PRINCIPIA'S SECOND EDITION

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Bernard Linsky. *The Evolution of Principia Mathematica: Bertrand Russell's Manuscripts and Notes for the Second Edition*. Cambridge: Cambridge U. P., 2011. Pp. vii, 407; 2 plates. ISBN: 978-1-10700-327-9. £96; US$159.

Russell and Whitehead's *Principia Mathematica* was a pioneering work which was in many ways overtaken by the success of the new discipline it helped found. There is little doubt that *Principia Mathematica* was one of the most important works in the development of symbolic logic, as even Quine, one of the critics of the foundations proposed in it, said: "This is the book that has meant the most to me." Yet few mathematicians or logicians follow the theory of types proposed in *Principia* as the foundation of mathematics, nor the logicist project of which it is a variant. While much of *Principia*'s notation has been incorporated into modern formal logic and set theory, much of it also is dated and not easy for contemporary mathematicians and logicians to read. Among those who were dissatisfied with the philosophical foundations of the first edition were the authors themselves.[1] During 1923 and 1924 Russell revised and updated *Principia*, and the second edition of the first volume appeared in 1925 with the other two volumes appearing in 1927. What resulted was a new edition with the philosophical underpinning modified, but with the superstructure, that is, the resulting theorems of the three volumes, unchanged. Russell wrote a new Introduction and added three Appendices, but other than that made only minor changes. The new philosophy of logic had as its core idea a principle of extensionality which Russell expressed as the view that "functions of propositions are always truth-functions and a function can only occur in a proposition through its values" (*PM*2 1:

[1] Russell wrote the new material for the second edition, but it is clear from a letter to him by Whitehead on 24 May 1923, quoted by Linsky, that he, too, thought the foundational system needed tweaking: "I don't think that 'types' are quite right" (p. 16).

xiv). It is interesting that at times Russell seems to have been hesitant about even endorsing this change.[2]

The second edition of *Principia* has not been well received; by and large it has either been ignored or scorned. Linsky cites especially negative remarks by Monk, but others have dismissed the second edition additions and it has generally been thought a failure since 1944, when Gödel pointed out an error in the Appendix B proof that mathematical induction can be "rectified" even without the axiom of reducibility. *The Evolution of Principia Mathematica* makes available to scholars Russell's notes and manuscripts of the new material added to the second edition. Linsky also seeks to redress the indifference to the second edition, to set the record straight about several misunderstandings, and to give an account of the new material. The book he has produced will be very valuable to scholars of Russell and to anyone interested in the development of type theory and indeed of logic as a whole during this time. This is also a difficult book, both because the material can get quite technical at times and also because the new material included in the introduction to the second edition is somewhat sketchy, and there are philosophical and technical points about which Russell himself is less than clear. As a result there have been disagreements about how the second edition is best to be interpreted. Linsky recognizes that the interpretive issues are under-determined by the text and wants to stay relatively neutral with respect to the different alternatives about the various controversies. At times it can be difficult to keep the various positions straight. The difficulty lies with the material, and Linsky does a very good job of explaining the issues to the layman.

The work is divided into eight chapters, and then devotes 180 pages to a printing of the manuscripts both for the additional material that was published and for several of Russell's notes, including a manuscript entitled "The Hierarchy of Propositions and Functions" and notes on an amended list of propositions. The opening chapters discuss the manuscript material found in the Russell Archives, explain what was added to the second edition, and discuss the roles of Whitehead and Ramsey. The third chapter discusses the advances in logic since the first edition, particularly with reference to those works discussed by Russell in the new Introduction. Linsky feels this is important not only because some have suggested that Russell had not kept up on logic but

[2] See, for example, his remark in the second edition (*PM*2 1: xiv) where he says of the difficulties of the new foundations, "perhaps they are not insurmountable". The remarks at the end of the new introduction also suggest some hesitancy where he says the new primitive propositions will not yield the theorems concerning the Dedekindian and well-ordered relations. On the other hand, he has no hesitation adopting the Sheffer stroke, nor with the revised quantificational theory of ∗8.

also because Russell seems to downplay the enormous advances in logic that had taken place, focusing instead on the Sheffer stroke as the most important advance in mathematical logic since the first edition.[3] This chapter contains valuable information about the authors Russell mentioned in the new Introduction and also about the developments in logic of the time. Issues concerning the difficulties with the earlier system and the development of the new one come up in the discussions of Chwistek and Wittgenstein. The fourth chapter gives an overview of Russell's symbolism and the basic understanding of Russell's theory of types. Those less familiar with *Principia* will find this chapter extremely valuable. The fifth and sixth chapters discuss the content of the new material and the various disputes which have arisen in interpreting it. The first reactions to the second edition are given in the seventh chapter with particular emphasis on Ramsey and Carnap. The eighth chapter is an annotated edition of a 35-page manuscript Russell sent to Carnap in 1922, setting out the key definitions of *Principia*, and including some discussion from Russell as well as examples of theorems proved in some of the later sections.

