SUBSTITUTION'S UNSOLVED "INSOLUBILIA"

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Russell's substitutional theory conferred philosophical advantages over the simple type theory it was to emulate. However, it faced propositional paradoxes, and in a 1906 paper "On 'Insolubilia' and Their Solution by Symbolic Logic", he modified the theory to block these paradoxes while preserving Cantor's results. My aim is to draw out several quandaries for the interpretation of the role of substitution in Russell's logic. If he was aware of the substitutional (p_0a_0) paradox in 1906, why did he advertise "Insolubilia" as a solution to the Epimenides? If he was dissatisfied with the solution, as his correspondence suggests, why did he go on to publish it? Why did substitution reappear with orders in "Mathematical Logic as Based on the Theory of Types" if he had rejected a hierarchy of orders as intolerable? I offer the following as possible explanations: he construed the "logical Epimenides" as a version of the p_0a_0 paradox; his dissatisfaction with the "Insolubilia" solution was philosophical, not technical; and substitution re-emerged because he hoped for a new philosophical gloss on orders. Whether or not my explanations are correct, these issues must be addressed in accounting for Russell's reasons for ramification.

There remains just one volume in the philosophical and logical series of *The Collected Papers of Bertrand Russell* yet to be published. It is the much-anticipated *Toward "Principia Mathematica", 1905–08*, which will contain the published papers and unpublished manuscripts from the period following "On Denoting" and leading to *Principia Mathematica*.[1] The volume will include crucial documents concerning the "substitutional theory" to which Russell adhered in the intervening period and is sure to generate new interest in the controversies surrounding the interpretation of the role the substitutional theory played in Russell's logicist project.[2] The aim of the substitutional theory, as it was first formulated in Russell's late November/early December 1905 paper "On Some Difficulties in the Theory of Transfinite Numbers and Order Types",[3] was to provide a "no-classes" theory which emulated an impredicative simple type theory (and, hence, classes) by the use of multiple individual variables. The theory would thereby secure Cantor's work and circumvent the paradoxes without introducing a hierarchy of types of functions. In his April 1906 paper "On the Substitutional Theory of Classes and Relations", Russell heralded the no-classes (substitutional) theory as "a complete solution of all the hoary difficulties about the one and the many; for, while allowing that there are many entities, it adheres with drastic pedantry to the old maxim that 'whatever is, is one'."[4] The advantage conferred by the substitutional theory was that it provided a solution to the paradoxes while, at the same time, preserving the conception of logic Russell propounded in *The Principles of Mathematics*, namely, a logic of propositions in which the variable is unrestricted and ranges over all entities.[5]

In "On the Substitutional Theory of Classes and Relations" Russell showed how Cantor's work can be recovered and how the Russell paradox, Cantor's paradox of the greatest cardinal, and Burali-Forti's

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1 *Papers* 5, forthcoming 2014.

2 GRATTAN-GUINNESS has aptly called the substitutional theory the "missing link" between the theory of descriptions and *PM*'s ramified theory of types (*Dear Russell—Dear Jourdain*, p. 94).

3 The paper was received by the London Mathematical Society on 24 November 1905 and read to it on 14 December.

4 *EA*, p. 189; *Papers* 5: 261. (Page references to *Papers* 5 texts were supplied editorially.)

5 In taking the notion of proposition and constituent as basic, the substitutional theory is more consistent than simple type theory with Russell's view in the *Principles* that the logic of propositions is more basic than that of classes, relations, or propositional functions. Russell identified the proposition as the basic "data" of the substitutional theory, more basic than relations in intension ("Substitutional Theory of Classes and Relations", *EA*, p. 175; *Papers* 5: 250).

paradox of the greatest ordinal can all be solved by the techniques of substitution without placing restrictions on the independent entity-variable. However, Russell's notes from April/May 1906 reveal that he had discovered that new paradoxes of propositions could be generated within his substitutional logic of propositions.[6] In his letter to Hawtrey of 22 January 1907,[7] Russell reported that the substitutional theory was "pilled" by a version of this propositional paradox unique to the theory, which Landini has called the $p_0 a_0$ paradox,[8] and it is well known that Russell subsequently abandoned the substitutional theory in favour of the final theory of logical types put forth in *Principia*. Importantly, Russell's initial response to the newly discovered paradoxes was not to abandon the substitutional theory, but rather to modify it, which he endeavoured to do in his 1906 paper "On 'Insolubilia' and Their Solution by Symbolic Logic".[9]

Despite Russell's claim that the modified theory propounded in "Insolubilia" was intended to solve the Epimenides paradox, Landini adamantly maintains that the solution put forth there was intended to solve the $p_0 a_0$ paradox. On Landini's account, Russell construed the Epimenides as a semantic paradox of propositions and sharply separated it from the logical Cantor-style diagonal paradoxes of which the $p_0 a_0$ paradox is an instance. Russell made mention of the Epimenides in "Insolubilia", says Landini, simply to illustrate the power of his solution to the $p_0 a_0$ paradox for a popular audience unfamiliar with the details of the substitutional theory.[10] Moreover, on Landini's interpretation, Russell remained satisfied with the solution put forth in that

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[6] See "On Substitution" (*Papers* 5: **5a**).

[7] The letter is printed in an appendix in Linsky, "The Substitutional Paradox in Russell's 1907 Letter to Hawtrey [Corrected Reprint]" and as *Papers* 5: **4e**.

[8] The name of the paradox was coined by Landini in *Russell's Hidden Substitutional Theory*. Since there are several paradoxes of the form of the $p_0 a_0$ paradox, I shall refer to this as the "Hawtrey version of the $p_0 a_0$ paradox" ("Hawtrey version" for short).

[9] "On 'Insolubilia' and Their Solution by Symbolic Logic" was the English manuscript which Russell sent to Couturat on 16 June 1906 to be translated into French. The paper was published in September 1906 under the French title "Les paradoxes de la logique". In *Papers* 5, Gregory H. Moore uses the English title "The Paradoxes of Logic"—a direct translation of "Les paradoxes de la logique", which was the title suggested to Russell by Couturat in late June 1906 (Schmid, p. 609).

