13] "Modern logic began in 1879, the year in which Gottlob Frege (1848–1925) published his *Begriffsschrift.*" In explanation, he expressed the concept of the *Begriffsschrift* as the *fons et origo* of modern logic, writing: "In less than ninety pages this booklet presented a number of discoveries that changed the face of logic," and adding that: "Frege's contribution marks one of the sharpest breaks that ever occurred in the development of a science."[14]

James Van Evra readily concedes that Frege indubitably deserves credit for priority for developing a quantification theory for logic, as well as other crucial innovations. He nevertheless also reminds us that Frege's work had little significant influence until brought to the fore by the ministrations of Russell; that in the last two decades of the nineteenth century and into the third decade of the twentieth, it was rather the influence of Charles Peirce, particularly as presented and developed in Schröder's *Vorlesungen über die Algebra der Logik*, that was the source of inspiration for the work, for example, of Löwenheim and Skolem. Thus, Van Evra complained that Frege is unfairly given more credit than his work warranted. He complains that:

> Accounts of the origin of modern quantification theory often let Frege steal the show. They point out that he got there first (*Begriffsschrift* (1879)), with a full rendering of quantificational logic which contained, within a single theory, both the quantification of individuals in first-order, as well as second-order quantification of functions. Given the additional (often implicit) assumption that our current version of quantification stems uniquely from him, it seems reasonable to ask why we should be concerned with another version of something which had already been discovered.[15]

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[13] Jean van Heijenoort, "Historical Development of Modern Logic", *Modern Logic* 2 (1992): 242–55 (at 242).

[14] Van Heijenoort, p. 242.

[15] James Van Evra, review of Christian J. W. Kloesel, ed., *Writings of Charles S. Peirce: a Chronological Edition*, Vol. 5: *1884–1886*, *Modern Logic* 6 (1996): 216–19 (at 216–17). For the influence of Peirce (and Schröder) on logical research, and in particular the development of model theory, from *c.* 1880 to *c.* 1920, see, e.g., Geraldine Brady, *From Peirce to Skolem: a Neglected Chapter in the History of Logic* (Amsterdam and New York: North-Holland/Elsevier Science, 2000), and Calixto Badesa, *The Birth of Model Theory: Löwenheim's Theorem in the Frame of the Theory of Relatives*, trans. M. Maudsley (Princeton and Oxford: Princeton U. P., 2004).

Gilbert Ryle (1900–1976) attempted to answer the question of why and how Russell's work proved to be pivotal in consummating the Fregean revolution. His explanation is found in the assertion that once the idea of *relation* which was "made respectable" by De Morgan and the resulting relational inferences were codified by Russell in *The Principles of Mathematics*, then:

> The potentialities of the *xRy* relational pattern, as against those of the overworked *s–p* pattern, were soon highly esteemed by philosophers, who hoped by means of it to bring to order all sorts of recalcitrances in the notions of knowing, believing....[16]

On the question of Russell's role in familiarizing philosophers and logicians with Frege's work, there are those who suggest that, while some credit is due Russell, he was not the sole, or even the most influential, disseminator of knowledge of Frege's work. Quine at one point clearly gives abundant credit to *Principia* for familiarizing us with Frege's work, writing: "It was not until Whitehead and Russell's great *Principia Mathematica* (1910–1913) that Frege's influence perceptibly enters the mainstream...."[17] But in his autobiography he seems to minimize the significance of Russell's role in bringing full realization of Frege's importance in contrast with his own role, asserting first that there was "no discoverable copy"[18] of the *Begriffsschrift* in America, and that he had to glean its contents instead from an "old review" by Philip Edward Bertrand Jourdain (1897– 1919),[19] and then declaring that:[20] "My celebration of Frege" in *Mathematical Logic* (1940) "and in the classroom must have helped to bring people to see Frege as the father of modern logic." True, "Russell had introduced him to us long ago, but we remained unaware of how much had been done first by Frege." For example, "I think

[16] Gilbert Ryle, "Introduction", in Ayer *et al.*, *The Revolution in Philosophy*, pp. 1–12 (at 9–10).

[17] Quine, "Peirce's Logic", p. 24.

[18] Quine, *The Time of My Life: an Autobiography* (Cambridge, MA, and London: MIT Press, 1985), p. 144.

