

Reviews

A CENTENARY COMPANION TO *PRINCIPIA MATHEMATICA*

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A famous anecdote of Russell's neatly summarizes the impact of *Principia Mathematica* in Russell's lifetime:

I used to know of only six people who had read the later parts of the book. Three of these were Poles, subsequently (I believe) liquidated by Hitler. The other three were Texans, subsequently successfully assimilated. (MPD, p. 86)

That the later parts (dealing with purely mathematical concerns) were somewhat neglected, was a disappointment to Russell and Whitehead. However, the same could not be said for the early parts of the work (dealing with philosophical and mathematical logic). The influence of these parts has been profound.

Logicism is a simple enough thesis: mathematics (to a greater or lesser degree, depending on which version of logicism—Frege's or Russell's—we are talking about) is part of logic. Every mathematical truth is really just a logical truth. This deceptively simple philosophical claim, however, can only be taken seriously if confirmed by supporting evidence. Logicism is a philosophical thesis that requires logical demonstration. Both Frege and Russell began by stating the philosophical case for logicism, and then attempting to demonstrate the thesis formally in subsequent work. Frege's attempt famously floundered after Russell discovered its inconsistency. After the best part of a decade spent trying to remove that inconsistency, Russell and Whitehead produced *Principia Mathematica*, which was their attempt to demonstrate logicism on, in the words of the editors of this volume, "a truly epic scale" (p. xvi).

Much appeared to have changed during those years in which *PM* was constructed. Russell's original philosophical statement of logicism in *The Principles of Mathematics* was a paradigm of elegance, at least with regard to its reduction of number theory to the calculus of classes. But in *PM*, things look very different. The logic of *PM* is stratified into a theory of types, something only tentatively considered in an appendix to the *Principles*; the classes to which numbers were reduced in the *Principles* are absent from the "no-classes" theory of *PM*; the much celebrated theory of quantification first presented in "On Denoting" is incorporated into *PM*. Additionally, Russell and Whitehead claim at some points to have rejected any commitment to propositions as entities, replacing them with a new theory of judgment to explain "propositional" content. After protracted attempts to digest these many changes and innovations to Russell's logic, most commentators remained unconvinced that the demonstration of logicism was successful.

It is a powerful testament to the lasting importance and value of *PM* that, in the century following its publication, the fact that the orthodox view of the work was that it had failed in its intended purpose posed no obstacle to its dramatic impact on the development of both mathematical logic and analytical philosophy. *PM*'s enormous influence in mathematical logic is largely due to subsidiary achievements made in the service of the logicist enterprise, rather than a reflection of that enterprise. *PM* gave the first accessible axiomatization of propositional and predicate logic (Frege had previously given an arguably more rigorous axiomatization, but his notation made the work far less accessible) which became the point of reference for subsequent work in the development of metamathematics, culminating in Gödel's famous incompleteness theorems which proved the incompleteness of any consistent formalization of arithmetic based on *PM*'s formal system. Furthermore, for generations of logicians, *PM* was the only book available in which to study mathematical logic. From our current perspective, at a time when the market for logic textbooks is at saturation point, it is easy to forget that *PM* was the only thing remotely resembling a textbook for many years.

While the mathematicians and logicians had obvious reason to study *PM*, philosophers also found much more to capture their interest than just the logicist thesis in *PM*. Russell's theory of descriptions was quickly established as one of the defining contributions to analytical philosophy, and it received a far clearer and more detailed statement in *PM* than was provided in "On Denoting". Furthermore, Russell embedded his logicist enterprise within a wider conception of philosophy as a science constructed on formal principles. This caught the imaginations of Wittgenstein, the Logical Positivists, and Quine, all of whom saw *PM* as the foundation of this proposed project.

The *Palgrave Companion* neatly slices up the task of studying these diverse aspects of *PM*'s legacy by dividing the work three ways, covering the influence

of *PM*, the philosophical issues raised in *PM*, and the mathematical contribution made by the work. (The book is actually divided into four sections, the philosophical material being split between Russell's general approach to logicism on the one hand, and the ontological commitments of type-theory on the other.)

The editors provide a very useful and illuminating introduction before Alasdair Urquhart opens the discussion with an excellent overview of *PM*'s first century of influence. One of the most important influences was the impact *PM* had on the thought of David Hilbert and his school. This influence, particularly Hilbert's response to the logicist thesis, is carefully charted in Reinhard Kahle's chapter. Jan Woleński discusses the reception of *PM* in Poland. Certainly, during the heyday of Polish logic, the influence of *PM* was substantial there. Woleński carefully recounts the impact of the work on the Polish school, even going so far as to suggest candidate identities of the three readers mentioned above in Russell's anecdote.