[10] Landini, *Russell's Hidden Substitutional Theory*, p. 377.

paper until at least January 1907, when he discovered that a mitigating axiom introduced there to recover arithmetic brought back a more complicated form of the p_0a_0 paradox as well.[11] These claims are used to buttress Landini's larger argument that the p_0a_0 paradox was the cause of ramification and to support the nominalist interpretation of *Principia* informed by that argument.[12] In this paper, I want to bring to light two facts that are important for interpreting the role of the substitutional theory in Russell's philosophy of logic. The first is that in his April/May 1906 manuscript, "On Substitution", Russell was expressly concerned with three versions of the p_0a_0 paradox. In "On Substitution" he stated the familiar Hawtrey version of the p_0a_0 paradox. However, he also formulated the *Principles*' Appendix B paradox in substitutional terms, yielding another version of the p_0a_0 paradox. Importantly, Russell expressly formulated this latter paradox to involve negation in the antecedent and, rather surprisingly, called this the "purely logical form" of the Epimenides. This suggests that Russell was concerned with the Epimenides in "Insolubilia" precisely because he thought its formulation in substitutional terms yielded a certain version of the p_0a_0 paradox. The second fact, which also presents a challenge for Landini's interpretation, is that in his letter to Jourdain of 4 July 1906, Russell expressed dissatisfaction with his solution to the Epimenides.[13] We have telling evidence from his correspondence with Couturat that "Insolubilia" was completed in June 1906, though it did not appear until September of that year, which suggests that the solution with which Russell was dissatisfied in early July was that which he had proposed in "Insolubilia". In this paper, I shall draw out

[11] Landini, *op. cit.* and "Logicism's Insolubilia and Their Solution by Russell's Substitutional Theory". Landini holds that the paradox which I shall call "the Appendix B version of the p_0a_0 paradox", like the Liar, involves intensional contexts and does not recur with the introduction of Russell's mitigating axiom from "Insolubilia". However, on Landini's view, "The 1906 axiom schema of reducibility nullifies the effect of the abandonment of generalized propositions where extensional contexts are concerned" (*Russell's Hidden Substitutional Theory*, pp. 230–2). In the Hawtrey version of the p_0a_0 paradox, the context is extensional, and hence the paradox is brought back by Russell's mitigating axiom. This version of the p_0a_0 paradox and no other led Russell to ramification. For a counter view, see *Russell's Hidden Substitutional Theory*, p. 157.

[12] Landini, *Russell's Hidden Substitutional Theory*, p. 235.

[13] Grattan-Guinness, p. 91.

the significance of these two facts, and then I shall offer some speculation of how they figure into an account of the rise and fall of the substitutional theory.

Landini has maintained that "Insolubilia" was intended as a solution to the p_0a_0 paradox and not the Epimenides. On Landini's view, the p_0a_0 paradox is not mentioned in "Insolubilia" because the semantic paradoxes like Epimenides are more readily understood, which has the result that Russell's sharp distinction between the logical and semantic paradoxes is obscured.[14] Landini's interpretation contains some crucial insights. Russell's primary concern in that paper was not to solve such semantic paradoxes as the Berry paradox, the König–Dixon paradox, and the Richard paradox. Russell indeed "separated" such semantic paradoxes from the logical ones in "Insolubilia" in so far as he gave them an entirely different treatment, dissolving rather than solving the latter, which he thought were not genuine contradictions but traded on incoherent viciously circular notions of "naming" and "defining".[15] He again took up these semantic paradoxes (Berry, König–Dixon, and Richard) in his 1908 paper, "Mathematical Logic as Based on the Theory of Types", and again pointed out that "all names" and "all definitions" are illegitimate notions. However, he now did so on analogy to the notion that "all classes", "all relations", and "all propositions" were illegitimate notions.[16] His former "separation" of the paradoxes appears to be superseded by the fact that they all arise from a violation of the vicious-circle principle and find a

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[14] Landini, "Logicism's Insolubilia", p. 377. On the view I wish to offer, Russell believed the Epimenides was a propositional paradox with a logical form akin to that of the p_0a_0. It is, of course, still quite possible that he presented the Epimenides rather than the p_0a_0 because the former was more readily understood.

[15] Landini, "Russell's Separation of the Logical and Semantic Paradoxes", pp. 275, 378. Indeed, in both "On the Substitutional Theory of Classes and Relations" and "Insolubilia", Russell appears to hold that the semantic paradoxes do not require the no-classes theory for their solution. See Russell's remarks in the former (*EA*, pp. 184–5; *Papers* 5: 257–8) and in the latter (*EA*, pp. 209–10; *Papers* 5: 282–3). As Stevens points out in his review of Landini's *Russell's Hidden Substitutional Theory*, Russell credited Ramsey with making explicit the distinction between the logical and semantic paradoxes which Russell had regarded (at least in places) as similar ("Substitution and the Theory of Types", pp. 171–2).

[16] "Mathematical Logic", pp. 225–6; *LK*, pp. 62–3; *Papers* 5: 590–1.

common solution in the theory of types.[17] Moreover, while Russell's unpublished manuscripts and correspondence show that it was the p_0a_0 paradox which "pilled" the substitutional theory, his remarks in "Insolubilia" and surrounding correspondence strongly suggest that the substitutional theory was indeed modified to solve the Epimenides. The p_0a_0 paradox is not mentioned in "Insolubilia" at all, while, in his letter to Jourdain of 14 June 1906, Russell wrote:

> … the no-classes theory … shows that we can employ the *symbol* $\dot{x}(\phi x)$ without ever assuming that this symbol in isolation means anything. I feel more and more certain that this theory is right. In order, however, to solve the *Epimenides*, it is necessary to extend it to *general* propositions, i.e. to such as $(x) . \phi x$ and $(\exists x) . \phi x$. This I shall explain in my answer [i.e., "Insolubilia"] to Poincaré's article in the current Revue de Métaphysique.[18]

The motivations expressed by Russell on 14 June remained unchanged in the final version of "Insolubilia".[19] These motivations are further corroborated in the following letter from Lytton Strachey to his sister Pippa, dated 9 July 1906:

> Bertie informs me that he has now abolished not only "classes", but "general propositions"—he thinks they're all merely the fantasies of the human mind. He's come to this conclusion because he finds it the only

[17] In "Insolubilia", Russell acknowledged the importance of Poincaré's vicious-circle principle and gave it the following Peanistic formulation: "Whatever involves an apparent variable must not be among the possible values of that variable" (*EA*, p. 204; *Papers* 5: 289). He was explicit, however, that the vicious-circle principle is not itself a solution to the paradoxes involving impredicativity. In "Insolubilia", he tells us "[i]t is important to observe that the vicious-circle principle is not itself the solution of vicious-circle paradoxes, but merely the result which a theory must yield if it is to afford a solution of them" (*EA*, p. 205; *Papers* 5: 289).

[18] Grattan-Guinness, p. 89. Importantly, Russell was not concerned with the liar paradox of the form "this proposition I am asserting is false", for this is not relevant to assertions made concerning all propositions and, hence, to the need for a solution which produces the vicious-circle principle as a result.

[19] In what follows, I claim that Russell's correspondence with Couturat confirms that the substance of "Insolubilia" was written by around 14 June 1906.