[19] Probably actually meaning the relevant section of Philip E. B. Jourdain's "The Development of Theories of Mathematical Logic and the Principles of Mathematics", *Quarterly Journal of Pure and Applied Mathematics* 41 (1910): 324–52; 43 (1912): 219–314; 44 (1913): 113–28, and most specifically 44 (1912): 237–69.

[20] Quine, *The Time of My Life*, p. 144.

Church first learned from my book that his functional abstraction was in Frege." And Church "returned the favor three years later, pointing out that my notion of referential position and even my example of the Morning Star and the Evening Star were in Frege," although "I may have got the example through Russell."

At the very most, one can say, as Ivor Grattan-Guinness has,[21] that the publicity which Russell gave to Frege's work, starting with the *Principles of Mathematics* in 1903, brought a "higher level of attention" to Frege and his work than it had hitherto received, and that, as a consequence, he helped launch what has been called the philosophers' "Frege-industry",[22] rather than that Russell *discovered*, or even merely *re*discovered, Frege and brought it to the attention of logicians. Nor can we overlook the importance in this regard of Jourdain who, though he may first have become cognizant of Frege's work through references by Russell, himself played a crucial role in broadcasting and expounding the features of Frege's work for mathematicians, through his account of Frege in his multi-part history of logic.

Whether we adopt the view that it was Frege's work that actually initiated a revolution in logic or the view that Russell, either through the *Principles* or *Principia*, or both, launched the revolution that Frege conceived, we are still left with the question, suggested by Van Evra's remarks, as well as by later confessions by Quine and others, of whether Peirce was more influential in establishing such innovations as a theory of quantification for his algebraic logic of relations than was Frege, even though Frege deserves credit for priority in establishing a quantification theory for a function-theoretic logic, and of whether what occurred was a *revolution* in any proper sense. Quine, for example, came to alter his youthful view that Frege, helped by Russell and himself, initiated a Fregean revolution in logic, when, for example, he wrote:

> General quantification theory is the full technique of "all", "some", and pronomial variables, and it is what distinguishes logic's modern estate. Charles

[21] Ivor Grattan-Guinness, *The Search for Mathematical Roots, 1870–1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel* (Princeton and London: Princeton U. P., 2000), p. 468.

[22] I trace this expression to Joong Fang, in a letter of 11 April 1986; see Irving H. Anellis, "Joong Fang of Jaean—a Retrospective", *Modern Logic* 3 (1993): 145–55 (at 147).

Sanders Peirce arrived at it independently four years after Frege. Peirce's work did indeed take off from that of Boole, De Morgan and Jevons. Ernst Schröder and Giuseppe Peano built in turn on Peirce's work, while Frege continued independently and unheeded.

The avenue from Boole through Peirce to the present is one of continuous development, and this, if anything, is the justification for dating modern logic from Boole; for there had been no comparable influence on Boole from his more primitive antecedents. But logic became a substantial branch of mathematics only with the emergence of general quantification theory at the hands of Frege and Peirce. I date modern logic from there.[23]

He likewise confessed that "Peirce and not Frege was indeed the founding father" of quantification; he had reminded us that the reason for this is that "Peirce's influence was continuous through Schröder's work, with side channels into Peano, culminating in *Principia Mathematica*", while Frege still "had been a voice crying in the wilderness."[24] Quine, like Van Evra, would thus come to assert that there was a continuity of development; that the more significant differences between the algebraic logicians such as Peirce and Schröder on the one hand and the "mathematical" logicians, such as Frege and Russell on the other, were (1) a choice of an algebraic structure for logic rather than a function-theoretic structure; and (2) the underlying philosophy, in particular the logicism of Frege and Russell, together with the Russello-Fregean concept of a *logica docens* as compared with the "Boolean" preference for a *logica utens*, or, as Jean van Heijenoort characterized it, absolutism and logic as language rather than relativism and logic as calculus.[25]

Our next question, therefore, must be whether or not there actually *was* what could legitimately be called a *revolution* at all, or instead what

[23] *The Time of My Life*, p. 144.

[24] Quine "Peirce's Logic", in Quine, *Selected Logic Papers*, enl. edn. (Cambridge, MA: Harvard U. P., 1995), pp. 258–65 (at 259); "Peirce's Logic", in Kenneth Laine Ketner, ed., *Peirce and Contemporary Thought: Philosophical Inquiries* (New York: Fordham U. P.), pp. 23–31 (at 24).

