THREE UNPUBLISHED MANUSCRIPTS FROM 1903

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I present and discuss three previously unpublished manuscripts written by Bertrand Russell in 1903, not included with similar manuscripts in Volume 4 of his *Collected Papers*. One is a one-page list of basic principles for his "functional theory" of May 1903, in which Russell partly anticipated the later Lambda Calculus. The next, catalogued under the title "Proof That No Function Takes All Values", largely explores the status of Cantor's proof that there is no greatest cardinal number in the variation of the functional theory holding that only *some* but not all complexes can be analyzed into function and argument. The final manuscript, "Meaning and Denotation", examines how his pre-1905 distinction between meaning and denotation is to be understood with respect to functions and their arguments. In them, Russell seems to endorse an extensional view of functions not endorsed in other works prior to the 1920s. All three manuscripts illustrate the close connection between his work on the logical paradoxes and his work on the theory of meaning.

Attached below are three previously unpublished manuscripts written by Bertrand Russell. They are not explicitly dated, but internal evidence leaves little doubt that they were written in 1903, probably near the middle of the year. They are of the same general nature as the manuscripts published in Volume 4 of Russell's *Collected Papers* (Parts I–III), and most closely relate to its Paper 3, "Functions". The reader should not read them in isolation, but consider them in the context of the other manuscripts published in that volume. They were known when the volume was published and are mentioned in one of its appendices (4: 633–4). They were perhaps considered too short, too redundant, or too unpolished for inclusion, though the difference between them and the material that was published is at most a matter of degree. The first is a single sheet on logical principles governing functions (RA1 230.030920). The second is a 25-leaf manuscript catalogued under the title "Proof That No Function Takes All Values" (230.030850). It is rather rough, and is written mostly in symbols. It does not display a clear paragraph structure and large portions are crossed off or marked by Russell himself as wrong. It was clearly never intended for public consumption. The final manuscript contains two leaves and is titled "Meaning and Denotation" (230.030950), and deals with the application of Russell's pre-1905 distinction between meaning and denotation to functions and their arguments. The three manuscripts together demonstrate the close interconnection between, on the one hand, Russell's work on logical first principles governing classes and functions (driven by the need to solve the logical paradoxes) and, on the other, his work on the theory of meaning, including his evolving views on the nature and identity conditions of propositions and other complexes.[1]

1. HISTORICAL BACKGROUND

In 1903, Russell's chief philosophical occupation was the attempt to solve certain logical paradoxes, especially the antinomy of classes now known as "Russell's paradox", which he simply referred to as "the Contradiction". In his journal of 23 May 1903, Russell wrote, "Four days ago, I solved the Contradiction—the relief of this is unspeakable" (*Papers* 12: 24). What he discovered was the first version of a "no classes" theory of classes, one in which the role played by classes in symbolic logic was given over to propositional and other functions. If there are no classes at all, then there is certainly no such class as the

[1] Because these manuscripts are likely to be of interest primarily to those interested in charting the development of Russell's thought, his own deletions have been retained, but struck out, except where completely illegible in the manuscripts. Angled brackets ⟨ ⟩ are used for editorial insertions meant to improve readability or grammaticality. Underlining has been changed to italics when it appears to be used merely for emphasis, but has retained when used for other purposes. For the sake of consistency, certain minor alterations were made, such as changing lowercase "sim" and "cls" to "Sim" and "Cls" throughout. Various abbreviations frequently used by Russell have been silently expanded.

class of all classes not members of themselves, and *that version* at least of Russell's paradox no longer threatens. Russell wrote excitedly to Frege the following day (24 May 1903):

> I believe I have discovered that classes are entirely superfluous. Your designation $\acute{\varepsilon}\,\phi(\varepsilon)$ can be used for ϕ itself, and $x \frown \acute{\varepsilon}\,\phi(\varepsilon)$ for $\phi(x)$. (I write ε instead of ⌢, like Peano.)[2]

Russell's inspiration here was Frege's notation for the "value-range" (*Werthverlauf*) of a function. The notation consists of a bound variable written first with a "smooth-breathing" (*spiritus lenis*) accent mark followed by an expression containing the variable. In particular, "$\acute{\varepsilon}$ … ε …" is to be taken as a name for the value-range of a function whose value for ε as argument is … ε…. Russell rejected Frege's rigid type distinction between functions and objects, and with it, Frege's distinction between a function and its value-range, and so his intention, as he makes it clear above, is to use this notation for the function itself. Unlike Frege, Russell used his notation, "$\acute{x}$ … x …", with any letter as a variable, not just Greek vowels. It was thus a notation for function abstraction, a precursor to Church's Lambda Calculus notation, "λx … x …".

At least as it comes across in the letter to Frege, Russell's strategy seems fairly simple.[3] In his symbolic logic, class variables such as α and β would be replaced by function variables ϕ and ψ, which would be allowed in subject position. Class abstracts would be replaced by function abstracts. He would continue to use the membership sign "ε" but reinterpret it to mean the application of a function to argument (similar to Frege's "⌢"). He also wrote "$\phi|x$" for the value of ϕ with x as argument, so that, at least if ϕ is a function, "$x\ \varepsilon\ \phi$" and "$\phi|x$" would be notational variants.

In the letter to Frege, Russell goes on to define relations and

[2] See Frege, *Philosophical and Mathematical Correspondence* (1980), pp. 158–9. Russell's correspondence with Couturat is also relevant; see Russell, *Correspondance sur la philosophie, la logique et la politique avec Louis Couturat (1897–1913)* (2001).