> way in which to get around the Cretan who said that all Cretans were liars. [20]

It would seem that Landini's contention that the Epimenides was not the chief target of Russell's solution in "Insolubilia" cannot be entirely correct. This, however, presents a quandary. If he was aware of the $p_0 a_0$ paradox in May 1906, then why did his ultimate defence of logicism against Poincaré's influential objections not address this paradox of propositions? Did he think that by solving the Epimenides paradox he had solved the $p_0 a_0$ paradox as well, and, if so, what did these paradoxes have in common?

In his review of Landini's *Russell's Hidden Substitutional Theory*, Stevens claims that there is a relevant similarity between the $p_0 a_0$ paradox and the Epimenides. The $p_0 a_0$ paradox, on one formulation offered by Landini, derives from the following substitution:

$$(\exists p, a)\left(a \text{ in } p \,.\, \& \,.\, (z)\left(p\,\frac{z}{a}!\right)\left\{(\exists r, c)\left(z = \left\{r\,\frac{y}{c} \,.\, \supset_y \,.\, y\right\} \& \sim\left(r\,\frac{z}{c}\right)\right)\right\}\right).$$

Stevens writes: "The Epimenides is almost a mirror image.… [I]n place of the resultant of the above substitution, we will have:

$$(\exists r, c)\left(z = \left\{r\,\frac{y}{c} \,.\, \supset_y \,.\, \sim y\right\} \& \left(r\,\frac{z}{c}\right)\right)\text{''}.$$ [21]

Stevens' comparison suggests, *contra* Landini, that the Epimenides was not only a logical paradox but shared a common diagonal structure with the $p_0 a_0$ paradox and the Appendix B paradox. The conclusion Stevens draws, however, is difficult to interpret. He writes:

> … the similarity in structure is evident and, surely, unlikely to have escaped Russell's notice. The paradox shows, as the substitutional paradox [$p_0 a_0$] and Appendix B paradox do, that the assumption of propositions

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[20] Holroyd, *Lytton Strachey* 1: 290.

[21] Stevens, p. 175. I have translated both Landini and Stevens' notation into the more familiar style which Russell uses in his January 1907 letter to Hawtrey. Except where Russell's own notation is cited, I have followed them in using curly brackets as a nominalizing operator to show where formulas are used as singular terms representing propositions.

> as logical objects turned out to introduce problems just as severe as … those introduced by the assumption of classes.… What I do think it shows, however, is that Russell was interested in the Epimenides and variants of the Liar because he thought they shared important features with the other paradoxes of propositions.[22]

The notion that the Epimenides and variants of the Liar merely have important features in common with other propositional paradoxes lends itself to the view that the Epimenides is similar to the p_0a_0 paradox, the Appendix B paradox, and other propositional paradoxes including variants of the Liar, merely in that it involves propositional quantification. Stevens' remark that the Epimenides is almost a "mirror image" of, and structurally similar to, the p_0a_0 paradox suggests the stronger claim that the Epimenides, like the p_0a_0 paradox, is a diagonal paradox generated by a violation of Cantor's power-class theorem. This is just what Landini denies as a part of his larger argument that the p_0a_0 paradox was the sole cause of ramification. With these considerations in view, I shall put forth an explanation of why Russell, who was aware of the p_0a_0 paradox which "pilled" the substitutional theory, seemed to be exclusively concerned with solving the Epimenides in "Insolubilia". To do so, it will be helpful, first, to show that Russell had expressly formulated more than one version of his p_0a_0 paradox prior to proposing his "no general propositions" solution in "Insolubilia" and, second, to introduce the distinction between entity and propositional substitution which he adopted in his manuscripts to block these propositional paradoxes and in terms of which he gave his logical construction of the Epimenides.

Sometime in April or May 1906,[23] at most a few weeks after writing "On the Substitutional Theory of Classes and Relations", Russell discovered that a propositional paradox is generated within the substitu-

[22] *Ibid.*

[23] The first mention of the p_0a_0 paradox as a genuine contradiction is in Russell's April/May manuscript "On Substitution", where he also tried to solve the Epimenides. "On the Substitutional Theory of Classes and Relations" was received by the London Mathematical Society on 24 April 1906, and he read the paper on 10 May, which suggests it was sometime after this and before completing "On Substitution" that that he realized the p_0a_0 "oddity" was a genuine contradiction.

tional theory by the same diagonal method involved in Cantor's *reductio* proof of the power-class theorem.[24] Cantor's power-class theorem showed that there could not be a function from objects onto classes of objects (or functions of objects). The identity conditions for propositions, however, readily afford such functions and are thus in violation of Cantor's result, i.e., they generate genuine contradictions. Indeed, the paradoxes which beset the transfinite all have this form, which Russell realized as early as 19 September 1902, when he told Frege that "… from Cantor's proposition that any class contains more subclasses than objects we can elicit constantly new contradictions."[25] According to Cantor's power-class theorem, the number of classes of entities is always greater than the number of entities,[26] which is the result that guarantees the transfinite numbers, which Russell was adamant about preserving. However, if propositions are entities, i.e., values of entity variables, an entity can be correlated with every class of propositions emulated by the matrices of substitution. Otherwise stated, the identity conditions for propositions give rise to functions from propositions onto the classes of propositions emulated by the matrices of substitution. The result is that the number of entities (propositions) is at least equal to the number of pairs of entities (classes emulated by matrices), violating Cantor's result. The paradox which results, when stated in substitutional terms, is the $p_0 a_0$ paradox. Within the substitutional theory, it is easy to find a function from matrices emulating classes, e.g., p/a, p'/a', p''/a'' etc., to propositions, e.g., $\left\{p \frac{b}{a} ! \, q\right\}$, $\left\{p' \frac{b}{a'} ! \, q\right\}$, $\left\{p'' \frac{b}{a''} ! \, q\right\}$ etc. Here the problematic function is:

[24] Cocchiarella, in "The Development of the Theory of Logical Types and the Notion of a Logical Subject in Russell's Early Philosophy", first suggested a violation of Cantor's power-class theorem by the $1 \to 1$ correlation of matrices to entities (propositions). According to Pelham and Urquhart in "Russellian Propositions", the difficulty arises in stating truth conditions for quantified propositions. The authors attempt to reconstruct a substitutional theory based on partial truth-assignments, in order to provide an alternative to ramified type theory that is in keeping with Russell's philosophical commitments while, at the same time, solving the $p_0 a_0$ and the Liar paradoxes and preserving the Axiom of Infinity which is derivable in substitution.

[25] Frege, *Philosophical and Mathematical Correspondence*, p. 147.

