# THE *GRUNDGESETZE*

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Gottlob Frege. *Basic Laws of Arithmetic. Derived Using Concept-script*. Volumes I and II. Translated and edited by Philip A. Ebert and Marcus Rossberg with Crispin Wright. Oxford: Oxford U. P., 2013. Pp. xxxix + XXXII + 253 + XV + 285 + A–42 + I–II. ISBN 978-0-19-928174-9. £60; US$110.

Given the steadily rising interest in Frege's work since the 1950s, it is surprising that his *Grundgesetze der Arithmetik*, the work he thought would be the crowning achievement of his career, has not previously been fully translated into English. Other Frege works have long been available in English. The earlier *Grundlagen der Arithmetik* was nicely, if not always entirely accurately, translated by J. L. Austin and published in 1950.[1] Translations of Frege's other philosophical writings began to appear in influential editions about the same time,[2] initially a small corpus that has been gradually added to over the years to include works from his *Nachlass* and letters from his correspondence.[3] Somewhat later, the *Begriffsschrift,* Frege's momentous first

[1] *The Foundations of Arithmetic* (Oxford: Blackwell, 1950; 2nd edn. 1953). The later translation under the same title by Dale Jacquette (London: Longman, 2007) is much less satisfactory.

[2] The first importance source for these was Peter Geach and Max Black's *Translations from the Philosophical Writing of Gottlob Frege* (1952; 2nd edn. 1960; 3rd edn. 1980), though it included reprints of some much earlier translations.

[3] The two most comprehensive English sources for the shorter writings are now Brian

contribution to modern logic, the work which Quine[4] said had made logic a great subject, was translated in full despite the difficulties of Fregean notation.[5] But through all this the *Grundgesetze* remained largely (though not entirely) untranslated.

The view seems, for a long time, to have been that Frege's contributions to logic were to be found in the *Begriffsschrift* and his contributions to philosophy in the *Grundlagen* and a small number of seminal articles. The *Grundgesetze*, on the other hand, was widely thought not to have added much to these magnificent achievements, but to have employed them in a project that was doomed by the discovery of Russell's paradox; namely, the attempt to show that arithmetic could be rigorously derived from purely logical principles. In the bleak aftermath of Russell's paradox it came to seem as if whatever was of value in the *Grundgesetze* had already been published elsewhere, while what was new in it was mistaken.[6] That assessment of the book's value, combined with the huge difficulties of typesetting Frege's idiosyncratic two-dimensional notation, and the sheer scale of the work (even without the anticipated third volume, which was abandoned in the face of Russell's paradox, it was, by far, Frege's largest work) made it seem as though a translation of the whole *Grundgesetze* was not only a prohibitively large undertaking, but also one which was not really necessary. And so, through much of the twentieth century, the *Grundgesetze* vied with *Principia Mathematica* as the world's best-known but least-read philosophical book.

But this understanding of the *Grundgesetze*'s importance could not withstand the development of neo-logicism, brought to prominence by Crispin Wright's *Frege's Conception of Numbers as Objects.*[7] The key insight of neo-

McGuinness, ed., Frege, *Collected Papers on Mathematics, Logic, and Philosophy* (1984); and Michael Beaney, ed., *The Frege Reader* (1997).

4 At least in early editions of *Mathematical Logic*. The remark was removed in later ones in deference to Boole.

5 By Stefan Bauer-Mengelberg in Jean van Heijenoort, ed., *From Frege to Gödel* (1967), pp. 5–82.

6 Not surprisingly the one part of the *Grundgesetze* which has appeared most frequently in translation is the "*Nachwort*" responding to Russell's paradox which Frege added as the book was in press. Geach and Black (2nd edn., pp. 234–44) include parts of it, but so great was the difficulty of typesetting Frege's notation that it was replaced by Geach's English paraphrases. This perhaps speaks well to Geach's command of the English language, but it offers a good demonstration of why a "concept-script" was needed in the first place. Even someone with only a basic knowledge of Frege's notation will surely find it easier to follow the original than Geach's paraphrases.