[3] Whether Russell's intention was really so simple, and what led him to this view, is a complicated matter, and somewhat controversial. For a variety of positions, see Cocchiarella, "Wither Russell's Paradox of Predication" (1973); Landini, "Russell to Frege 24 May 1903: 'I Believe I Have Discovered That Classes Are Entirely Superfluous'" (1992); and Klement, "The Origins of the Propositional Functions Version of Russell's Paradox" (2005).

properties of functions similar to the usual relations and properties of classes (subset, etc.), defining the number of ϕ's as the function $\acute{\psi}(\phi \text{ Sim } \psi)$, i.e., the function satisfied by a function ψ just in case a $1 \rightarrow 1$ correlation exists between the ϕ's and ψ's. Russell's imagined system of function abstraction came several steps closer to modern Lambda Calculi than did Frege's system of value-ranges.[4] For example, Russell anticipated Schönfinkel's method of treating multi-argument functions as single-argument functions whose values are themselves functions. Thus rather than thinking of "+" as representing a function with two arguments, writing "$2 + 3$", one might write "$(+|2)|3$". Here "+" represents a function; its value for 2 as argument is the function written here as "$(+|2)$", which itself takes an argument and yields as value two more than that argument. Hence "$(+|2)|3$" stands for five.

At least early in this period, Russell did not distinguish logically between propositional functions and other functions (later called "denoting functions"). Thus the notation could be used for both functions such as $\acute{x}$(the center of mass of x), whose value for an occupied physical region as argument would be a point in space, as well as a function such as $\acute{x}$(Plato loves x), whose value for something as argument is the proposition that Plato loves it. Relations then could be understood as functions with functions as value, and "xRy" be used as shorthand for "$(R|x)|y$", just as with "+". Abstracts for functions with functions as value, doing the work of relations and other multi-argument functions, could be written with successive smooth-breathing abstractors. E.g., "$\acute{y}\acute{x}(y$ loves $x)$" would represent the function whose value for Plato as argument is the function $\acute{x}$(Plato loves x), and so "$[\acute{y}\acute{x}(y$ loves $x)\,|\,$Plato$]\,|\,$Socrates" would be another name for the proposition that Plato loves Socrates.

At this time, Russell's logic embraced an all-encompassing logical type of "entity" (*PoM*, p. 43), and the variables of his logic were taken as unrestrictedly quantifying over all entities whatsoever. Without further qualifications then, there is nothing preventing a function from taking other functions, or even itself, as argument. As Russell soon discovered, this meant that the simple form of his functional theory did not escape Russell's paradox after all, as it could be formulated in

[4] For further comparison between Russell's approach at this time and later systems, see KLEMENT, "Russell's 1903–05 Anticipation of the Lambda Calculus" (2003).

terms of a function W satisfied by all and only functions not satisfying themselves. Russell reported his 1903 discovery of this in a 1906 letter to Jourdain as follows:

> Then, in May 1903, I thought I had solved the whole thing by denying classes altogether; I still kept propositional functions, and made ϕ do duty for $\dot{z}(\phi z)$. I treated ϕ as an entity. All went well till I came to consider the function W, where
>
> $$W(\phi) \,.\!\equiv_\phi.\, \sim\phi(\phi)\ .$$
>
> This brought back the contradiction, and showed that I had gained nothing by rejecting classes.
>
> …
>
> Gradually I discovered that to assume a separable ϕ in ϕx is just the same, essentially, as to assume a class defined by ϕx, and that non-predicative functions must not be analysable into a ϕ and an x.[5]

Russell understood a function as something got by differentiating one element in a complex from the remainder, where the remainder is the function. The realization then seems to have been that not every complex containing an entity x can be divided cleanly into that entity and a function as "remainder". Russell began to speak of "functional complexes" and wrote:

$$X \text{ Focp } x$$

to mean that X is a functional complex of x and remainder. Technically this means that Russell would reject the usual higher-order comprehension principle for functions:

> $(\exists\phi)(x)(\phi|x = \ldots x \ldots)$, where $\ldots x \ldots$ is an expression containing x but not ϕ free

Instead, he would accept only a weaker version, perhaps along the lines of:

$$(\exists x)(X \text{ Focp } x) \supset (\exists\phi)(x)(\phi|x = X)$$

Russell seems to use the uppercase letter "X" for a complex containing

[5] See GRATTAN-GUINNESS, ed., *Dear Russell—Dear Jourdain* (1977), pp. 78–9.

the entity represented by the lowercase "x", and "Y" for a complex containing the entity represented by lowercase "y". In modern terminology, it is perhaps a bit unclear, however, whether "X" and "Y" are best interpreted as object-language variables for complexes, or as metalinguistic schematic letters to be replaced in particular instances with complex expressions containing "x" or "y". Nonetheless, it appears that when "X" in such a context is replaced by a particular instance containing "x", the "x" in question can be bound by quantifiers with a scope narrower than the whole formula.

Russell does not seem to have discovered a way of stating a general criterion for the functionality of complexes, nor even a partial demarcation that allowed what was necessary for his logicist project to proceed. By 1904 at least, Russell concluded that abandoning classes in favor of functions did not have the advantages he initially imagined, and returned to a realism about classes, employing certain variations of what he called the "zigzag theory" (see *EA*, pp. 145–51) to address the paradoxes involving classes. Nonetheless, some of the core themes and ideas of the 1903 "functional theory" remained. As late as early 1905, Russell spoke of what he then called "modes of combination" of an entity x within a complex ($C \between x$) such that the remainder of the complex over and above x cannot always be thought of as a separate entity, or function (see, e.g., *Papers* 4: 366), still without giving conditions for when it can. Stability on these issues was not reached until after his discovery of the theory of descriptions in 1905, which made non-propositional functions unnecessary, and which allowed him to develop a "substitutional theory" on which the notion of a propositional function was replaced by the notion of the substitution of one entity within another.[6]

II. THE FUNCTIONS SHEET AND PROPOSITIONAL IDENTITY

The first manuscript, the one-page sheet on "Functions", appears to be one of the first written for the "functional theory". The first numbered principle claims that *every* complex containing x can be divided into function and argument, which indicates that he did not yet differentiate between "functional" and "non-functional" complexes.