[26] Or, more precisely, a class must have more subclasses than members.

$$f(p,a) = \left\{p\frac{b}{a}!\,q\right\}.$$

Some of these are members of their correlated classes, while others are not. The ones which are not form a class, namely, the class $p_0 a_0$, where

$$p_0 = \left\{(\exists p,a)\,.\,a_0 = \left(\left\{p\frac{b}{a}!\,q\right\}\ \&\sim\left(p\frac{a_0}{a}\right)\right)\right\}.$$

This yields a contradiction:

$$(\exists p,a)\left(p_0\frac{b}{a_0}!\,q = p\ \frac{b}{a}!\,q\ \&\ p\frac{p_0\frac{b}{a_0}!q}{a}\right)\,.\equiv.$$
$$\sim(\exists p,a)\left(p_0\frac{b}{a_0}!\,q = p\frac{b}{a}!\,q\ \&\ p\frac{p_0\frac{b}{a_0}!q}{a}\right).$$

Or, for convenience:

$$p_0\frac{p_0\frac{b}{a_0}!q}{a_0}\,.\equiv.\sim\left(p_0\frac{p_0\frac{b}{a_0}!q}{a_0}\right).$$

This is a contradiction on assumption by a substitutional axiom that $p = p_0$ and $a = a_0$, so that

$$p\frac{p_0\frac{b}{a_0}!q}{a}\,.\equiv.\,p_0\frac{p_0\frac{b}{a_0}!q}{a_0}.$$

The $p_0 a_0$ paradox, then, is a full-blown logical contradiction generated by a violation of Cantor's power-class theorem.[27] The $p_0 a_0$ paradox formulated above is the version which Russell presented in his 1907 letter to Hawtrey and which had made its first appearance in "On Substitution". I call this the Hawtrey version of the $p_0 a_0$ paradox. In "On Substitution" Russell also discovered that a $p_0 a_0$ paradox which is the analogue of the Appendix B paradox from the *Principles*, now stated in terms of substitution, could be generated by the function:

[27] For LANDINI's articulation of the paradox, see *Russell's Hidden Substitutional Theory*, p. 202.

$$f(p,a) = \left\{p\frac{s}{a} \supset_s s\right\}.$$

The above function is from matrices emulating classes, e.g., p/a, p'/a', p''/a'', etc., to the propositions $\left\{p\frac{s}{a} \supset_s s\right\}$, $\left\{p'\frac{s}{a'} \supset_s s\right\}$, $\left\{p''\frac{s}{a''} \supset_s s\right\}$. The members not in their correlated class form the class $p_0 a_0$, where

$$p_0 = \left\{(\exists p, a)\left(a_0 = \left\{p\frac{s}{a} \supset_s s\right\} \,\&\, \sim\left(p\frac{a_0}{a}\right)\right)\right\}.$$

I call the paradox which results from the substitution of $\left\{p\frac{s}{a} \supset_s s\right\}$ for a_0 in p_0, the Appendix B version of the $p_0 a_0$ paradox.

While Russell, thinking that all of the paradoxes of logic were solved in his substitutional theory, had set aside the Epimenides, the new conflict with Cantor's power-class theorem (that every class emulated by a substitutional matrix can be correlated with a distinct proposition) rekindled his interest in the Epimenides.[28] In "On Substitution" Russell formulated the problematic proposition as follows:

$$p_0 = (\exists p, a) : a_0 = p\frac{b}{a}!\,q : p\frac{a_0}{a}!\,r. \supset r\,.\sim r \; {}^{29}$$

where p_0 is defined as the proposition "there is some p and some a such that a_0 is the proposition '*q results from substituting b for a in p*', and (for all r) r's resulting from the substitution of a_0 for a in p implies $\sim r$." Russell supposed that what is to be substituted for a_0 in p_0 is the proposition $p_0\frac{b}{a_0}!\,q$. The result is contradictory on the assumption that, since $\left\{p_0\frac{b}{a_0}!\,q\right\}$ is identical to the proposition $\left\{p\frac{b}{a}!\,q\right\}$, they must have identical constituents. Hence, $p_0 = p$ and $a_0 = a$. In "On Substitution" he attempts to circumvent the contradiction by adopting the principle that a proposition cannot be substituted for an entity. This requirement blocks the problematic substitution of the proposition $\left\{p_0\frac{b}{a_0}!\,q\right\}$ for a_0 in p_0. The function which here violates Cantor's

28 "On Substitution", fos. 32, 33 (*Papers* 5: 139–40). "Logic in Which Propositions Are Not Entities", fo. 13 (*Papers* 5: 265).

29 When nominalized, this is: $p_0 = \left\{(\exists p, a) : a_0 = p\frac{b}{a}!\,q : p\frac{a_0}{a}!\,r\,. \supset r\,.\sim r\right\}$.

power-class theorem is the inverse of the $1 \rightarrow 1$ function $(p, a) = \left\{p \frac{b}{a} ! \, q\right\}$. By the requirement stipulated in "On Substitution" that a proposition cannot be substituted for an entity, $1 \rightarrow 1$ functions such as $f(p, a) = \left\{p_0 \frac{b}{a_0} ! \, q\right\}$ are blocked,[30] so that, for example, we may have $\left\{p \frac{b}{a} ! \, q\right\} = \left\{p_0 \frac{b}{a_0} ! \, q\right\}$ without having $a = a_0$ and $p = p_0$.[31] We may now turn to a consideration of whether Russell construed the Epimenides as akin to the $p_0 a_0$ paradox and, if so, what sort of substitution it might involve.

What the manuscripts reveal is that Russell entertained the notion that, when the Epimenides was stripped of such psychological features as "assertion", or "belief",[32] what remained was a logical paradox with a form akin to that of the $p_0 a_0$ paradox. Working within the distinction he had introduced earlier in "On Substitution" between propositional and entity substitution, Russell presented the Epimenides in its "purely logical form". First, he supplied the function from which the paradox is derived:

$$fp \,.\!=\!.\, (\exists q, b) : p = \sim \vee \text{‘} q/b \,.\, {\sim} q \frac{p}{b} \,.$$

The notation involved is that which is appropriate to "propositional substitution", which he had first applied to the $p_0 a_0$ paradox above. Now, $\vee$ ‘q/b is defined as

$$q \frac{s}{b} \supset_s {\sim} s \qquad \text{Df.}$$

Next, the function is applied to itself:

[30] "On Substitution", fo. 57 (*Papers* 5: 148).

[31] *Ibid.*, fo. 83 (*Papers* 5: 160).