Part II of the book turns to the directly philosophical questions concerning Russell's logicism. A somewhat vexed question in the interpretation of *PM* has been that concerning Russell's apparent disregard for metatheoretical questions. This can seem a little puzzling for, although exact treatments of metatheory did not emerge until sometime after the publication of *PM*, some of the basic metatheoretical questions seem to be so obvious that one wonders why Russell and Whitehead paid such little heed to them. For example, we know that Russell gave more than a passing thought to the question of which axioms to draw on, including the question of minimizing their number. Why then did he not take the obvious further step of finding some way to determine the logical independence of his axioms? In recent years, an influential answer to this question has been offered which maintains that this absence is in fact necessitated by Russell's philosophy of logic. On this view, Russell was committed to *universalism* about logic, according to which there is only one universal logic and hence no distinction between a formal system and its metatheory. Thus, there can be no vantage point external to the system from which to assess that system's logical features. Patricia Blanchette demonstrates convincingly that this interpretation is flawed in two major ways: firstly, it is flawed as an interpretation of Russell's view, and secondly, it is internally flawed as a claim about the consequences of universalism. Universalism does not automatically impose any obstacle to metatheory, Blanchette argues. That obstacle only arises when universalism is strengthened to a position she terms "exclusivism", according to which the "derivations [of a formal system] offer the *only* way of presenting compelling or scientifically-acceptable arguments" (p. 63). Neither Frege nor Russell was an exclusivist, she maintains. This leaves us without an answer to the original question of why Russell paid such little attention to metatheory. Blanchette goes on to provide her own answer to this

question by suggesting that Russell mistakenly thought that the inapplicability of geometric models to the question of the independence of logical axioms led him to think there was no prospect of interesting metatheoretical results about *PM*.

Another point on which Russell is often seen to have lacked the clarity enjoyed by current logicians is the nature of the variable. Edwin Mares contributes a fascinating discussion of Russell's treatment of free variables ("real" variables as Russell called them) and how Russell construed them semantically. Particularly interesting is the link that Mares proposes between real variables and Russell's 1919 theory of vague denotation as situated in his imagistic conception of propositional content. This is an important contribution to the study of Russell's philosophical development as Russell's account of propositional content from 1919 onwards is normally treated wholly independently of his work on logic (he had, after all, repeatedly declared that meaning had more to do with psychology than logic at this time). However, Mares suggests that the conception of vague denotation, although clearly signalling Russell's increasing engagement with empiricist conceptions of content, dovetails deliberately with the logical system of the second edition of *PM*.

The final two contributions to this second section of the book are concerned with the "no-classes" theory of *PM*. Byeong-uk Yi gives a detailed reconstruction of *PM*'s no-class theory which firmly reiterates the orthodox view of *PM* as failing to achieve its logicist enterprise, by revealing numerous problems with the theory which, despite some innovative suggestions from Yi as to how it might be repaired, he concludes is ultimately still dependent on a logically incoherent notion of class. Jolen Galaugher focuses on disentangling the "Frege–Russell" definition of number to show that the two versions of logicism being assimilated here are in fact quite distinct in a number of fundamental ways. Of course, it is commonly acknowledged that *PM*'s definition of cardinals as classes of similar classes has to be taken with more than a pinch of salt, bearing in mind that, according to their contextual elimination at * 20, there *are* no classes to *be* similar, but Galaugher shows that the differences run deeper than this. Charting the development of the logic of *PM* through Russell's manuscripts on denoting, substitution, and so on, as well as drawing on correspondence between Couturat and Russell, Galaugher shows that Russell and Whitehead operated with a very different notion of a class to Frege's notion of the value-range of a concept and, therefore, built their version of logicism on quite different foundations.

Part III remains focused on philosophical issues regarding Russell's logicism but now specifically narrows the topic to the question of *PM*'s theory of types. This theory has always been the most controversial aspect of the work. Perhaps the most prominent topic among scholars of Russell's philosophical logic in recent years has been the debate over the relationship between the

logicism of the *Principles* and that of *PM*. Early interpretations of Russell's logical development simply understood *PM* as abandoning key components of the philosophy of the *Principles* in response to the contradictions. As more and more of Russell's manuscripts and correspondence from the period of *PM*'s construction has come to light, however, a number of commentators have mounted vigorous challenges to this interpretation. The *Principles* defends a doctrine according to which no restrictions can be placed on the range of a variable in a genuinely logical proposition. This doctrine of unrestricted variation seems incompatible with type-theory. However, study of Russell's substitutional theory (most of the writings on which have only very recently been published in *Papers 5*), show that Russell at one time saw type-theory and unrestricted variation as mutually compatible. The question is, did he still think that by the time he wrote *PM*?