7 (1983). Neo-logicism, in fact, goes back quite a way before that (some might argue even to Frege) but at least to Charles Parsons' 1965 article "Frege's Theory of Number" in Max Black, ed., *Philosophy in America* (1965). George Boolos's important later contributions to neo-logicism should also be mentioned. See the papers in his *Logic, Logic, and Logic* (1998), pp. 183–236, 275–314.

logicism is that Russell's paradox is less deadly to Frege's system than initially appeared. As Wright (pp. 154–69) outlines in some detail, the Peano postulates can be derived in second-order logic from what is now known as Hume's Principle, namely the thesis that the number of $F$'s is identical to the number of $G$'s iff there is a 1–1 mapping of the $F$'s onto the $G$'s, without appeal to the notorious Basic Law v, which allowed the derivation of Russell's paradox. The question therefore arose: if Basic Law v were dropped and replaced by Hume's Principle, would the resulting system be consistent? And indeed, as Burgess and Hazen quickly noted[8], what is now known as Fregean arithmetic (i.e., second-order logic plus Hume's Principle) has a model and is consistent if analysis is, which is a pretty strong result.[9]

Ironically, however, the initial forays into neo-logicism did not draw immediate attention to the *Grundgesetze*, for Wright took up the point of view of the *Grundlagen*. There, in Part IV, Frege gives pride of place to Hume's Principle (stated explicitly in §73) and, by means of informal arguments only, suggests that arithmetic might be developed on that basis. He does not attempt there anything as ambitious as even a sketch of how the Peano postulates themselves might be derived—and not surprisingly, since there is no formal theory in the *Grundlagen* from which to derive them. That was Wright's contribution, again done (as Wright notes) without full formal rigour, but in enough detail to show how full proofs might be constructed. As Michael Dummett complained: "He could have achieved the same result with less trouble by observing that Frege himself gives just such a derivation of those theorems."[10]

Frege's derivations, however, are to be found in *Grundgesetze*. But there the formal development is based on Basic Law v and Hume's Principle is much less prominent than it was in the *Grundlagen*, though both halves are stated early in Part II: right-to-left in Vol. I, §53 and left-to-right in I, §69. Frege's proofs of the two halves, of course, depend on Basic Law v, but it turns out that that is the only essential service that Basic Law v provides in the *Grundgesetze*. One might wonder, therefore, why Frege himself didn't become a neo-logicist, when confronted with Russell's paradox. The philosophical reason is that Hume's Principle is an abstraction principle for numbers and thus should itself be regarded as an arithmetic proposition in need of derivation in the logicist project. This is related to the famous Julius Caesar problem, that Hume's Principle cannot ensure that Julius Caesar is not a number. It was a key purpose of Frege's logicism to explain what numbers were rather than to postulate them as reified abstractions from equivalence relations on concepts.

8 John Burgess (1984), review of Wright; Allen Hazen (1985), review of Wright.

9 Boolos calls this "consistent, with moral certainty" (p. 291). He gives details of the modelling on pp. 187–91.

10 Dummett, *Frege: Philosophy of Mathematics* (1991), p. 123.

Basic Law v is also an abstraction principle, but for value-ranges, and thus does not, by Frege's lights, need a logicist elimination, for he treated value-ranges as *bona fide* logical objects. There are also technical reasons which make Basic Law v convenient even where not essential. Given Basic Law v, he is able to make use of value-ranges and these prove a very useful technical device in almost all the proofs that constitute the development of arithmetic in Part II of the *Grundgesetze*. But, though convenient, value-ranges are dispensable and the development of arithmetic, including the Peano postulates, can go ahead without them. The only place where value-ranges are essential is in the proof of Hume's Principle.[11]

All this makes an English translation of the complete *Grundgesetze* an urgent necessity. The *Grundgesetze*, of course, has remained available in German. The first edition, naturally, is very rare. Russell's copy survives in his library: he had its two volumes bound as one (despite their slightly differing page sizes).[12] But a facsimile reprint of both volumes appeared in 1962 and another in 1998, the second with a list of corrections by Christian Thiel.[13] More surprisingly, a new edition was produced in 2009 by Thomas Müller, Bernard Schröder and Rainer Stuhlmann-Laeisz in which Frege's original concept-script is replaced by modern formal notation. I have not seen this edition, but I can imagine that users of the present translation might find it useful to have at hand, for Frege's notation takes some getting used to.