[32] By contrast, in his manuscripts, chiefly "On Substitution" and "The Paradox of the *Liar*", Russell also attempted solutions to versions of the Epimenides which emphasized its psychological features, as well as the fact that it seemed crucially to involve the concepts of falsity and negation.

$$f(\sim \vee `f\hat{p}) \,.=:.\, (\exists q, b) :.\, (fp_0)\frac{p}{p_0} \,.\supset_p .\, \sim p :=: \\ q\frac{s}{b} .\supset_s .\sim s :.\sim q\frac{\sim\vee`f`\hat{p}}{b}.$$ [33]

That is:

$$fp =:.\, (\exists q, b) :.\, p \,.=: q\frac{s}{b} .\supset_s .\sim s :.\sim q\frac{p}{b}.$$

Stated in the familiar language of the usual substitutional theory, that is, without distinguishing propositional from entity substitutions, this is:

$$fx = \left\{(\exists q, b)\left(x = \left\{q\frac{z}{b} \supset_z \sim z\right\} \,\&\, \sim(q\frac{x}{b})\right)\right\}.$$

Importantly, it is different from the Hawtrey version of the $p_0 a_0$ paradox given above, and matches more exactly the pattern we would expect from a transcription of the Appendix B paradox into the language of the substitutional theory (given above as the Appendix B version of the $p_0 a_0$ paradox). What the manuscript "On Substitution" shows, then, is that immediately prior to offering his solution to the propositional paradoxes in "Insolubilia", Russell had formulated not only the Hawtrey version of the $p_0 a_0$ paradox, but also a substitutional paradox analogous to his Appendix B paradox, and regarded it, when stated with a negation in the antecedent, as the purely logical form of the Epimenides.

Russell's task, in "Insolubilia", was to block the paradoxes from within a logic of propositions without blocking general induction, that is, induction for all properties rather than a restricted set of them or, in a logic of propositions, induction for all propositions. He articulated his solution as follows:

> Hence to reconcile the unrestricted range of the variable with the vicious-circle principle, which might seem impossible at first sight, we have to construct a theory in which every expression which contains an apparent variable (i.e. which contains such words as *all*, *any*, *some*, *the*) is shown to be a mere *façon de parler*…. And such expressions include all descriptive

[33] "On Substitution", fo. 82 (*Papers* 5: 159).

> phrases (the so-and-so), all classes, all relations in extension, and all *general* propositions, i.e. all propositions of the form "ϕx is true for all (or some) values of x".[34]

The result is that statements concerning "all propositions", and indeed any statement containing a bound variable, do not give rise to some new definite proposition. Russell wrote:

> … a statement about *all* is really an affirmation of an ambiguous one of the several propositions got from particular cases. E.g. if we state: "Whatever x may be, $x = x$", we are stating an ambiguous one of the propositions of the form $x = x$; thus, though we have a new *statement*, we do not have a new proposition.[35]

In "Insolubilia", then, he denied that an assertion about all propositions gives rise to a new distinct proposition, without denying that the unrestricted variable ranges over all entities. He then introduced a mitigating axiom to recover arithmetic: while there is no proposition expressed by a statement containing a bound variable, there is an equivalent statement in which no bound variable appears. As we have seen in the Appendix B version of the $p_0 a_0$ paradox, which is the analogue of Russell's purely logical Epimenides, we have

$$p_0 = \left\{(\exists p, a)\left(a_0 = \left\{p\,\frac{s}{a} \supset_s s\right\} \;\&\; \sim\left(p\,\frac{a_0}{a}\right)\right)\right\}.$$

In this case, the problematic function which produces a violation of Cantor's power-class theorem is $f(p, a) = \left\{p\,\frac{s}{a} \supset_s s\right\}$, which involves a general proposition. In the case of the Epimenides, we have: $f(p, a) = \left\{p\,\frac{s}{a} \supset_s \sim s\right\}$, which involves a general proposition. In the Hawtrey version of the $p_0 a_0$ paradox, the function which violates Cantor's power-class theorem does not involve a general proposition. It is the inverse of the $1 \to 1$ function

$$f(p, a) = \left\{p\,\frac{b}{a}\,!\,q\right\}.$$

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[34] "Insolubilia", p. 206 (*Papers* 5: 290).

[35] *Ibid.*, p. 204 (*Papers* 5: 289).

Nevertheless, the definition of $p_0 a_0$ does involve a general proposition.[36] We have:

$$p_0 = \left\{(\exists p, a)\left(a_0 = \left\{p\,\frac{b}{a}\,!\,q\right\} \;\&\sim \left(p\,\frac{a_0}{a}\right)\right)\right\}.$$

While Russell attempted, in his manuscripts, to block the functions which violate Cantor's power-class theorem by denying that a proposition can be substituted for an entity, the solution which he adopted in "Insolubilia" targeted quantification over propositions. By dispensing with general propositions, Russell intended to solve the "purely logical" Epimenides, the Appendix B version of the $p_0 a_0$ paradox which shared its form, and the familiar Hawtrey version of the $p_0 a_0$ paradox—all of which, as we have just seen, involved general propositions. In "On the Substitutional Theory of Classes and Relations" and "Insolubilia", he gave semantic paradoxes of naming and defining a separate treatment, and there is, in this sense, a basis for Landini's claim that he distinguished them from the logical paradoxes.[37] However, on my view Russell was concerned with the Epimenides paradox in "Insolubilia" precisely because it was an analogue of the Appendix B version of the $p_0 a_0$ paradox and, on his own construction of it, "purely logical", so that its solution could be extended to the other propositional paradoxes which afflicted his theory. Recall that Landini also holds that Russell remained committed to the "Insolubilia" solution until at least January 1907. I shall turn next to a consideration of the data which appears to controvert this view.

It is important to recognize that a great deal hinged on the success of the modified substitutional theory propounded in "Insolubilia". In his influential paper, "Les mathématiques et la logique", Poincaré had raised various objections to the logicist project which Russell had proposed to carry out in terms of the substitutional theory in his November/December 1905 paper "On Some Difficulties in the Theory of

[36] *Papers* 5 will contain the data needed to determine whether Russell recognized the different ways in which the propositional paradoxes involve general propositions in their formulation, and whether he realized that the Hawtrey version alone was brought back by his mitigating axiom of "Insolubilia" as Landini urges.

[37] For a counter view see PELHAM AND URQUHART, p. 12.

Transfinite Numbers and Order Types". When Poincaré first published his criticisms,[38] Russell was confident about the substitutional theory and had concluded, in "On the Substitutional Theory of Classes and Relations", that all of the paradoxes were solved within it. In the light of the newly discovered propositional paradoxes, he set out to modify the theory before giving his official reply to Poincaré. In a letter to Couturat dated 15 May 1906, Russell wrote: "I shall follow your advice by responding to Mr. Poincaré … I will not respond right away, because I would like to put in order what I have to say about the solution to the contradictions."[39] By the time "On the Substitutional Theory of Classes and Relations" was accepted for publication in October 1906, Russell had already published his proposed solution to the paradox, and, around 14 October 1906, he withdrew the earlier article from publication. His proposed solution to the newly discovered forms of the contradiction was published in French, under the title "Les paradoxes de la logique", in September 1906. The reply was intended to give Russell's official defence of the logicist project carried out in the terms of the no-classes theory against Poincaré and his followers, who held that the logicist project could not succeed without the theory of classes.