[25] Jean van Heijenoort, "Logic as Calculus and Logic as Language", *Synthèse* 17 (1967): pp. 324–30; reprinted in Robert S. Cohen and Marx W. Wartofsky, eds., *Boston Studies in the Philosophy of Science*, Vol. 3: *Proceedings of the Boston Colloquium for the Philosophy of Science 1964–1966; in Memory of Russell Norwood Hanson* (Dordrecht: D. Reidel, 1967; New York: Humanities Press, 1968), pp. 440–6; reprinted in Jean van Heijenoort, *Selected Essays* (Naples: Bibliopolis, 1986), pp. 11–16; and "Absolutism and Relativism in Logic" (1979), *Selected Essays*, pp. 75–83.

Grattan-Guinness might perhaps call a *convolution*.

From the writings of Peano, it is evident that he saw a link between the work of the "Booleans", including Peirce and Schröder, to his own work. Moreover, he saw Russell's work as filling a gap between their work and his own. In a letter to Russell of 19 March 1901, Peano explicitly declared that Russell's paper on the logic of relations of 1901 "fills a gap between the work of Peirce and Schröder on the one hand and the *Formulaire* on the other."[26]

This transition from the algebraic to the function-theoretic model for mathematical logic is a subject which I continue to investigate.

But the other aspect of the presumptive "revolution", as characterized, for example, by Gillies is the rejection of the Aristotelian paradigm in favor of the Fregean, and one of the salient factors of the paradigm shift is the replacement of the subject-predicate syntax for propositions with the function-theoretic syntax. Here, I suggest, there remain difficulties. One, which I wish to mention, but on which I will not spend much time, is Gillies' "textbook argument", in which he compares a handful of select logic textbooks of the pre- and post-revolutionary period. There are two flaws in his argument: (1) the argument is statistically invalid, since Gillies picks only a handful of examples from each group; and (2) the comparison is unfair insofar as the textbooks selected were intended for different audiences—the pre-revolutionary textbooks having been designed for philosophy undergraduate students, whereas the post-revolutionary samples were written for mathematics graduate students.

The difficulty which I should like to devote some attention to is that, rather than reject out-of-hand the Aristotelian syllogism, the revolutionaries ought to incorporate the syllogism into their logical theory. Frege himself noted that only *Barbara*, Formula 65 in the *Begriffsschrift*, is universally valid when rewritten in his system. Frege also singled out what amount to the syllogisms *Felapton* ("No M is P; all M is S: therefore some S is not P" (Frege's Formula 59) and *Fesapo* ("No P is M; all M is S: therefore some S is not P" (Frege's Formula 62) in the *Begriffsschrift* for translation. Whereas five of the seven syllogisms that Frege considered turn out to be invalid in his system, nevertheless, not only Frege, but Peano and Russell as well, sought to translate the Aristotelian syllo-

[26] Hubert C. Kennedy, "Nine Letters from Giuseppe Peano to Bertrand Russell", *History and Philosophy of Logic* 13 (1975): 205–20 (at 206).

gisms into their respective systems, thereby preserving, albeit in altered form, the Aristotelian syllogism. As Quine would notice,[27] these constitute the fragment of Russello-Fregean first-order logic that we recognize as the monadic predicate logic; thus for example, we have

All A are B	$(\forall x)(Ax \supset Bx)$
Some A is B	$(\exists x)(Ax \wedge Bx)$.

This idea was hardly new, and Peirce, MacColl, Peano, and Russell were among those who, through the 1880s and into the 1910s, translated *Barbara* (each in their own notation) as equivalent to: $((S \supset M)\ \&\ (M \supset P)) \supset (S \supset P)$.

Moreover, in the 1910s to 1930s, considerable attention was given to translating Aristotelian and Boolean syllogisms into the new logistic as found in *Principia*, and *vice versa*. The best example, Norbert Wiener's comparison of the classical Boole–Schröder calculus with the logic of *Principia*, has been thoroughly discussed by Grattan-Guinness.[28] I would like to mention here one of a number of other less well remembered efforts from this period.[29]

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[27] Quine, *Methods of Logic* (New York: Holt, Rhinehardt & Winston, 1972 [1st edn., 1950]), pp. 72–4.

[28] Grattan-Guinness, "Wiener on the Logics of Russell and Schröder: an Account of His Doctoral Thesis, and of His Discussion of It with Russell", *Annals of Science* 32 (1975): 103–32.