[6] For further discussion of the post-1905 view, see Landini, *Russell's Hidden Substitutional Theory* (1998), and the related manuscripts in *Papers* 5.

Russell describes a function as something got by "taking away" one constituent of a complex, which suggests that functions are structurally simpler than their values. This contrasts with Russell's earlier views from *The Principles of Mathematics*, in which he criticized similar views found in Frege. There he held instead that what remains when a constituent of a proposition is removed is, in most cases, "no discoverable kind of entity" (p. 107; see also Ch. VII and p. 508), and that the entity to be varied must instead be replaced by a variable. In his mid-1903 manuscripts, however, Russell gives no indication that he thinks of functions as containing variables or any other such "placeholder" objects in their argument places. Russell's use of the Greek letter "ξ" in the symbolic rendering of thesis 1 to represent the argument spot of the function comes from a similar device used by Frege.[7]

It seems likely that the sheet was written at the same time as the 24 May 1903 letter to Frege. On the back is written a single formula "$\sim\phi\{\phi(p)\,.\supset_p.\,p\}$". This pertains to the functional reworking of the Cantorian paradox of propositions Russell discusses in §500 of *PoM*, which Russell had broached in an earlier letter to Frege.[8] By Cantor's power-class theorem, if propositions can be members of classes, there ought to be more classes of propositions than propositions. However, it seems possible, for each class of propositions, m, to form a distinct proposition, viz., the proposition that all members of m are true. Cantor's diagonal method leads us to a contradiction. Reworked to involve functions rather than classes, the difficulty is that for each function ϕ satisfiable by propositions, we can generate a distinct proposition, the proposition that all propositions satisfying ϕ are true: $\phi(p)\,.\supset_p.\,p$. Some propositions of this form will satisfy the function ϕ they are "about"; others will not (in which case, $\sim\phi\{\phi(p)\,.\supset_p.\,p\}$). Consider then the function ψ satisfied by all and only propositions of this form that do not satisfy the proposition they are about:

$$\psi = \acute{q}[(\exists\phi)(q =:\phi(p)\,.\supset_p.\,p:.\sim\phi(q))]$$

as well as the proposition of this form "about" ψ itself:

$$r\ =:\psi(p)\,.\supset_p.\,p$$

7 See Frege, *Basic Laws of Arithmetic* (2013), p. 6.

8 See Frege, *Philosophical and Mathematical Correspondence*, p. 147.

It seems likely that we will get the following contradiction:

$$\psi(r) \equiv {\sim}\psi(r)$$

Notice, however, that the fact that for each function ϕ, we have a proposition $\phi(p) \,.\supset_p.\, p$, only entails that there are as many propositions as functions if the propositions so generated are always distinct for distinct functions. Similarly, if one works out the proof of the left-to-right half of the above contradictory biconditional, it will be seen that it requires the assumption that if r (i.e., $\psi(p) \,.\supset_p.\, p$) is identical to a proposition of the form $\phi(p) \,.\supset_p.\, p$, and r does not satisfy ϕ, r must also not satisfy ψ. This seems plausible on the grounds that identical propositions must have identical constituents. If r is both the proposition $\psi(p) \,.\supset_p.\, p$ as well as the proposition $\phi(p) \,.\supset_p.\, p$, then surely ϕ and ψ must be the same function. However, at the end of his 24 May 1903 letter to Frege, Russell addresses this functional version of the paradox, and makes note of a theorem proven by Frege in the appendix on Russell's paradox added to Volume II of Frege's *Grundgesetze*, according to which there is no function from functions to objects that always yields distinct objects for non-coextensive functions as argument. Considering $\acute{\phi}(\phi(p) \,.\supset_p.\, p)$ as such a function Russell concludes that it must be possible for $\psi(p) \,.\supset_p.\, p$ to be the same proposition as $\phi(p) \,.\supset_p.\, p$ even while this single proposition satisfies the function ψ but not the other function ϕ. This blocks the left-to-right half of the contradictory biconditional, and, apparently, solves the paradox.

Principle 3 of the one-page function sheet arguably sheds light on this. Consider the proposition (complex) $8 \geq 8$. There are three ways to "take away" the number 8 from this complex to arrive at a function, i.e., $\acute{x}(x \geq 8)$, $\acute{x}(8 \geq x)$ and $\acute{x}(x \geq x)$. By principle 3, it would seem to follow that if I resupply 8 as argument to any one of these functions, I arrive back at the original proposition. Hence, it seems to entail the following identities:[9]

[9] Whether or not the symbolic representation of 3 lower on the sheet is sufficient by itself to derive these results depends a bit on how the "replacement rule" for function variables such as ϕ would be handled in Russell's logic at the time, which is not altogether clear. Yet I think Russell intended results such as these to follow, directly or indirectly; this is confirmed by a principle in another period manuscript (*Papers* 4: 53).

$$8 \,\varepsilon\, \acute{x}(x \geq 8) = (8 \geq 8)$$
$$8 \,\varepsilon\, \acute{x}(8 \geq x) = (8 \geq 8)$$
$$8 \,\varepsilon\, \acute{x}(x \geq x) = (8 \geq 8)$$
and therefore, $8 \,\varepsilon\, \acute{x}(x \geq 8) = 8 \,\varepsilon\, \acute{x}(8 \geq x)$, etc.