In the fourth chapter, where he explains the theory of types, Linsky adopts Alonzo Church's interpretation of the "ramified" theory of types.[4] Linsky is aware that Church's formulation has been criticized as an historical account of the logic of *Principia*, particularly by Gregory Landini, who has emphasized that bound variables in *Principia* are restricted to predicative functions, that circumflexed variables only occur in the discussion of the system and so should not be thought of as term-forming operators, and that Church himself recognized that he was deviating in some respects from the actual presentation in *Principia*. Linsky is not persuaded by Landini's arguments although he does devote several footnotes to statements of Landini's alternative nominalist reading of *Principia*'s higher-order variables and his critique of Church's interpretation.[5] While the reasons for his disagreement with Landini could per-

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[3] See p. 40. The remark about the Sheffer stroke is in *PM*2 1: xiii. See also Goldfarb, "Logic in the Twenties", p. 353. On p. 6 Linsky cites Ray Monk's disparaging remarks about Russell having "neither the time nor inclination" to master the recent technical literature on logic.

[4] Church, "Comparison of Russell's Resolution of the Semantical Antinomies with That of Tarski". The system of what Church calls *r*-types here has been widely used in discussions of Russell's ramified theory of types.

[5] Landini's name surfaces many times in this work and that is no surprise, as he had written a response to John Myhill's discussion of the indefinability of the natural numbers in the second edition, and he devoted a chapter of his work on Wittgenstein and Russell to the second edition of *Principia*. Early on Linsky mentions that other than Gödel's 1944 discussion and Myhill's 1974 paper, there was little written on the

haps have been made clearer, Linsky's discussion in this chapter of the notation in *Principia* is probably the clearest in the literature.

The philosophical heart of this book is its discussion of the new material, given in Chapters 5 and 6. Chapter 5 focuses on Russell's adoption of extensionality in the second edition, dealing with the Introduction and Appendices A and C; Chapter 6 is devoted to the problematic Appendix B, where Russell attempted to recover induction without the axiom of reducibility.

Russell's new account of logic rested on adoption of the principle of extensionality. The only functions of propositions are truth functions and functions can only occur in a proposition through its values, so there can be "no logical matrix of the form $f!(\phi!\hat{z})$" (*PM*2 1: xxxi). Russell credited this position to Wittgenstein. In Chapter 5, Linsky examines the various courses this view could take. The chapter covers much of the material in the new Introduction, including the introduction of the Sheffer stroke, Appendix A on the new quantification theory ∗8, and Appendix C where Russell goes beyond the question of the extensionality of mathematics to try to handle the two apparent counter-examples to the claim that all functions of propositions are truth functions: "A believes p" and "p is about A". Linsky's discussion of these topics is very clear. With respect to the overall issue of extensionality, Linsky has previously argued that the first edition of *Principia* was an intensional logic based on a hierarchy of propositional functions which is distinct from a hierarchy of universals.[6] He thus finds the imposition of extensionality on the type-theory of *Principia* deeply troubling and ultimately not successful. He is especially concerned with Russell's claim that there is no matrix of the form $f!(\phi!\hat{z})$, i.e., that propositional functions themselves cannot occur as logical subjects. Cocchiarella has argued that such a foundation will not be enough to generate mathematics as it will restrict the logic to a fragment of second-order logic.[7] Linsky is concerned about this, yet he himself provides a way out of Cocchiarella's concern about the restriction of logic (p. 118). Linsky has a hard time taking Russell at his word that propositional functions only occur through their values, and cites one part of the "Hierarchy of Propositions and Functions" manuscript which did not make it in to the second edition and which, Linsky suggests, indicates that perhaps Russell would have included matrices involving higher-level universals (p. 120).

In contrast to Linsky, Landini has proposed a nominalist interpretation of

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logic of the second edition except by Landini and Allen Hazen.

[6] See Linsky, *Russell's Metaphysical Logic*.

[7] See Cocchiarella, "The Development of the Theory of Logical Types", esp. pp. 109–11, and "Russell's Theory of Logical Types and the Atomistic Hierarchy of Sentences".

all higher-order variables and argued that the second edition's remarks about extensionality are captured by an axiom he calls EXT, which licenses the substitution in all formulas of functions of different order/types which apply to all and only arguments of the same simple type.[8] Landini argued that this axiom succeeds in repairing the defect in Russell's proof of induction in Appendix B and is a way of understanding Russell's various remarks about extensionality. Linsky resists this view in part because he sees Russell as committed to a hierarchy of universals of different types and in part because he thinks the EXT principle is too strong since a version of the Axiom of Reducibility can be derived from it.