[29] One could also offer other examples of efforts, both from the 1910s, 1920s and 1930s, and much more recent vintage, not merely to find accommodation between Aristotelian and Fregean paradigms, but examples of outright rejection of the Fregean paradigm. Of the former, Heidegger's questioning of the value of teaching symbolic logic, or logistic, which he dismissed as mere "*Rechnen*", is well known (see, e.g., "Neuere Forschungen über Logik", *Literaturische Rundschau für das Katholische Deutschland* 38 (1912): 465–72, 517–24, 565–70). Of the latter, the examples of Frederic Sommers (e.g. "On a Fregean Dogma", in Imre Lakatos, ed., *Problems in the Philosophy of Mathematics* [Amsterdam: North Holland, 1967)], pp. 47–62) and George Englebretsen (e.g., *Three Logicians: Aristotle, Leibniz and Sommers and the Syllogistic* [Assen: Van Gorcum, 1981] have gained the greatest traction. Edward A. Hacker and William Tuthill Parry (e.g., Hacker, "Number System for the Immediate Inferences and the Syllogism in Aristotelian Logic", *Notre Dame Journal of Formal Logic* 8 [1967]: 318–20; Hacker and Parry, "Pure Numerical Boolean Syllogisms", *ibid.*, 321–4; Parry and Hacker, *Aristotelian Logic* [Albany: State U. of New York P., 1991]) had also devised an Aristotelian arithmeticization of syllogistic logic that is closer in style and spirit to that of Leibniz than that of Sommers (also developed in Sommers' "The Calculus of Terms", *Mind* 79 [1970]: 1–39, and put in

In 1932, for example, Archie J. Bahm (1907–1996) debated with Henry Bradford Smith (1882–1938) the question of the translatability of Aristotelian syllogistic into algebraic logic and into the language of *Principia Mathematica*. Smith took his start in the class calculus. He undertook to show, in his *Symbolic Logic*, how to deduce the postulates of Aristotle's system directly from the Boole–Schröder calculus,[30] and then set out to prove, in "On the Relation of the Aristotelian Algebra to That of Boole-Schroeder", the consistency of Aristotelian "algebra" by showing how to deduce the postulates of the Boole–Schröder calculus from Aristotelian syllogistic, using respectively the definitions for Aristotelian inclusion $a \angle b$ and for the Boolean inclusion $a \supset b$.[31] Finally, after developing the Hamiltonian set of forms from the properties of the Boole–Schröder calculus in "On the Derivation of the Aristotelian Algebra from the Properties of a Hamiltonian Set",[32] and employing these to establish the characteristic features from Aristotle's logic of obversion, contraposition, and simple conversion where they occurred, and subalternation and the valid moods of syllogisms and using the forms of Aristotelian logic thus defined, and expressing in terms of Boolean inclusion and deduced the fundamental properties of Hamilton's logic, he undertook to prove the invalidity of the equivalent of *Barbara* given in *Principia Mathematica*.[33]

Smith's aim, as described by Kattsoff, was to determine: "What happens to modern logical theories if a new set of forms can be found by which Aristotle is vindicated?"[34] Smith's attitude towards the new sym-

textbook form in Sommers' *An Invitation to Formal Reasoning: the Logic of Terms* [Aldershot, UK: Ashgate, 2000]). For a brief sketch of Sommers' system, see Irving H. Anellis, *Van Heijenoort: Logic and Its History in the Work and Writings of Jean van Heijenoort* (Ames, IA: Modern Logic Publishing, 1994), p. 51.

[30] Henry Bradford Smith, *Symbolic Logic, Method and Development* (New York: F. S. Crofts & Co., 1927).

[31] *The Moni*st 42 (1932): 282–9.

[32] *The Monist* 42 (1932): 290–3.

[33] Smith, *Symbolic Logic*, p. 132. Smith's efforts in this regard can be traced back to at least his "Aristotle's Other Logic", *American Journal of Psychology* 29 (1918), 431–4, where he first raises the question of the validity of the moods and figures of the traditional syllogisms in light of the postulational method when taking account of the universal class and the empty class, and "Non-Aristotelian Logic", *Journal of Philosophy, Psychology and Scientific Methods* 15 (1918): 453–8, where he begins to develop his system. In this work, Smith used "∠" instead of "<".