The "=" here represents *identity* of propositions, and is stronger than mere material equivalence (≡). In the vocabulary of the contemporary Lambda Calculus, Russell's principle 3 seems tantamount to the claim that $\beta\eta$-convertible formulas represent identical complexes. Notice, however, that $\acute{x}(x \geq 8)$, $\acute{x}(8 \geq x)$ and $\acute{x}(x \geq x)$ are not even coextensive, much less identical functions. It is therefore possible for expressions of the forms "$x \,\varepsilon\, \phi$" and "$x \,\varepsilon\, \psi$", or, equivalently, "$\phi|x$" and "$\psi|x$", to represent the same proposition (or other complex) without "ϕ" and "ψ" representing the same function, or even coextensive functions. This leaves room similarly that $\psi|p\,.\supset_p.\,p$ and $\phi|p\,.\supset_p.\,p$ could be the same proposition without ϕ and ψ being the same, apparently vindicating Russell's suggestion regarding the propositional paradox in the letter to Frege. Unfortunately, matters are not quite so simple, as we shall discover below.

III. FUNCTIONAL COMPLEXES AND CANTOR'S THEOREM

The second manuscript, by far the longest presented here, derives from later in 1903, and clearly represents a more nuanced position. Russell has recognized the need to distinguish functional and non-functional complexes, and the manuscript is largely dedicated to exploring this distinction and its consequences for the logicist project. The manuscript is filed in the Russell Archives under the title "Proof That No Function Takes All Values", most likely because that is what is written at the top of the first sheet. The title is misleading, however, as that makes up the topic of discussion only for that page, and Russell does not in the end endorse the so-called "proof", noting ways to resist its conclusion already six lines down. Perhaps a better name for the manuscript would be "Functional Complexes and Cantor's Disproof of a Greatest Cardinal". Russell is concerned throughout with the notion of a functional complex and, especially in the first half of the manuscript, with the ramifications of this notion for the applicability of Cantor's theorem in the resulting system. As mentioned above, the manuscript is quite rough. It is reproduced below with the sheets in the order in which they were filed in the Archives. It is not certain whether or not this is the order of composition, or even if the various leaves were in any sense meant as a single discussion.

Cantor's power-class theorem states that the number of subclasses of a given class a is always greater than the number of members of that class. For *reductio*, suppose instead that there were as many members of a as subclasses of a; it would then be possible to map subclasses of a to members such that each subclass is correlated with a distinct member. However, Cantor then entreats us to consider the subclass w of a containing all those members of a correlated with a class in which they are not included. As w is a subclass of a, it must be correlated with some member, a_w of a. But now a_w is a member of w just in case it is not. Contradiction. The power-class (class of all subclasses) of a given class a with n members has 2^n members, as it results from making all possible combinations of yes-no, in-or-out, choices for the n members of a. So the result is typically taken to show that $2^n > n$ even when n is infinite, and hence that there is no greatest cardinal number. No class can be of the largest possible size: its power-class is larger.

However, when these results are transferred from a class theory to Russell's functional theory, they become doubtful because of the potential inapplicability of the "diagonalization" procedure at the heart of Cantor's proof. At first glance, it would appear possible to provide a parallel argument showing that there must be more "sub-functions" of a given function ϕ (functions satisfied only by arguments also satisfying ϕ) as there are arguments that satisfy ϕ. An alleged correlation from all sub-functions of ϕ to arguments satisfying ϕ would seem to omit the "diagonal" function W satisfied by all and only arguments satisfying ϕ correlated with a sub-function of ϕ that they do not satisfy. However, an issue arises as to whether or not there is such a diagonal function, as the formula defining it may not represent a functional complex. Suppose for example, we represent the alleged correlation between functions and arguments as f; W would then be defined as the function satisfying those values of x such that $(\exists\psi)(x = f|\psi\,.\sim\psi|x)$. However, the complex $(\exists\psi)(x = f|\psi\,.\sim\psi|x)$ is quite close in form to the kind of complexes Russell needed to rule out as "non-functional" in order to solve Russell's paradox. (This should come as no surprise as Russell arrived at Russell's paradox itself by means of a diagonal argument when attempting to reconcile Cantor's theorem with the existence of such "large" classes as the universal class and class of all classes; see *PoM*, pp. 101, 367.) Hence, the identification of certain complexes as "non-functional" threatens to undermine the functional equivalents of Cantor's main results.

Recall that on the proposed treatment of mathematics, the cardinal number of ϕ's is identified with the function satisfied by those functions with which ϕ is "similar" (Sim, i.e., the relation of standing in a $1 \rightarrow 1$ correlation with). Recall also that relations are here treated as functions having functions as value, so that "a Sim b" is more properly written "$(\mathrm{Sim}|a)|b$". Hence "$\mathrm{Sim}|a$" by itself represents the cardinal number of the "class" (really, function) a. Russell writes the "power-class" of a as Cls'a, and the issue comes down to, as it is written several times in the manuscript, whether or not $\mathrm{Sim}|\mathrm{Cls}`a > \mathrm{Sim}|a$. (By this time Russell differentiates propositional from "denoting" functions, writing "$\phi|x$" with the vertical bar for the application of a propositional or relational function to its argument, and an inverted comma $f`x$ for the application of a denoting or "other" function to its argument, as mentioned later in the same 1906 letter to Jourdain.)