The discussion of Landini's views is developed in Chapter 6, which is devoted to the issues involved in Appendix B. Appendix B, with its new section ∗89, was designed to show that some results concerning induction (specifically some theorems of ∗120 and ∗121 which depended on ∗90) could be recovered in the new system even without the axiom of reducibility. In 1944 Gödel pointed out an error in the proof of ∗89.16 and also noticed that the grammar for the second edition seemed to allow that "functions of a higher order than a predicate itself … can appear as arguments for a predicate of functions".[9] As Linsky points out, none of the earlier reviewers of the second edition noticed the error. Linsky guides the reader through the details of the error Gödel saw and why it would be easy to overlook the error. With respect to Gödel's second point, that Russell was violating the type strictures of the first edition, Linsky brings up the suggestions of Hazen and Landini that the second edition requires a reformulation of the system of types. He mentions Hazen's BMT (for Appendix B Modified Types), and points out that while Hazen's system would allow for Landini's principle EXT to be well formed, Hazen himself doesn't endorse it. Linsky hesitates on whether either Hazen's or Landini's new system of types is needed. While discussing Zermelo's proof of the Schröder–Bernstein theorem, Russell said that "$p`\kappa \in \kappa$" (*Principia*'s notation for $\cap\, \kappa \in \kappa$) was "impossible, since $p`\kappa$ is not of the same order as members of κ" (*PM*2 1: xxxix–xl). Linsky points out that Russell doesn't quite say that "$p`\kappa \in \kappa$" should be admitted, but rather that the effect of it can still be achieved by considering the defining expressions for the classes (p. 161). The issue here is

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[8] LANDINI presents this first in "The *Definability* of the Set of Natural Numbers in the 1925 *Principia Mathematica*", and discusses the issues again in Chapter 6 of *Wittgenstein's Apprenticeship with Russell*. Hazen's discussion is found in HAZEN and DAVOREN, "Russell's 1925 Logic", and HAZEN, "A 'Constructive' Proper Extension of Ramified Type".

[9] GÖDEL, "Russell's Mathematical Logic", p. 134. Gödel says that in Appendix B such things occur constantly.

whether we can have a higher-order class as a member of a lower one. Linsky is resisting this conclusion which is at the core of Landini and Hazen's reformulation of the theory of types for the second edition. I find Linsky's reading of Russell here strained, for Russell in this passage seems to be giving a justification for the admissibility of the proposition, and then is concerned with whether the claim concerning its truth in this instance could be justified. On this issue of an implicit new type theory, Linsky also mentions a line in the "Hierarchy of Propositions and Functions" manuscript where Russell explicitly allowed for a higher order class to be a member of a lower-order class. This part of the "Hierarchy of Propositions and Functions" manuscript was not included in the final version, and Linsky remarks that "it is only in the material that was deleted from the final version of Appendix B that this issue is settled conclusively" (p. 163). But given Linsky's overall reluctance to accept the conclusion that a different theory of types is required for the second edition, I wasn't sure whether he thought the issue was only tentatively settled for Russell because he did not include that material in the final version, or whether he thought the passage from "Hierarchy of Propositions and Functions" really settled the issue conclusively for the second edition.

One obvious place where it might seem the issue was conclusively settled is with

$$*89.12 \quad \vdash : \rho \in \mathrm{Cls\ induct}_3 \,.\supset .\, (\exists \mu_2) \,.\, \rho = \mu_2$$

where Russell identifies a third-order inductive class with a second-order function. This leads Linsky to a discussion of the role of identity and his and Hazen's objection to $\phi x \equiv_x \psi x \,.\supset .\, \phi \hat{x} = \psi \hat{x}$, which Russell explicitly concludes from his considerations of extensionality (*PM*2 1: xxxix). The discussion is inconclusive, although at the end (p. 166) Linsky seems to be suggesting that a new theory of types is endorsed in the second edition.

Linsky also gives an account of Myhill's argument that what Russell was trying to do in Appendix B could not be done.[10] The explanation of Myhill's argument is very clear. Myhill's result applies to the system of ramified types of the first edition, including a principle of extensionality (but confined to

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[10] Myhill adopts the formalization of K. Schütte (Myhill, "The Undefinability of the Set of Natural Numbers in the Ramified *Principia*", p. 21) and then adds his comprehension and extensionality axioms A and B in place of the axiom of reducibility (p. 22). The reason why Landini and Myhill can come up with opposite results concerning the recovery of induction and the definability of the natural numbers and can both be correct is that they are working with different formal systems for the second edition.