[34] Kattsoff, "Concerning the Validity of Aristotelian Logic", *Philosophy of Science* 1 (1934): 149–62 (see p. 149).

bolic logic, the motivation for his "rescue" effort for syllogistic logic, and his choice of techniques in undertaking his rescue, as he himself expressed it, is found in his review of Clarence Irving Lewis and Cooper Harold Langford's *Symbolic Logic*.[35] In view of our question of the role of *Principia* in establishing the Fregean revolution in logic, it is worth quoting *in extenso*:

> For more than two milleniums [*sic*] the queen of the sciences had been viewed as a discipline begun and completed by a single man within the span of his own life. As late at the eighteenth century Immanuel Kant says in effect: "While it is indeed remarkable that no one has been able to detect any flaw in the logic of Aristotle, it is still more significant that no one has been able to add an important word to what the Stagirite has said." This vast expanse of time corresponds to what we might term the "first age of fixation." The work of the scholastics and their successors, notably Leibnitz, so far as it concerns logic, had left only ripples on the current of human thought.
>
> In 1846 Sir William Hamilton published the prospectus of *A New Analytic of Logical Forms*, a solvent which was soon to make its power felt. No one has ever denied the immense scope as well as the thoroughness of Hamilton's learning, but nearly all historians have disparaged his critical sense. This criticism none the less inaugurated a development, and for years that immediately succeed, the ancient science is in a state of flux. Two names, De Morgan and Boole, tell most of the story of this time of flux. It is brought to an end by the labors of Peirce and Schroeder. The few years that follow might be termed the "second age of fixation."
>
> Meanwhile other solvents were being prepared. Frege had published his *Begriffsschrift* in 1879, Peano had begun to issue his *Formulaire* in 1895, Whitehead produced his *Universal Algebra* in 1898, Russell his *Principles* in 1903. Finally, 1910–13, appears the *Principia Mathematica* of Russell and Whitehead.[36]

Looking at the details, we see that Smith considered the fact that there were syllogisms valid in Aristotelian logic that were invalid in the logic of *Principia*, and set about to rescue syllogistic. His first step was to provide a new interpretation for the four categorical propositions, and then to demonstrate, with the aid of this rewriting, that every inference valid

[35] Smith, review of *Symbolic Logic* by Lewis and Langford, *Philosophy of Science* 1 (1934): 239–46.

[36] Smith, *ibid*., p. 239. Smith's comments upon Leibniz and his immediate followers can be readily be excused; he could not, of course, have been expected to anticipate the work of Sommers and Engelbretsen or of Parry and Hacker.

in traditional logic is also valid in symbolic logic. His interpretation treats the terms of the traditional proposition as classes, and the copula as expressing a relation between these classes. For classes a and b, the A, E, I, and O propositions concern relations between a, b, and b', where b' is the class of all not-b. Thus, in *Symbolic Logic* the A-proposition "All A are B", expressed in Smith's notation as $A(a, b)$, asserts that $a < b$, to be understood as the relation whose full meaning is

$$A(a, b) = (a < b)\ [(b < a) + (a < b')'\ (b' < a)'],$$

according to which we have that a implies b is to be interpreted to mean that a is included in b and either b is included in a or it is false that a is included in not-b and also false that not-b is included in a. Here, "<" (also written, e.g. in an early work, as "$\angle$") is understood in the usual sense for the Boole–Schröder algebra as either class inclusion or implication, or the copula, with concatenation (and easily replaced notationally by using ".") doing duty for logical multiplication (i.e. conjunction or intersection), "+" as logical addition (i.e. disjunction or union), and "$'$" as complementation (negation). With this, he interprets "No A is B" as the relation equivalent to "All A is non-B", so that we have $a < b'$, which we can similarly expand to express the full relation. The I and O propositions are then obtained by taking the contradictories of A and E, respectively. Neither Bahm nor Smith directly or indirectly mentions ∗10.26 or ∗10.3 of *Principia* in their debate with one another, despite its obvious relevance.

In "On the Relation of the Aristotelian Algebra to That of Boole–Schroeder", Smith argues that "$a < b$" represents the Aristotelean proposition "a included in b", while "$a \supset b$" represents the Boolean proposition "a included in b", and then argues that the two are logically equivalent, giving the definition

$$a < b = a \supset b\,(b \supset a + a \supset b' \,.\, b' \supset a),$$

or, by substitution,

$$a \supset b = a + b + a'a < a + bb' < b',$$

to permit easy passage between the Aristotelian and Boolean systems.