This relates to the issue as to whether or not there is a function having all values in the following way. If a function had all entities whatever among its values, it would thereby have all functions among its values. Since functions are always one-valued, a function having all functions as values would itself be a correlation between some entities and all functions, thereby violating the functional version of Cantor's theorem. If F were such a function, diagonalization would lead us to consider the function satisfied by all those entities x correlated by F with some function they do not satisfy, i.e., all those x such that $\sim\{(F|x)|x\}$. If there were such a function—call it f—and it were itself a value of F (as all entities are) then there would be a y such that $F|y = f$; however, we have that $f|y$ just in case $\sim\{(F|y)|y\}$, but also, since $F|y = f$ that $f|y$ just in case $(F|y)|y$. As Russell notes, the contradiction can be avoided by denying that $\sim\{(F|x)|x\}$ is a functional complex of x, and hence that there is any such function as f.

Russell assumes that a complex X is functional just in case its negation is. On this assumption, if $\sim\{(F|x)|x\}$ is not a functional complex of x, then neither is $(F|x)|x$. Notice, however, that since xRx is a notational variant of $\{(R|x)|x\}$, holding that *no* complex of the form $(F|x)|x$ is functional is tantamount to denying all functions of the form $\dot{x}(xRx)$, i.e., denying the existence of reflexive relational properties. While this is perhaps welcome in the case of, e.g., self-membership, overall, Russell seems to think it too harsh a conclusion in other cases. We find him trying to make room for, e.g., "suicide" (as he puts it on fol. 14), i.e., $\dot{x}(x \text{ kills } x)$. Yet, he fails to provide a general criterion for which complexes of the form $(F|x)|x$ are to be regarded as functional.

He does explore certain suggestions for *sufficient* conditions under which a complex X containing x is functional, such as when the expression "X" contains no instances of the vertical bar "|" (fol. 2), or in which the function which it would define would always be satisfied only by entities of a "lower order" in some kind of hierarchy (fol. 18). This last suggestion is perhaps an early forerunner of his mature ramified hierarchy, though note what he has in mind here does not seem to be the kind of "type theory" that puts restrictions on *meaningfulness*. The lowest order of functions is those only *true* of individuals, not those only meaningfully assertable of individuals. In addition to such sufficient conditions, Russell provides certain principles (fol. 22) guaranteeing that a complex is functional (or non-functional) in case its parts are functional (or non-functional). For instance, the disjunction of two functional complexes is also a functional complex. Russell hoped that this would be enough to establish the existence of the functions needed for his mathematical project, even without a generic and effective criterion to settle all cases. Without Russell's having provided a definite criterion for functionality, one cannot determine in any definitive way what the fate of Cantor's results is within the system. It is worth noting, however, that Nino Cocchiarella has provided certain working reconstructions of Russell's views during this period, employing a notion of a stratified formula similar to that employed in Quine's "New Foundations" as a criterion for "functionality". As one might expect, Cantor's theorem fails in this reconstructed system for much the same reason it fails in Quine's system.[10]

[10] See COCCHIARELLA, *Logical Studies in Early Analytic Philosophy* (1987), Chs. 1–2. The reader, however, may wish to judge for him/herself how closely Cocchiarella's reconstructions come to what Russell had in mind during this period. Notably, Russell took the notion of X being a functional complex of x as a property of the complex X in relation to its constituent x, and thought the notion could be represented in the object language itself. This at least appears to differ in important ways from Cocchi-

Although they receive less attention, the manuscript also contains some very interesting discussion of the Cantorian paradoxes of propositions. On folio 18, Russell notes that for every class, there is a class of propositions of the same cardinality, i.e., the class of all propositions of the form $y = y$ where y is a member of the class. This assumes that self-identity propositions are distinct for distinct entities. As Russell notes, because he countenances a universal class (universal function), this means that there are as many propositions as entities altogether, including as many propositions as classes of propositions. He seems not to regard this conclusion as problematic, as he has rejected Cantor's result suggesting the opposite. Given what has been written, it would seem natural for him to claim that the "diagonally defined" classes of propositions do not exist, as classes are treated in this system as functions, and the complexes that would define these functions are not functional.

However, later in the manuscript (fol. 21), Russell seems to suggest a different tack, one more akin to the suggestion in his May 1903 letter to Frege. Interestingly, Russell hints here at a version of the propositional paradox not solved even by the suggestion that $\beta\eta$-convertible formulas represent identical propositions. This version involves correlating each function ϕ with the proposition Russell writes as "$\exists\phi$". He is not here using "$\exists$" as the modern existential quantifier, and it does not bind the variable ϕ. When Russell uses "$\exists$" as the modern variable-binding quantifier, he always writes parentheses around it and the variable it binds, i.e., "$(\exists\phi)$". When no such parentheses are used, "$\exists$" is analogous to Peano's original usage of this symbol as a predicate which could be asserted of a non-empty class. In Peano's notation "$\exists a$" means that the class a is non-empty. Russell is using functions in place of classes in his logic, and "$\exists\phi$" represents the proposition that ϕ is satisfied by at least one argument. Let us consider all propositions of the form $\exists\phi$ which do not themselves satisfy the function ϕ which they assert to be non-empty. If the complex $(\exists\phi).\{x = \exists\phi \,.\sim \phi|x\}$ is a functional complex of x, then these propositions are those that satisfy the function $\acute{x}[(\exists\phi).\{x = \exists\phi \,.\sim \phi|x\}]$. Let us call this function f. Paradox threatens if we consider the proposition that

arella's approach on which "stratification" is a metalinguistic property of certain formulas, and is used to stipulate which values of certain schemata are taken as axioms. Whether or not these two approaches are compatible is worthy of further study.

f is non-empty, viz., $\exists f$, and ask whether or not:

$$f|\exists f \,.\!\equiv\!.\sim f|\exists f$$

The proof of the right-to-left half of this biconditional is straightforward. For the left-to-right, assume $f|\exists f$. There is then some ϕ such that $\exists f = \exists\phi$ and $\sim\phi|\exists f$. If we assume

$$\phi \sim= \psi \,.\!\supset\!.\, \exists\phi \sim= \exists\psi$$

it follows then that because $\exists f = \exists\phi$, it holds that $f = \phi$ and hence, because $\sim\phi|\exists f$, it also holds that $\sim f|\exists f$.