propositional functions of the same order/type) and without the axiom of reducibility. With respect to Landini's recovery of the proof of ∗89.17 and the recovery of the definition of the natural numbers, Linsky is more hesitant. He accepts Landini's proof, given Landini's new system of types for the second edition with his liberal axiom EXT, but argues that EXT is too strong to be philosophically acceptable to Russell, since a reducibility-like principle will be a consequence of it, and the whole point of Appendix B was to prove induction without reducibility. Linsky quotes Russell's remarks in the second edition that what was sought was an axiom "less objectionable" than the axiom of reducibility (p. 168, see *PM*2 1: xiv). In the first edition, though, Russell had remarked that "it is by no means improbable that it [the axiom of reducibility] should be found to be deducible from some other more fundamental and more evident axiom" (*PM*2 1: 60). There are very interesting issues here, discussed further in Hazen[11] and Landini.[12] In his work Landini seems to agree that his principle EXT may not best capture what Russell had in mind, as it allows for a recovery of Cantor's proof which Russell had not thought possible in the system of the second edition.[13]

In the last chapter before the reproduction of the manuscript material, Linsky discusses Ramsey's work and the reviews of the second edition. The most detailed discussion here is of Ramsey's view in "The Foundations of Mathematics". Linsky mentions Ramsey's famous separation of the two classes of paradoxes which occurs in the opening pages of Ramsey's work, but unlike others he correctly notes that Ramsey doesn't just adopt a simple theory of types, but rather changes the notion of a predicate function to include propositional functions derived from infinite disjunctions and conjunctions of atomic propositions. Ramsey retained the orders of propositional functions with respect to what he called the "subjective" presentation of the functions, and which Linsky distinguishes by calling them "predicates". The arguments to which the functions apply concern simply the type and not the order. The very loosening of the type theory suggested by Hazen and Landini is thus present in Ramsey's new account of types.

Linsky says that even with Ramsey's new notion of a predicative function, "the axiom of reducibility is not a logical truth" (p. 175). I am not quite sure why he says this. He discusses remarks Ramsey makes about identity, and suggests that Ramsey would object to Russell's notion of identity because of

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[11] "A 'Constructive' Proper Extension of Ramified Type Theory".

[12] *Wittgenstein's Apprenticeship with Russell*.

[13] See *Wittgenstein's Apprenticeship*, p. 214. At this point Landini suggests a weaker principle EXT⋆ which requires that the propositional functions being substituted in the contexts not be true of all their arguments or false of all their arguments.

the possibility of objects sharing all predicative properties but differing on properties of higher order (*ibid.*). In fact, Ramsey did bring up this possibility only to dismiss it.[14] Ramsey's real objection to Russell's definition of identity was that he thought it was not self-contradictory for two different things to share all their properties, and he specifically separated this issue from the question of the axiom of reducibility (Ramsey, p. 31). Thus he rejected the *Principia* account of identity, plumping instead for something like Wittgenstein's view. Ramsey thought predicative functions should be redefined as those functions derived from any conjunction or disjunction of atomic propositions, including infinite ones, and then concluded that "all the functions of individuals which occur in *Principia* are in our sense predicative and included in our variable ϕ, so that all need for the axiom of reducibility disappears" (Ramsey, p. 41). If we understand the axiom of reducibility to affirm that for every function of any order there is a coextensive predicative function, then the axiom is clearly true on Ramsey's new understanding of predicative function.

Perhaps Linsky thinks that Ramsey's somewhat cryptic counter-example to the axiom of reducibility applied to his own liberal predicative functions. Here is Ramsey's counter-example:

> … it is clearly possible that there should be an infinity of atomic functions, and an individual a such that whichever atomic function we take there is another individual agreeing with a in respect of all the other functions, but not in respect of the function taken. Then $(\phi) . \phi\hat{x}! x \equiv \phi! a$ could not be equivalent to any elementary function of x. (Ramsey, p. 57)

For this to be a counter-example to reducibility, we need to understand "predicative functions" as "elementary functions" which are at most finite conjunctions or disjunctions of "atomic functions", which are independent of each other. The counter-example doesn't apply to Ramsey's revised predicative functions, but only to these elementary functions.

The Evolution of Principia Mathematica had as its purpose to make the manuscripts and notes available to scholars and "to restore the reputation of the second edition of *Principia Mathematica* as a serious contribution to logic" (p. 3). Linsky has certainly succeeded in the difficult task of making the notes and manuscripts available. He has also shown that Russell's work on the second edition was not the work of a bumbler out of touch with the logic of his day,

[14] Ramsey, "The Foundations of Mathematics", p. 30.

and that this work sheds light on a host of issues in the philosophy of logic. But Linsky is in the end rather pessimistic about Russell's system in the second edition.[15]

[15] In writing this review, I have benefited from discussions with Gregory Landini and Nicholas Griffin.

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