Couturat was responsible for translating "Insolubilia" into what would become the French paper "Les paradoxes de la logique", and from the correspondence we know that the substance of "Insolubilia" was completed by 16 June 1906, for on 15 June 1906, Russell told Couturat he would send the paper the following day.[40] Recall that, according to Strachey, and his own letter to Jourdain of 14 June, Russell intended to solve the Epimenides in "Insolubilia" by dispensing with general propositions.[41] The crucial parts II and III, in which he proposed to dispense with general propositions to solve the Epimenides,

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[38] Poincaré, "Les mathématiques et la logique".

[39] Schmid, p. 604.

[40] *Ibid.*, p. 606.

[41] It is worth noting that in his letter to Jourdain of 14 June 1906 (Grattan-Guinness, p. 89), the solution Russell proposed was to dispense with general propositions, which is the solution proposed in "Insolubilia". General propositions are propositions which assert the truth (or falsity) of all values of a propositional function.

were certainly in Couturat's possession before 23 June 1906. Moreover, in his letter to Couturat of 23 June, Russell indicated that he intended to make only minor changes to the paper.[42] In his reply of 25 June 1906, Couturat sent him the French translation of the paper and indicated that he was awaiting corrections. In particular, he suggested that Russell include more on induction and the actual infinite in reply to Poincaré's objections, which, in his letter of 2 July 1906, Russell declined to do.[43] Indeed, subsequent correspondence makes it fairly clear that all changes were editorial and not substantive.[44] In the letter of 2 July, he sent only minor corrections to Couturat of what was the now complete French version of "Insolubilia", containing the solution to the Epimenides proposed in his official reply to Poincaré.[45] However, in a letter to Jourdain of 4 July 1906, Russell told him that he is "not very well satisfied" with his views concerning the solution to the Epimenides. He even inquired as to whether Jourdain knew of any way to solve the Epimenides![46] Since the Couturat correspondence confirms that his solution to the Epimenides in "Insolubilia" had been formulated prior to his letter of 4 July 1906, the solution with which he was now dissatisfied was that proposed in "Insolubilia". On Landini's interpretation, Russell remained satisfied with the solution proposed in "Insolubilia" at least until January 1907, when he discovered that a mitigating axiom introduced in "Insolubilia" to recover

[42] The most significant change is relatively minor. Russell had attributed to William of Ockham the view that paradoxes akin to the Epimenides arise from vicious circles and are solved by satisfying the principle that no proposition can assert anything of itself. In the published French text, he clarified: "The vicious circle is not mentioned explicitly, but it appears indubitable that the sense of the proposed solution is that which I attribute to it here" (*Papers* 5: 282 n.7; French text, 748 n.7).

[43] SCHMID, pp. 611–12.

[44] In his letter of 22 July 1906, Couturat told Russell that he had made the minor corrections Russell suggested and would send the proofs so he could make any further corrections (SCHMID, pp. 612–14). In his letter of 1 August 1906, Russell told Couturat that he had found no further changes to make (*ibid.*, p. 615). All of this suggests that, apart from a few minor corrections, the complete French version of "Insolubilia" was in Couturat's possession before or on 25 June.

[45] For instance, in the letter of 2 July 1906, Russell wrote: "P. 16, l. 11. I do not see why you questioned the word *apparent*. When I say 'I am lying', I say: '$(\exists p)$. I assert $p \,.\, {\sim}p$'. Here p is *apparent.* As for the title, I think you are right." He added that "'The Paradoxes of Logic' seems to me to be a very good title."

[46] GRATTAN-GUINNESS, p. 91.

arithmetic brought back a more complicated form of the $p_0 a_0$ paradox as well.[47] The letter to Jourdain of 4 July 1906 suggests, *contra* Landini, that Russell had grown dissatisfied, well before January 1907, with the solution proposed in "Insolubilia" to solve the propositional paradoxes.[48] What is especially puzzling, however, is that he went on to publish his solution in French in the *Revue de métaphysique* in September 1906. It is difficult to imagine that Russell, who would soon withdraw his earlier paper "On the Substitutional Theory of Classes and Relations" on account of the propositional paradoxes, would commit himself in print to a flawed theory, particularly in the paper designed to give a definitive defence of logicism and put to rest the debate with Poincaré and his followers. The nature of Russell's dissatisfaction wants explanation.

If Russell was dissatisfied in July 1906 with his solution from "Insolubilia", remarks in "On Substitution" suggest that his dissatisfaction was not technical but philosophical. Toward the end of "On Substitution" he set forth the solution proposed in "Insolubilia":

> We shall say that "$\vdash . \phi x$" asserts one, but an ambiguous one, of the values of $\phi \hat{x}$; there is not a new proposition in addition to these values, any more than there is a new entity x in addition to the values of x. And we shall no longer distinguish between "$\vdash . \phi x$" and "$\vdash . (x) . \phi x$".[49]

Noting that the advantage of this solution is that it was technically feasible and avoided contradictions, Russell added:

[47] Landini, *Russell's Hidden Substitutional Theory* and "Logicism's Insolubilia". In his book, pp. 232–3, Landini shows how the $p_0 a_0$ paradox is recovered by the introduction of Russell's mitigating axiom from "Insolubilia". There is not, to my knowledge, any documentary evidence of Russell's discovery of more complicated forms of the paradox in his attempt to work out his axioms of reduction in "The Paradox of the *Liar*" or subsequent texts.

[48] To be precise, Landini holds that it is solely the $p_0 a_0$ paradox, unique to the substitutional theory, which could not be solved without some modification of the axioms of "Insolubilia" (*Russell's Hidden Substitutional Theory*, p. 230). His view is that while the Appendix B paradox of propositions and the Epimenides involve intensional contexts and cannot be formulated without general propositions, the $p_0 a_0$ paradox is recovered by Russell's mitigating axiom of "Insolubilia".

[49] "On Substitution", fo. 250 (*Papers* 5: 229).