No one has made a more significant contribution to this debate than Gregory Landini, whose 1998 book *Russell's Hidden Substitutional Theory*¹ revolutionized our understanding of the development of *PM* and type-theory. Landini's contribution to this volume summarizes his interpretation of Russell's logical development in a precise and accessible form which effectively lays down the gauntlet to the traditional interpretation of *PM*. On Landini's interpretation, the things which are typed in type-theory, namely propositional functions, are not entities of any kind but simply open formulas of *PM*'s formal language. Central to his interpretation is his insistence that quantification appears in two distinct semantic forms in *PM*: first-order quantification is objectual and unrestricted (in the intended semantic theory, in other words, everything in the domain of first-order quantification is of the same unrestricted type), while high-order quantification is interpreted substitutionally, the range of quantified predicate variables being type-restricted by the structural features of the open formulas that are its substitution instances.

One point at which I find myself in disagreement with Landini is over Russell's treatment of different sorts of contradictions. Landini is at pains to insist that Russell was not directly concerned with the semantic paradox of the Liar and its variants during the time that he was constructing type-theory, taking the paradox of substitution, first recovered from Russell's manuscripts by Landini in his book² and now published in *Papers 5* (p. 125), to be the driving force behind ramification of the theory of types. This is directly contrary to the usual interpretation of type-theory which holds that ramification is needed

¹ GREGORY LANDINI, *Russell's Hidden Substitutional Theory* (1998). See my "Substitution and the Theory of Types" (2003), for extensive discussion.

² *Russell's Hidden Substitutional Theory*, where the 22 January 1907 letter to Hawtrej is the frontispiece and is discussed on pp. 234–5; Bernard Linsky published the full text in *Russell* (2002).

if the simple theory of types is to block paradoxes of quantification (including the version of the Liar that Russell considers). Landini insists that this is a mistake on the grounds that the formal language of the substitutional calculus contains no propositional-attitude expressions, nor any semantic expressions (e.g. truth and falsehood predicates, or predicates like “is lying”) a claim which, if true, will surely apply to later systems such as *PM*’s, too. The substitutional paradox, Landini notes, has a Cantorian diagonal structure and is, therefore, an apparently logical paradox, not a semantic one (pp. 208–9), and it is this paradox that drives Russell to ramification.

There are a number of problems facing Landini’s interpretation on this issue. For one thing, it makes many of Russell’s own remarks and writings rather hard to explain. He wrote a long manuscript “The Paradox of the ‘Liar’” (*Papers* 5: II) in 1906, right at the high-point of his work on substitution. He explicitly identifies the fact that the proposition quantifies over itself as the cause of the paradox, not the inclusion of semantic predicates in its formulation:

What seems plain from this paradox is, that a proposition about a set of propositions can never be a member of that set. This impossibility cannot, however, be simply decreed because of the paradox; we must find some reason in the nature of propositions which shows that the impossibility subsists. (*Papers* 5: 320)

The remainder of the manuscript seems to be seeking a solution of the paradox that justifies imposing restrictions on quantification. This is hard to square with Landini’s interpretation. Admittedly, as Landini points out, there are places where Russell does seem to recognize the distinction between semantic and logical paradoxes (p. 210), but this recognition only extends to paradoxes of notions such as “definability” or “nameability”.³ Furthermore, it is notable (and ignored by Landini) that these paradoxes are dealt with by a common solution in terms of their violation of the vicious-circle principle in *PM* (I: 66–7). Landini’s interpretation of Russell is unashamedly revisionist (pp. 163–4). Much of the revisions to the traditional interpretation of *PM* that he recommends have compelling textual support, but his insistence that Russell pre-empted the differentiation of semantic and logical paradoxes that is usually assumed to have been first recognized several years later by Ramsey is a point where his interpretation, it seems to me, is stretched to the limit of historical plausibility.

Landini’s interpretation is highly controversial, but it is also ingenious and (notwithstanding the concerns raised in the previous paragraph) based in a careful re-examination of published and unpublished materials of Russell’s

³ E.g. in “The Substitutional Theory of Classes and Relations” (*Papers* 5: 258).

that many other commentators ignore. Wherever one stands on the controversies that Landini's interpretation has generated in Russell scholarship, no current discussion of the interpretation of *PM* would be complete without considering it. Landini's contribution is, therefore, a real highlight of this book as it serves to crystalize his views perspicuously, and furthermore to provide an important reference point for two other excellent chapters which immediately follow it, the first by Kevin Klement, the second by James Levine.