There are two broad lines of response to this worry that might have been attractive to Russell at this time. The more obvious response, to my mind, would be to deny that $(\exists\phi).\{x = \exists\phi\,.\sim\phi|x\}$ is a functional complex of x, and hence to deny that there is any such function as f. Russell, however, seems attracted here to the supposition that if a complex X containing x only yields a truth when x is replaced by an individual (non-function)—or, equivalently, if the would-be function defined by X by removing x would only hold of individuals—then X is functional. Because Russell holds at this time that propositions are individuals, and $(\exists\phi).\{x = \exists\phi\,.\sim\phi|x\}$ would only hold for values of x that are propositions, he is disinclined to conclude that it is non-functional. Instead, he seems attracted to a response similar to that given to the version of the propositional paradox discussed at the end of the letter to Frege. This involves adopting a coarser-grained understanding of the identity conditions of propositions. Recall that, for the earlier paradox, Russell accepted that $\phi|p\,.\supset_p.\,p$ could be the same proposition as $\psi|p\,.\supset_p.\,p$ without ϕ and ψ being the same, or even coextensive, functions. The analogous move here would require denying the assumption that $\phi \sim= \psi\,.\!\supset\!.\,\exists\phi \sim= \exists\psi$ given above, i.e., to allow that $\exists\phi$ could be the same proposition as $\exists\psi$ even while ϕ and ψ are distinct. However, even the assumption that $\beta\eta$-convertible formulas express the same proposition does not make it clear how or why this could be. Indeed, allowing this seems to require an extremely coarse-grained understanding of the identity conditions of propositions, one that seems wholly at odds with Russell's general conception of a proposition as a structured complex of parts.

That Russell is willing to consider coarse-grained accounts of propositions is confirmed in the paragraph lower on the leaf, where he seems willing to accept the following principle:

$$\phi = \psi \,.\supset.\, \phi|x = \psi|x$$ [11]

This is stronger than it may appear. Russell uses "=" as a sign for strict identity only when flanked by names of propositions or other individuals; when flanked by signs for functions, coextensionality suffices. Russell uses another relation sign "1'" (from Schröder) for strict identity, and defines "=" in terms of it as follows:

$$u = v \,.=:.\, \mathrm{Indiv}(u) \,.\supset.\, u \; 1' \; v : \sim\mathrm{Indiv}(u) \,.\supset.\, \sim\mathrm{Indiv}(v) \,.\, u|x \equiv_x v|x \;\mathrm{Df}$$

Hence, if ϕ and ψ are propositional functions, the above principle means that if ϕ and ψ are coextensive, then $\phi|x$ and $\psi|x$ are *the same proposition*. Adopting such loose identity conditions for propositions may help with the propositional paradoxes—as Russell notes, it allows that there may be fewer propositions than functions. But it seems to open Russell up to arguments of the style of the so-called "Frege–Church slingshot", for identifying all materially equivalent propositions. Consider arbitrary propositions p and q that happen to have the same truth-value, i.e., $p \equiv q$. The functions $\dot{x}(x = x \,.\, p)$ and $\dot{x}(x = x \,.\, q)$ would be coextensive and thus "equal" according to the above definition of "=". Assuming $\phi = \psi \,.\supset.\, \phi|x = \psi|x$, this means that the proposition $\dot{x}(x = x \,.\, p)|a$ is the same as $\dot{x}(x = x \,.\, q)|a$, and we then get that $a = a \,.\, p$ and $a = a \,.\, q$ are the same proposition. To get from this that p and q are the same proposition, one need only make the plausible assumption that conjunctions are identical only when they have identical conjuncts. This would mean there are only two propositions: the true one, and the false one—which seems to destroy the very notion of a Russellian proposition.

Matters are complicated somewhat, however, by Russell's employment during this period of a distinction between meaning and denotation. Two expressions may differ in meaning and yet denote the same thing; perhaps two expressions may even differ in meaning and yet denote the same proposition. Perhaps "$\phi|a$" and "$\psi|a$", where

[11] Russell rejects this principle in other 1903 works: see *Papers* 4: 53.

"ϕ" and "ψ" represent merely coextensive functions, differ in meaning despite standing for the same proposition.

IV. MEANING AND DENOTATION AND FUNCTIONS

The issue of how the meaning/denotation distinction applies to propositions is naturally tied up with how the distinction applies to propositional (and other) functions, an issue raised explicitly in our last manuscript, "Meaning and Denotation". In the opening paragraph, Russell makes it clear that the meaning/denotation distinction applies to function expressions, and that what is called "the function" is the denotation. He even seems to endorse extensional identity conditions for functions, though this unfortunately is obscured somewhat by his choice of example. By modern sensibilities, we would not simply *define* "human" as "rational animal"; the coextensionality of these predicates is taken to be an empirical discovery. However, it was not so long ago that "human" and "rational animal" were routinely given as examples of predicates having not only the same extension, but the same intension.[12] This makes it a bit difficult to assess how radical to take Russell's claim. There are perhaps two reasons to think he does mean at this point to identify the denotation of all coextensive function expressions. First, he acknowledges a difference in meaning between "to be a rational animal" and "to be a man"; if these were taken as the same property only because being a man is simply defined as being a rational animal, i.e., if "man" were just a convenient abbreviation for "rational animal", it is unclear that there would even be a difference in meaning. Secondly, at one point in the second manuscript (fol. 24), Russell seems willing to forgo the distinction between strict identity "1‘" and equality for functions "=", there giving "=" the definition he had originally given to "1‘". Since Russell has all but identified materially equivalent propositions (as denotations), this too is tantamount to adopting an extensional view of functions. If he did in fact consider an extensional view of not only functions, but propositions, at this time, it was somewhat of an anomaly in his philosophical development, and one it seems did not last long. Indeed, he explicitly calls it into question in another (presumably, later) 1903 manuscript, apparently in part due to objections by Whitehead (*Papers* 4: 310).