> As philosophy, I am dissatisfied with the view that "all men are mortal" is an ambiguous assertion of the mortality of this or that man. What makes this view unsatisfactory is that, in $(x).\,\phi x$, we don't primarily have the *values* of ϕx, but we have primarily $\phi\hat{x}$; and our proposition is really about $\phi\hat{x}$ rather than about any of its values....
>
> The only possible view is that $(x)\,.\,\phi x$ or $(\phi)\,.\,f(\phi)$ does not assert something for "all arguments", but only for "all arguments which give a significant value of the function".[50]

To construe $(x)\,.\,\phi x$ as asserting something for all arguments which give a significant value of the function, is to place a restriction on the variable. Thus, prior to setting forth his solution in "Insolubilia", Russell had concluded that his solution, though technically viable, was philosophically unsatisfactory. If Russell thought that the chief advantage of substitution—the preservation of the unrestricted entity variable—had been lost, it is not altogether surprising that he went on to supply a solution technically equivalent to that proposed in "Insolubilia" from within a theory of types whose ranges of significance are restricted to avoid contradictions. In his letter to Jourdain of 10 September 1906, he wrote:

> I incline at present to the doctrine of types, much as it appears in Appendix B of my book. To this I add that propositions and functions can never be *apparent* variables, so that statements about *all* of them are meaningless. Statements about *any* of them are admitted; but these affirm ambiguously some one of a number of propositions, and do not state a new proposition. Take, e.g.,
>
> $$p\,.\supset.\,p \vee q$$
>
> which is a primitive proposition. This is a single formula, intended to state each separate case, not to state that each separate case is true. For purposes of deduction, this is necessary, quite apart from Epimenides.... This observation is due to Frege....
>
> If we want to state any case of ϕx, we write "$\vdash .\, \phi x$"; if we want to state that all cases are true, we write "$\vdash (x)\,.\,\phi x$". Thus there is to be no such thing as a statement that something holds of all propositions, though

[50] *Ibid.*, fo. 251 (*Papers* 5: 229).

> there are ambiguous statements applying to any propositions; but these do not state a new proposition.[51]

Russell's proposal here was intended, like that of "Insolubilia", to circumvent the paradoxes by dispensing with general propositions without blocking general induction. The solution was now stated, however, from within a theory of types. He vacillated more than once during this period between a substitutional theory and a theory which placed type indices on predicate variables, and it is possible that he simply intended to give a convenient expression of a solution which was, at bottom, to be carried out within the substitutional theory.[52] Indeed, there is much in the manuscripts to suggest that Russell regarded "Insolubilia" as programmatic.[53] While he entertained various solutions to the semantic version of the Epimenides in his September manuscript, "The Paradox of the *Liar*",[54] Russell again revisited the solution to the "purely logical" version of the Epimenides adumbrated in "Insolubilia" and attempted to work out his mitigating axioms (principles of reduction) in terms of substitution to recover induction. However, in so doing, he again faced philosophical dissatisfaction with the theory, particularly with the lack of philosophical motivation for his reduction principle. Once again, he entertained a propositional hierarchy in the hope that he would only require "relative types" (orders)[55]

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[51] Grattan-Guinness, pp. 91–2.

[52] Russell again explored the solution stated here in types from within substitution in "The Paradox of the *Liar*", but this time with orders of propositions (fo. 23; *Papers* 5: 328).

[53] Russell underscored the likelihood that the views proposed in "Insolubilia" would require modification (*EA*, pp. 198–9; *Papers* 5: 284). He pointed out that a careful analysis and mathematical reconstruction would be needed to decide on the absolutely best form in which to state the principles (*ibid.*, p. 214; *Papers* 5: 296). He took up these questions in "The Paradox of the *Liar*".

[54] Russell's opening remarks in "The Paradox of the *Liar*" begin with the peculiar features of the Epimenides, particularly with the question of whether true propositions alone subsist. In response, he entertained the possibility that belief is a relation between a thought and the constituents of a proposition rather than the proposition itself (fos. 4–5; *Papers* 5: 321–2).

[55] This is essentially the view Russell ultimately adopted in "Mathematical Logic as Based on the Theory of Types", on which orders (types) of propositions are relative to the arguments taken for propositional variables in particular cases.

to block the paradoxes, but he concluded:

> It rather looks as if the complications of the substitutional point of view were too great.
>
> If, when γ is a constant, $\phi\gamma$ is really a denoting expression, that seems to militate against the substitutional view. For it seems to demand that $\phi\hat{\gamma}$ should be taken into consideration, and $\phi\gamma$ regarded as
> $$(\phi\hat{\gamma})\frac{\gamma}{\hat{\gamma}}.$$
> But if $\phi\hat{\gamma}$ must in any case be admitted, the advantages of substitution are lost. The notation $\phi\gamma$ involves regarding $\phi\gamma$ as an *instance* of $\phi\hat{\gamma}$, and thus approaching it by a denoting expression. If we put down the actual value which $\phi\hat{\gamma}$ has in the case of the argument γ, ϕ has been swallowed up.[56]

The attempt to treat $\phi\gamma$ denoting all propositions of the form $\phi\gamma$ in which γ is not bound required putting $\phi\gamma$ in subject position $(\phi\hat{\gamma})\frac{\gamma}{\hat{\gamma}}$, so that the substitutional theory was incapable of avoiding the introduction of genuine propositional function variables (which stand for any value in the domain of the quantifier).[57] If Russell had supposed that the introduction of functions as apparent variables was unavoidable, there would have been no advantage to retaining substitution. His adoption of a hierarchy of orders of propositions in "The Paradox of the *Liar*" to block these propositional paradoxes within the theory, far from assuaging his philosophical dissatisfaction, amounted to precisely the view he had rejected as philosophically "intolerable" in "On Substitution".[58] Whitehead's letter to Russell of 7 October 1906 suggests that while Russell sought to avoid restrictions on the variable and hoped to arrive at a doctrine of logical types which could be translated into the terms of substitution, he came increasingly to believe that a substitutional theory which preserved the unrestricted variable could not alone solve the paradoxes. Whitehead wrote:

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[56] "Paradox of the *Liar*", fos. 86–7 (*Papers* 5: 359).

[57] Russell's concern was that the actual propositions would only be denoted by $\phi\hat{\gamma}$ when the value of ϕ was assigned ("Paradox of the *Liar*", fo. 88; *Papers* 5: 359–60).

[58] "On Substitution", fo. 140 (*Papers* 5: 185).