These definitions are then used to consider the relations from among those given by the traditional square of opposition and is asserted to allow one to treat empty classes as unproblematical within the Aristotelian system as they are for the modern (Boolean) square of opposition.

In "On the Derivation of the Aristotelian Algebra from the Properties of a Hamiltonian Set", Smith noted that Hamilton criticized Aristotelian logic for its failure to quantify over predicates. The purpose of "On the Derivation …" is therefore to derive the properties of a Hamiltonian set of forms from the properties of the Boole–Schröder logic and from these in turn to establish the characteristic inferential properties of Aristotelian logic, in particular obversion, contraposition, simple conversion in the cases where they obtain, subalteration, and the valid moods of the syllogism. Smith determines that all of the Aristotelian characteristics hold except for the reflective property for the "all are" relation, which alone fails, in the limiting case of the empty universe. The basic forms are immediately obtained:

$\alpha(ab)$ all a is all b | $\beta(ab)$ some a is some b
$\gamma(ab)$ all a is some b | $\delta(ab)$ all non-a is some b
$\epsilon(ab)$ all a is some non-b | $\eta(ab)$ all non-a is all b

From the standpoint of natural language, "some" is understood to exclude the possibility of "all". We can therefore understand the Hamiltonian forms in terms of Boolean inclusion, and the fundamental properties of Hamiltonian logic can be derived. For Boolean inclusion, we have

$a \supset b$ = a included in b = ab (abbreviated)
$a \supset b'$ = a included in non-b = ab (abbreviated)
$\overline{a' \supset b}$ = non-a is not included in b = $\overline{a'b}$ etc.,

where the overbar is the denial of the proposition. In Hamiltonian logic, we therefore obtain

$$\alpha(ab) = ab \,.\, ba \,.\, \overline{a\,b'} \,.\, \overline{b'a} \qquad \beta(ab) = \overline{a\,b} \,.\, \overline{b\,a} \,.\, \overline{a\,b'} \,.\, \overline{b'a} + (ab + ba)(ab' + b'a)$$
$$\gamma(ab) = ab \,.\, \overline{b\,a} \,.\, \overline{a\,b'} \,.\, \overline{b'a} \qquad \gamma(ba) = ba \,.\, \overline{a\,b} \,.\, \overline{a\,b'} \,.\, \overline{b'a}$$
$$\delta(ab) = b'a \,.\, \overline{a\,b'} \,.\, \overline{a\,b} \,.\, \overline{b\,a} \qquad \epsilon(ab) = ab' \,.\, \overline{b'a} \,.\, \overline{a\,b} \,.\, \overline{b\,a}$$
$$\eta(ab) = ab' \,.\, b'a \,.\, \overline{a\,b} \,.\, \overline{b\,a}$$

One of the major complaints that Smith had with the logic of *Principia* was with its interpretation of "x implies y" as "either not x or y". Much of the vast apparatus behind Smith's translations between Aristotelian, Boole–Schröder, and Hamiltonian forms was designed explicitly to answer to this equivalence, with the ultimate goal of justifying the relations between the propositions of the traditional square of opposition.

In response, Bahm argued, in "Henry Bradford Smith on the Equivalent Form of Barbara", that Smith failed to prove the invalidity of the equivalent of *Barbara* given in *Principia Mathematica* as Smith claimed to do by his method of translation.[37] Two years later, this discussion was also joined by Paul Henle (1902–1962) who examined Smith's system in greater detail than did Bahm.[38] Henle, examining the arguments of both Smith and Smith's defender Louis Osgood Kattsoff (1908–1979),[39] admits, in opposition to Kattsoff's assertion, that Smith's system is self-consistent. But Henle concludes that it is nevertheless "difficult to see" Smith's system as equivalent to Aristotelian logic.[40] Kattsoff concluded that it is totally obvious that "*the question of the existential import of propositions is solved negatively once and for all by a suitable definition of the four categorical relations.*"[41] Kattsoff bases his argument upon the consequences of the system of Smith's logic for the traditional square of opposition. Henle and Kattsoff examine the relation among propositions obtaining in Smith's system and note that there are those elements of the traditional square of opposition that do not hold in Smith's translation of Aristotle's logic. In particular, Henle argues that the Principles of Contradiction and Excluded Middle would have to be excised from Aristotelian syllogistic if Smith and Aristotle's system were to be considered to be equivalent, given that, on Henle's understanding, both principles are assumed to be false in Smith's system. Smith responded to Henle's arguments by asserting that, taking the Principle of Contradiction to assert that "No proposition of the form 'S is both P and non-P' can be true", we should be permitted to infer from it that "No entity is both red and non-red", which Smith's system holds to be true. Given the class of entities is the universe (1) and the class of both red and non-red

[37] Bahm, *The Monist* 42 (1932): 632–3.