[12] See, e.g., CARNAP, *Meaning and Necessity* (1956), p. 15.

The rest of the manuscript deals with how to understand the relationship between functions and their arguments when the argument or value of the function is a denoting complex. Russell himself is unable to answer all the questions he poses. Similar questions are raised in other manuscripts from 1903 and subsequent years leading up to his abandonment of the meaning/denotation distinction in 1905 (see, e.g., *Papers* 4, Parts II–III). Here, Russell suggests that $\dot{x}(x$ is a man$)|$Edward VII is the same "value" of the function $\dot{x}(x$ is a man$)$ as $\dot{x}(x$ is a man$)|$the King of England. Since the "values" of this (propositional) function are presumably propositions, this suggests that they are the same proposition. This is a deviation from Russell's earlier views according to which "The King of England is human" would be taken as expressing a distinct proposition from "Edward VII is human" in virtue of having a denoting concept as constituent as opposed to Edward himself. Notice, again, however, that the phrases "$\dot{x}(x$ is a man$)|$Edward VII" and "$\dot{x}(x$ is a man$)|$the King of England" could differ in meaning, even if they denote the same proposition.

We can see how providing an answer to difficult questions about meaning and denotation is necessitated by his view of functions in 1903. As we have seen, functions are described as what is got by removing a constituent from a complex. The value of a function for a given argument, therefore, should be the same as what is got by replacing or substituting the constituent of the original complex with the argument. In the second manuscript (fol. 13; *cf.* fol. 14), we find Russell writing:

$$(\phi|x)\ \text{Focp}\ x\,.\supset.\,\{\dot{x}(\phi|x)\}|y = (\phi|x)\,\frac{y}{x}$$

Can we form a function, however, by removing a constituent from a denoting complex which is not present in its denotation? If so, this greatly complicates the overall picture. Consider the following instance of the above:

$$\begin{aligned}&(\{\dot{y}[\dot{z}(z\ \text{is Catholic})|\text{the King of}\ y]\}|\text{England})\ \text{Focp England}\,.\supset.\\ &\qquad\{\dot{x}(\{\dot{y}[\dot{z}(z\ \text{is Catholic})|\text{the King of}\ y]\}|x)\}|\text{Italy} =\\ &\qquad\qquad(\{\dot{y}[\dot{z}(z\ \text{is Catholic})|\text{the King of}\ y]\}|\text{England})\,\frac{\text{Italy}}{\text{England}}\end{aligned}$$

Assuming the antecedent holds, by various $\beta\eta$-conversions, this entails:

$$(\text{the King of Italy is Catholic}) = [\acute{z}(z \text{ is Catholic})|\text{the King of England}]\frac{\text{Italy}}{\text{England}}$$

However, if $\acute{z}(z$ is Catholic$)|$the King of England is the same "value" of the function $\acute{z}(z$ is Catholic$)$, as $\acute{z}(z$ is Catholic$)|$Edward VII, then it would seem we may replace the one with the other above:

$$(\text{the King of Italy is Catholic}) = [\acute{z}(z \text{ is Catholic})|\text{Edward VII}]\frac{\text{Italy}}{\text{England}}$$

But presumably, substituting Italy for England in $\acute{z}(z$ is Catholic$)|$Edward VII leaves it unchanged, and thus we have:

$$(\text{The King of Italy is Catholic}) = (\text{Edward VII is Catholic})$$

This result is worse even than identifying all propositions with the same truth-value, for here we are identifying propositions with *different* truth-values—a clear absurdity.

Russell was not unaware of such issues; as he notes on folio 19 of the second manuscript, "$\acute{x}(X)$ has to do with the *meaning* of X, whereas $X = X'$ has to do with the denotation". Nonetheless, respecting this difference would require fine-tuning the logic to disallow substitution of expressions with identical denotations but distinct meanings in contexts that "have to do" with the meaning. As Russell acknowledges in another period manuscript, one should perhaps write "$\acute{x}$("… x …")" rather than simply "$\acute{x}$(… x …)" (*Papers* 4: 195). Russell never provides such a fine-tuned logic, and no doubt it was in part his failed attempts to do so that began him down the road to the conclusion given in "On Denoting"'s infamous Gray's Elegy Argument that the attempt to disambiguate between a denoting complex and what it denotes inevitably leads to unsolvable problems and "inextricable tangles".

What is perhaps worse is that from the standpoint of the functional theory of 1903, had Russell succeeded in developing a method for speaking of meanings or denoting complexes as opposed to their denotations, it likely would have undermined certain other commitments of his views at the time. Recall for example that his suggestion in the letter to Frege and elsewhere as to how to solve the Cantorian paradoxes of propositions was to identify apparently different propositions. For instance, perhaps $\exists\phi$ can be the same proposition as $\exists\psi$ even when $\phi \neq \psi$, and hence perhaps there are fewer propositions than functions. However, suppose we pose the problem not in terms of generating a distinct proposition for every function, but in terms of generating a distinct *denoting complex which denotes a proposition* for each function. The analogous solution would require allowing that "$\exists\phi$" could be the same *meaning* as "$\exists\psi$" even when $\phi \neq \psi$, which is far less plausible.