The needs of English readers have been much less well served, although there have been translations of bits of the *Grundgesetze*. The earliest of these was by Johann Stachelroth and Russell's former student, Philip Jourdain.[14] The most extensive was by Montgomery Furth,[15] which includes the Foreword, the Introduction and the whole of Part I (Frege's exposition of his "Concept-script") as well as the Appendix from Volume II on Russell's paradox and, a nice touch, a small fragment from §§54–5, where Frege uses Basic Law v to prove the fateful proposition "$\vdash f(a) = a \frown \dot{\varepsilon} f(\varepsilon)$" (prop. ($\phi$), I: 73), more familiar to modern readers as "$\vdash f(a) \equiv a \in \{x : fx\}$" the unrestricted

11 For details see Richard G. Heck, "The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*" (1993).

12 A record of Russell's marginalia in them is found in Bernard Linsky, "Russell's Marginalia in his Copies of Frege's Works" (2004).

13 Both published at Hildesheim by Georg Olms. I have not seen the 1998 printing, but the one from the 1960s included a paperback version, dated 1966 on the copy in the McMaster Library.

14 *The Monist* (1915–17). The translation covered the Foreword, Introduction and §§1–7, all from Volume I, stopping just before the typesetting starts to get interesting. (Frege's quantifier notation appears in the next section.) Geach and Black reprinted most of this, with minor changes, in their collection.

15 *The Basic Laws of Arithmetic. Exposition of the System* (1964).

comprehension principle of naive set theory. Russell's paradox follows immediately.[16] Furth's abridgement has certainly been useful, but, breaking off as it does just at the point where Frege starts actually deriving arithmetical results, it resembles a long, elaborate overture to an opera that never happens. And since the only bit of the formal development of the *Grundgesetze* which is included is the bit at which it breaks down, it tends to reinforce the view that the main interest of the *Grundgesetze* lies in its failure.

The present translation reveals that this is far from being the case. It is not even the case that the material that has not been translated until now covers only the formal development of Frege's system, something of interest perhaps to logicians and historians of logic, but not to philosophers. There is for example, in the second volume, a very long philosophical disquisition on the real numbers (at the start of Part III, pp. 69–162), including a sharp, polemical critique, reminiscent of the *Grundlagen*, of competing theories. I seem to recall that Dummett somewhere said that it was not very good, certainly less good than the philosophical critiques in the *Grundlagen.* At any rate, it is useful to have it available for examination.

The present translation, work on which began over ten years ago when the translators were both completing doctorates at the Arché research centre at St. Andrews, seems particularly well thought through. It has been assisted by a virtual galaxy of Frege scholars and discussed in detail in the Frege community. The translators discuss the translation of various technical terms in their Introduction, sometimes at length where the matter is controversial, and there is an extensive glossary of technical terms (pp. xxvi–xxviii). Ebert and Rossberg are to be commended on the uniformity of their translation. Where Frege uses a particular technical term they translate it uniformly by the same English word. Where Frege's or English idiom makes this impossible, the deviation is recorded in the translators' notes at the end keyed to the text by alphabetical markers. (See, for example, note *a* on "numbers" on p. 70 which indicates that there Frege used "*Nummern*" rather than the usual "*Zahlen*": he was referring to the numbers assigned to his inference rules.) Where Frege uses different technical terms, Ebert and Rossberg respect his usage by choosing different English words, even where the implied distinction Frege is making is not clear. Thus, for example, Frege's "*Definition*" is always translated by "definition", whereas his "*Erklärung*" is (with two exceptions) always translated by "explanation" and "*Erläuterung*" by "elucidation", even though it is not clear what distinction Frege intended to draw between a *Definition* and an *Erklärung*. (*Erläuterungen* are always informal elucidations, aids to the reader's understanding, outside of the formal system.) Finally, where Frege uses words with the same root, Ebert and Rossberg use English cognates: e.g. their use