[38] Paul Henle, "A Note on the Validity of Aristotelian Logic", *Philosophy of Science* 2 (1935): 111–13.

[39] Louis Kattsoff, "Concerning the Validity of Aristotelian Logic" (see n.34).

[40] Henle, p. 113.

[41] Kattsoff, p. 162 (emphasis in original).

is the empty class (0), $E(1, 0)$ is true in Smith's system. The difficulty, Henle continues, is that any further specification yields a false statement. He gives such examples as "No chair is both red and non-red" as false while their contradictories are true. That is, $E(a, b)$ is false for $b = 0$ and $a \neq 1$ once we particularize the universe of discourse. Moreover, once a singular proposition is regarded as a universal proposition, as it is in traditional logic, the two assertions—that some particular chair is both red and non-red, and that some particular chair is not both red and non-red—are true. Henle concludes: "This not only involves a breakdown of one of the laws of logic most firmly established by tradition, but also is repugnant to common sense" (p. 112). In reply, Smith notes that Henle's critique relies in these instances upon ontological and psychological, rather than purely logical, considerations of the Principle of Contradiction.[42] With the basic definitions and translations established, Smith examines syllogistic relations for a, b, c, d, ac, and bd, including those under which condition any one of the four terms is represented as i, where i is either the empty class (0) or the universe (1). Thus, for example, $E(a, b)$ is false if $b = 0$ and $a \neq 1$. So, although Henle tells us that he fails to see how Smith's system is equivalent to Aristotle's, Smith's response is that Henle is here confusing the logical version of the Principle of Contradiction with the ontological or psychological version.

The literary debate regarding the relation between traditional logic and the new logistic that was raised by the question of existential import of propositions in which Russell and MacColl had engaged was thus continued a generation later in light of results of *Principia*. Thus, for example, Filmer Stuart Cuckow Northrop (1893–1992) and Andrew Paul Uchenko (or Ushenko; 1900–1956) debated the existential import of universal affirmative propositions in Aristotelian categorical logic.[43]

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[42] Smith, reply to Henle, "A Note on the Validity of Aristotelian Logic", *Philosophy of Science* 2 (1935): 113–14. The distinction between logical, ontological, and psychological versions of the Principle was introduced by Jan Łukasiewicz, "Über den Satz des Widerspruchs bei Aristoteles", *Bulletin International de l'Académie des Sciences de Cracovie*, Cl. d'histoire et de philosophie (1910): 15–38 (trans. Vernon Wedin as "On the Principle of Contradiction in Aristotle", *Review of Metaphysics* 24 [1970–71]: 485–509); and his book *O zasadzie sprzeczności u Aristotelesa. Studium krytyczne* (Cracow: Akademia Umiejętności, 1910; trans. Jacek Barski, *Über den Satz des Widerspruchs bei Aristoteles* [Hildesheim and New York: G. Olms, 1993]), on which Smith's reply was based.

[43] See, e.g., F. S. C. Northrop, "An Internal Inconsistency in Aristotelian Logic", *The Monist* 38 (1928): 193–210, and "A Reply, Emphasizing the Existential Import of Prop-

Smith dealt with this issue by arguing that 0 and 1 are limiting conditions on the translation between the Aristotelian and Boolean arrangements concerning the relations between the *A*, *E*, *I* and *O* propositions given in the traditional square of opposition, for terms *a*, *b*, *c*, *d*. Thus, for example, letting $\kappa(ab)$ and $\lambda(ab)$ be any of the Hamiltonian forms, then, provided $\kappa(ab)$ and $\lambda(ab)$ are not identical (and taking $\gamma(ab)$ and $\gamma(ba)$ as distinct), then $\kappa(ab) \cdot \lambda(ab) = 0$, since, Smith explains (*Symbolic Logic*, p. 291), each of the forms (α)–(η) contains a term that contradicts a term in one of the other forms (α)–(η). Adding, then, the forms *U* and *V* to the traditional *A*, *E*, *I*, and *O* forms, where, now