Klement's paper, while voicing general sympathy for Landini's outlook, differs in his interpretation of how the circumflex operation behaves in *PM*. The circumflex, taken at face value, seems to be a term-forming operator which allows us to turn propositional functions into logical subjects. This is hard to square with the view defended by Landini (and endorsed by others, including Klement and myself) according to which propositional functions are not entities in *PM*. Landini takes what Klement calls "the heroic course" of arguing that the circumflex is not used as a term-forming operator in actual applications of the formal language of *PM*, but only in informal discussion of the language. Thus, Landini effectively reads the circumflex as a metalinguistic device. I think it is fair to say that the jury is still out on Landini's proposal—mainly because the jury is still out on the question of whether *PM* has the resources to support a coherent distinction between object and metalanguage at all, which makes it notoriously difficult to arrive at any agreement as to which parts of the book are formal applications of the system, and which parts are informal explanations of it. Klement prefers to avoid such difficulties by conceding that the circumflex is used to a limited extent within the formal language, but that this is far more restricted than the commonly assumed interpretation (which tends to construe the circumflexed expressions somewhat anachronistically as more or less equivalent to λ -abstracts).

Despite these (admittedly important) differences of detail between Klement and Landini, they are in firm agreement that the theory of types is not intended as an ontological theory in *PM*, and thus does not signal the wholesale rejection of the core metaphysics of the *Principles* that it is traditionally taken to provide. It is this fundamental claim that Levine seeks to challenge in his contribution. Although a number of people have voiced sympathy with this interpretation of *PM*, Levine identifies Landini, Klement, and myself as its main proponents. In fact, despite the agreement as to the general landscape of Russell's ontology at the time of *PM*, there are significant differences between the three of us (as demonstrated in Klement's disagreement with Landini over the circumflex discussed above, for example). Levine is admirably sensitive to these differences and develops a string of objections, carefully grounded in a detailed study of Russell's published and unpublished works, to a number of claims that emerge from these varying interpretations. Responding to the objections Levine raises against the interpretation in question,

and to my own contributions to it in particular, is a task best saved for another time when it can be carried out in detail. While there are (unsurprisingly) several points on which I disagree with Levine's interpretation, they do not detract from my admiration for his piece. It is the most detailed critical response yet to the recent interpretation of *PM*, and the first to fully engage with both the textual and philosophical arguments that support it. Taken together, the contributions made by Landini, Klement, and Levine are evidence that philosophical interest in the foundations of *PM* is stronger than ever before.

The remaining two chapters in this section are concerned with more formal aspects of the logic of *PM*. Harold Hodes offers an illuminating overview of systems of ramified-type logics which, unlike the pioneering but rather informal version embodied in *PM*, are presented in accordance with the standards we now expect of formal systems, both in terms of the statement of grammatical rules and model-theoretic interpretation of the system. Dustin Tucker tackles the question of how best to handle quantification in a ramified type-theory, developing a new theory of quantification which is designed to apply the vicious-circle principle only when the threat of paradox arises, thus allowing non-paradoxical statements that are outlawed by *PM*'s ramified type-theory back into the system.

The final section of the book is concerned with the mathematics of *PM*. As Nicholas Griffin shows in his contribution, it is surprisingly hard to specify exactly which parts of mathematics were actually covered in *PM*. It is well known that the work never encompassed geometry, this being saved for an intended fourth volume which was never completed. However, there are other notable absences, such as group theory. Griffin delves into Russell's interest in the theory of groups in an attempt to understand its absence from *PM*. Arie Hinkis examines the treatment of the "Cantor–Bernstein Theorem" (which states that a bijection holds between an infinite set A and an infinite set B only if A and B are equinumerous) in *PM*. This theorem is proved in four different versions in *PM*, each of which reveals interesting features of the formal apparatus of *PM* (not least the difficulty that adopting a no-classes theory poses for set-theoretic reasoning). Finally Sébastien Gandon offers a discussion of the theory of measurement and real analysis contained in the third volume of *PM*. This is a fitting conclusion to a volume that stands as a wonderful monument to the first century of *PM*'s influence. As the editors say in their introduction, thanks to Gandon "[w]e join the three Poles and three Texans of the anecdote in knowing what goes on in those later sections" (p. xxvii).

PM is an extraordinary intellectual achievement, despite its apparent failure to achieve its intended purpose of demonstrating of the truth of logicism. Most (though not all) now think that logicism buckled to the point of failure under the combined weight of Russell's paradox and subsequent metalogical results such as Gödel's incompleteness theorems, even when fortified by *PM*'s

type-theory. But what it did achieve was remarkable. *PM* showed how vast portions of mathematics could be reduced to set-theory. It established a new discipline of mathematical logic and placed it at the heart of both mathematics and philosophy. It played a pivotal role in establishing the dominant tradition in philosophy for the next hundred years and beyond. *PM* is a difficult book, but one that still deserves attention. If a reminder of that fact is needed, then this superb *Companion* will certainly provide it.

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