[16] Set "$f(x)$" as "$x \notin x$", then "$a \notin a \equiv a \in \{x: x \notin x\}$". Substitute "$\{x: x \notin x\}$" for "$a$".

of "reference" for "*Bedeutung*", "co-referential" for "*gleichbedeutend*", etc. Their choices are so well thought through that one hopes that they will become the industry standard in English-language Frege scholarship. The only one I feel at all inclined to question is their use of "ordinary language" for Frege's "*Wortsprache*". Frege uses "*Wortsprache*" as a contrast to formal languages, in particular to his own concept-script. "Natural language" is very often used for this purpose in logic texts, but Ebert and Rossberg object that Esperanto is not a natural language but is a *Wortsprache*. I confess Esperanto had never occurred to me in this context, but even so I prefer "natural language" to "ordinary language". To me the latter suggests a contrast with "technical language", or what Frege in some of his philosophical writings called "scientific language", which was not the contrast Frege had in mind in the *Grundgesetze*.

The typesetting of the book is another wonder. There has been a basic Frege module in LaTeX for some time, but it was not equal to the demands imposed by the *Grundgesetze*. So new symbols had to be created and, since the amount of symbolism over the two volumes is so great, new means of inserting symbols into the text had to be found. The development of LaTeX became a research project in its own right—one is reminded of Cambridge University Press having to cut new type to print *Principia*. The typesetting has been done so cleverly, using the intermittently columnar approach of Frege's original publisher (a great space-saver when so many formulae are taller than they are wide), that the translation manages to follow the pagination of the original. (This explains the apparently bizarre pagination details in the header of this review.) It has two great advantages. It makes comparison with the original easy and also preserves over a century's worth of page references in the secondary literature.

The present volume includes a complete translation of both volumes of the *Grundgesetze*, including all front and end material (including Frege's indexes). There is a short Foreword by Crispin Wright and a much longer Introduction by the two translators. The translators have taken the opportunity to correct errors and misprints in the original. These are listed at the end of the volume, after the notes on the translation, and the two lists are paginated continuously with the second volume. The translators provide their own index to both volumes, much fuller and more useful than the two brief lists of topics provided by Frege, and paginated separately right at the end.

Frege's notation is so unfamiliar that it might have been useful to have an index of notations. This is compensated for partly by the carefulness of Frege's own explanations, compendiously assembled at the beginning of the first volume, but also by a 42-page appendix (also separately paginated), "How to Read the *Grundgesetze*", by Roy T. Cook, which works systematically through all the idiosyncrasies of Fregean notation. I am apt, like, I suspect, most modern readers of Frege, to render his formulae mentally into a rough approximation in modern logical notation. Cook's appendix reveals how rough this approximation normally is. On almost every point, Frege's notation is a little bit stranger than one had previously thought. There is much less symbolism to learn than in *Principia Mathematica*, which, if anything, suffers from notational overload, with variant notations and definitional equivalences proliferating beyond what is necessary. Frege's notation is comparatively sparse, has a great deal of integrity and a certain rugged charm. One can't imagine actually using it, but it is certainly worth learning how to read it properly. Cook's Appendix is the fullest, most systematic and by far the best guide to it that I have come across.

It has been a long wait for a good, complete translation of the *Grundgesetze*. I have already mentioned several reasons for the delay, but both the translators and Wright mention an additional one. Astonishingly, the UK's inaptly named Research Excellence Framework, which seems to have the country's universities in a near-Stalinist grip, does not consider such work to be research. This, apparently, not only makes it difficult to get financial support for such projects, but makes it risky for scholars beginning their academic careers to embark on such work which doesn't have bureaucratic approval. All Frege scholars, and indeed all historians of analytic philosophy, have cause to be grateful to the Arché research centre in Scotland for having supported the project. The attitude of the Research Excellence Framework looks like a strong argument for Scottish independence!

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