> The nastiness which you wanted to avoid is the Frege bugbear of propositional functions becoming unmeaning when certain terms are substituted. According to the doctrine of types we have got to put up with this. Thus certain things (such as functions) which can be named and talked about won't do as arguments in some propositional functions. The result is that we have to use the restricted variable. The doctrine of substitution was on stronger ground here; for it did without the function entirely, and simply brought in p/a as a typographical device…. Hence if you want the unrestricted variable, the doctrine of substitution is the true solution. But then this doctrine won't work, will it? Nor do I see the necessity for its complications.[59]

The philosophical merits of the substitutional theory were vitiated by the requirements of a solution to the propositional paradoxes.[60] Russell's inability to discover any philosophically satisfactory substitution-theoretic alternative to the solution set forth in "Insolubilia" led him to reject substitution in the late fall of 1906 or early winter of 1907. In January 1907, Russell famously remarked to Hawtrey that the substitutional theory has been "pilled" by the $p_0 a_0$ paradox. Moreover, in his letter to Jourdain of 1 June 1907, he reported that he had abandoned the "no general propositions" solution of "Insolubilia" and the proof of infinity dependent upon it, attributing this to the fact that "… the liar and its analogues has led me to be chary of treating propositions as entities."[61] Nevertheless, the substitutional theory appeared alive and well in his 1908 paper "Mathematical Logic as Based on the Theory of Types", where he again introduced a hierarchy of propositions to obviate the propositional paradoxes. Whereas Russell doubted, in "The Paradox of the *Liar*", that he could avoid types of propositional functions, he now held that type-regimented propositional functions were merely variables limited by internal significance conditions, analyzable into substitution with orders. While the theory can be stated most conveniently in terms of propositional functions

[59] Whitehead to Russell, 7 October 1906, RA1 710.057398.

[60] Or, if Landini's view proves correct, the requirements of a solution which blocks the Hawtrey version of the $p_0 a_0$ paradox.

[61] Grattan-Guinness, p. 105.

regimented by ramified type indices, it is essentially a ramified substitutional theory, which is, of course, type free.[62] The manuscripts suggest that the substitutional theory re-emerged precisely when Russell came to reconsider the source of his earlier philosophical dissatisfaction.

In June 1907 Russell added a note in the margin of the very page which contains the argument which "militate[s] against the substitutional view": that $\phi\gamma$, which denotes any proposition in which γ is free, must be an instance of $\phi\hat{\gamma}$. He now wrote: "I doubt whether this argument is valid against substitution. $\phi\gamma$ denotes q, where q is the actual value for the argument γ. If p is the value for the argument α, $p\frac{\gamma}{\alpha}!\,q$. The point wants reconsidering."[63] The note was added before "Mathematical Logic" was completed in late June or early July 1907. There the substitutional theory was resurrected precisely to avoid quantification over propositional function variables (functions as apparent variables). Russell may well have been content, from a technical point of view, with the theory from "The Paradox of the *Liar*" that translated the convenient language of predicate variables into the underlying grammar of substitution, for it was this theory he adopted in "Mathematical Logic", now with a new philosophical gloss of limitations on (bound) predicate variables.[64] Whitehead's letter to Russell of 16 June 1907 confirms, in any case, that around the time he added the marginal note, Russell had again embraced substitution as the "proper explanatory starting-point" for type theory, but this time with a hierarchy of propositions.[65] Perhaps his return to substitution as the

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[62] In "Mathematical Logic" Russell explained how a hierarchy of functions was derived from propositions of various orders by means of substitution, and explicitly told us the advantage of regarding the theory as, at bottom, substitutional (p. 239; *LK*, pp. 77–8; *Papers* 5: 603–4).

[63] Marginal note dated June 1907 added to fo. 87 of "The Paradox of the *Liar*" (*Papers* 5: 805).

[64] This possibility comports with Russell's claim: "In 1906 I discovered the theory of types. After this it only remained to write the book out.... I worked at it from ten to twelve hours a day for about eight months in the year, from 1907 to 1910" (*Auto.* 1: 152). On this account, he had arrived at the technical theory in his 1906 "The Paradox of the *Liar*" manuscript, but had yet to arrive at his philosophical gloss on the theory. See also section "IV. Functions as Variables" in Russell's "Introduction to the Second Edition" of *Principia* (1: xxviii).

[65] Whitehead to Russell, 16 June 1907. Though he adopted a hierarchy of orders of

explanatory starting-point for types was occasioned by his envisioning, in the weeks before "Mathematical Logic" was completed, a solution to his earlier philosophical qualms. If this was so, then perhaps Russell regarded his own chariness of propositions as entities as compatible with his attempt to give a proxy for higher-order quantification in substitution.

I have offered an account of why Russell, who had discovered the p_0a_0 paradox in his April/May 1906 manuscript "On Substitution", purported to be concerned with solving the Epimenides paradox in "Insolubilia". Since the p_0a_0 paradox had been discovered prior to "Insolubilia", it is difficult to imagine that he would have neglected its solution in the paper intended to vindicate logicism by showing how the substitutional theory could obviate all of the logical paradoxes while recovering Cantor's work. I have suggested that Russell construed the Epimenides as a logical paradox analogous to the Appendix B version of his p_0a_0 paradox and that the "no general propositions" theory which he proposed in "Insolubilia" was intended to solve the Epimenides precisely because he thought its solution could be extended to both versions of the p_0a_0 paradox. Russell did not, then, exclude a solution to the p_0a_0 paradox from the paper intended to defend logicism against Poincaré's objections. There is not, to my knowledge, any manuscript material to corroborate Landini's view that Russell only abandoned his "Insolubilia" solution due to a discovery that the mitigating axiom introduced to recover Cantor's work also resuscitated the Hawtrey version of the p_0a_0 paradox. Rather, the fact that Russell had expressly entertained a syntax of orders of propositional variables to solve the propositional paradoxes in "The Paradox of the *Liar*", together with his expression of philosophical dissatisfaction with the solution to the Epimenides propounded in "Insolubilia", presents a challenge for Landini's view that Russell remained committed to his "Insolubilia" view until at least January 1907. I have suggested that he may have believed that "Insolubilia" provided a technically valid solution to all three versions of the p_0a_0 paradox, but that the account of propositional quantification as well

propositions in "Mathematical Logic", he may also have hoped that propositional symbols would be regarded as disguised descriptions of complexes—a view he had begun to entertain in "Fundamentals" (*Papers* 5: 19).

as the principle of reduction given there were beset with philosophical difficulties from the outset.[66] Russell explored other substitution-theoretic solutions, but could find no philosophically satisfactory alternative and abandoned substitution in the late fall of 1906 or early winter of 1907. His correspondence with Whitehead, along with his marginal comment added in June 1907, suggests that he reconsidered some of his philosophical reservations before again embracing substitution as the proper explanatory starting-point for types in "Mathematical Logic". It remains for Volume 5 of the *Collected Papers* to present the materials needed to settle the particular issues raised here.[67] The great lengths to which Russell went, in his published papers and unpublished manuscripts of the period, to supply a proxy for higher-order quantification from within substitution, must surely be weighed in the controversy surrounding the correct interpretation of *Principia Mathematica*'s ramified type theory.

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[66] After writing "Insolubilia", Russell became increasingly concerned with the lack of a philosophical motivation for his principle of reduction. See "The Paradox of the Liar" and "Fundamentals", fo. 29 (*Papers* 5: 552).

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