$A(ab) = \alpha(ab) + \gamma(ab)$	all *a* is *b*
$E(ab) = \alpha(ab') + \gamma(ab'),\ \eta(ab) + \epsilon(ab)$	no *a* is *b*
$U(ab) = \alpha(a'b) + \gamma(a'b),\ d(ab) + \eta(ab)$	all non-*a* is *b*
$O(ab) = \alpha(ab) + \beta(ab) + \gamma(ba) + \delta(ab) + (ab) + \eta(ab)$	some *a* is not *b*
$I(ab) = \alpha(ab) + \beta(ab) + \gamma(ab) + \gamma(ba) + \delta(ab)$	some *a* is *b*
$V(ab) = \alpha(ab) + \beta(ab) + \gamma(ab) + \gamma(ba) + \delta(ab)$	some non-*a* is not *b*,

we obtain the contradictory pairs *A* and *O*, *E* and *I*, and *U* and *V*,

so that $$A(ab) \cdot E(ab) = 0$$

and $$A(ab) \cdot U(ab) = 0,$$

and, with the proper algebraic computations, we satisfy the traditional square of opposition even in the face of the empty class.

What Bahm argued is that Smith failed in his attempt to demonstrate that the Russellian equivalent of *Barbara* is invalid, despite all the translation apparatus that he had devised for the purpose.

Examining the details, Smith wrote in *Symbolic Logic* that Russell's definition of implication is erroneous, and (reading "$<$" now as implication) depends upon establishing that $a < (b < c)$ is equivalent to $ab < c$. Smith's argument, Bahm asserts, depends upon showing that Russell's

ositions", *The Monist* 39 (1929): 157–9, and A. P. Uchenko, "Aristotelian Logic and the Logic of Classes. A Reply to Professor Northrop, *The Monist* 39 (1929): 153–6.

For a survey of the history of the discussions on existential import of propositions, see, e.g., Joseph S. Wu, "The Problem of Existential Import (from George Boole to P. F. Strawson)", *Notre Dame Journal of Formal Logic* 10 (1969): 415–24.

equivalent form of *Barbara* is valid. But, Bahm argues, there is a serious error in Smith's proof, as follows. Granted, Smith is correct in assuming that any proposition A does not imply $(A')'$, where the latter is true for all meanings of its terms. Smith's representation of *Barbara*, as presented by Bahm, is: $A(ba) < (A(cb) < A(ca))$, which implies $A(bc) < A(bb) < A(ba)$ if $b = c$. Smith's error, Bahm holds, is to assume the identity of b and c; and this supposed identity is the source of all of Smith's failures to establish the infelicitude of Russell's rendition of *Barbara*. Bahm notes, correctly, that, for any term t, $A(tt)$ in Smith's system, which Smith calls the multiplier, is the universe; i.e. $A(tt) = 1$ (or i). Bahm then argues that Smith improperly identifies i with $(A')'$. The difficulty for Smith, Bahm explains, then, hinges upon Smith's allowing $b = c$. Bahm therefore finds the error in Smith's system to be based upon his violation of the condition that the referential values of all the terms are to be retained throughout. This insistence by Bahm that Smith's system is inconsistent, and his critique of Russell's system consequently misguided, because of the meanings of the terms, is the basis for Smith's rebuttal that Bahm's critique confuses the logical with the ontological versions of the Principle of Contradiction.

It is also worth noting that, for all of his criticisms of Smith's arguments, Bahm himself did not argue in opposition to the Aristotelian system, but couched his critique in the same traditional terms as did Smith.

What is clear from examples such as these, and from Smith's animadversions in regard to his endeavour to establish the reintegration of the logic of *Principia* with traditional logic through the medium of Boole, Peirce and Schröder on the one side and Hamilton on the other, is that logicians, in the presumably post-revolutionary era, rather than reject outright the Aristotelian paradigm, continued to seek to integrate traditional logic within the framework of the new logistic, or Fregean paradigm. In some cases they did so by rewriting the syllogisms as formulas of propositional logic; in other cases, in terms of a logic of relations; in yet other cases, in terms of first-order logic as the monadic predicate calculus; and, in the case of those like Henry Bradford Smith, undertaking also to accommodate the traditional logic to the new by establishing a correlation between the traditional and the Boolean squares of opposition. Logicians such as Smith sought to make this accommodation through reformulating the traditional propositions and overriding the claims of traditional logic for the existential import of propositions.