In any case, these manuscripts clearly demonstrate the close connection between Russell's work on paradox solving and his interests in the theory of meaning and the structure and identity conditions of propositions. They provide a few missing puzzle pieces necessary to fit together a complete picture of the development of Russell's thought. Indeed, as philosophy has, as far as I know, not yet provided an unproblematic account of propositions as entities that solve such paradoxes, nor an uncontroversial understanding of functions or properties and their mode of occurrence within states of affairs, facts or propositions, these manuscripts may even be instructive for current research.

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1. "FUNCTIONS" (230.030920[1])

Functions.

1. In every complex of which x is a constituent, we can distinguish x and the rest of the complex, which is the function.
2. Given a function, we may wish to indicate what this becomes when the argument x is supplied to it.
3. If from a complex we take away x and so obtain a function, and if we then supply x as argument to this function, we obtain again the original complex.
4. If we supply an argument to a function, and then take it away, we get the function again.

Symbolically.

1. Given $\phi|x$, any complex containing x, we can form the function $\acute{\xi}\phi|\xi$.
2. If u is a function, we indicate by $\underline{x \,\varepsilon\, u}$ what u becomes when x is supplied to it as argument.

3.		$\vdash x \,\varepsilon\, \acute{\xi}\phi|\xi = \phi|x$	~~Df~~ Pp
	or	$\vdash : u = \acute{\xi}\phi|\xi \,.\supset.\, x \,\varepsilon\, u = \phi|x$	Pp
4.		$\vdash . \acute{\xi}(\xi \,\varepsilon\, u) = u$	Pp
	or	$\vdash : \dot{x}(x \,\varepsilon\, u) = \dot{x}\phi|x \,.\supset.\, \dot{x}\phi|x = u$	
5.		$?\ \phi = \acute{\xi}\phi|\xi$	Df? (Not worth it.)

1 ⟨On verso, rotated 180°: ⟩ $\sim\phi\{\phi(p) \,.\supset_p .\, p\}$

2. "PROOF THAT NO FUNCTION TAKES ALL VALUES" (230.030850[1])

Proof that no function takes all values. *folio ⟨1⟩*

$$\{(F|x)|x\}\ \mathrm{Focp}\ x\,.\,\acute{x}\{\sim(F|x)|x\} = f\,.\supset.\sim(F|x)|x = f|x\,.$$
$$\supset.\sim\{(F|x)|x = f|x\}\,.\supset:(F|x)\ \mathrm{Focp}\ x\,.\supset.F|x \sim= f:$$
$$\supset:(x).F|x \sim= f$$

i.e. if $\acute{x}(\sim xFx)$ is admissable, $F|x$ can't take all values.
But this by no means proves that $F|x$ can't take all values. We have

$$\mathrm{Cls}|f\,.\supset_f.(\exists x)\,.\,(F|x = f) : \supset.\sim[\not\acute{x}\{\sim(F|x)|x\}]\ \mathrm{Focp}\ x$$

It would seem, then, that $\acute{x}(x)$, $\acute{x}(\iota x)$, $\acute{\phi}(-\phi)$ etc. may be admissable.
We have to remember that $(\iota\breve{R}|x)|x$ may not do.
Can $F|x$ ever take all values?
Put $F = \acute{z}\acute{y}(p)$. Then $F|x = \{\acute{y}(p)\}\frac{x}{z}$
It seems $\acute{x}(xRx)$ is sometimes admissable, sometimes not.
Put $F = \acute{z}\acute{y}((y \times z))$. Then $F|x = \{\acute{y}((y \times z))\}\,\frac{x}{z} = \acute{y}(y \times x)$
$(F|x)|x = (x \times x)$
Thus all turns on whether $(x \times x)$ is admissable.
The rule is: $\acute{x}(xRx)$ is admissable if $R = \acute{z}\acute{y}((y \times z))$ and $\acute{x}((x \times x))$ is admissable.

Consider $x|x$. Suppose $x = \acute{y}(p)$ and p Focp y. Then *folio ⟨2⟩*

$$x|z = p\frac{z}{y},\ x|x = p\frac{x}{y} = p\frac{\acute{y}(p)}{y}$$

We want a Pp:

$$R = \acute{y}\acute{z}(y \times z)\,.\{(y \times z)\tfrac{x}{y}\}\tfrac{x}{z}\ \mathrm{Focp}\ x\,.\supset.\{(R|x)|x\}\ \mathrm{Focp}\ x\ \ \mathrm{Pp}$$

[1] ⟨On verso, and foliated 7:⟩ 11.6 $\vdash :.\ \mathrm{Cls}|u\,.\supset.(\exists y)\,.\,X\frac{y}{x} = u : \supset.\sim\{(X|x)\ \mathrm{Focp}\ x\}$

We ought to begin with: If no term of the form $X|X'$ comes in $p, \acute{x}(p)$ is all right: thus:

$$(\theta|z)\,\text{const}\,X\,.\supset_{\theta,z}:x\sim\text{const}\,\theta\,.\vee.\,x\sim\text{const}\,z:.\supset.\,X\ \text{Focp}\ x \quad \text{Pp}$$

This leaves out $\imath x|x$, $x|\imath x$, $\text{Ɑ}R|R$, $\sim x|x$, $\phi|(f|\phi)$, and so on. It shows that *all* complexes in which | does not occur are functional. But we need Pp's admitting some cases where $X|X'$ does occur.

First case where $X|X'$ occurs: in proof of $\text{Nc}|\beta\,.\supset.\,\beta < 2^{\beta}$, we have $\imath|\{p \cap -\imath|(\breve{S}|p)\}\ \varepsilon\ \acute{x}\{\exists a \cap \acute{p}(x = \imath p \cap -\imath\breve{S}|p)\} \cap -\breve{S}|p$. This is a complex of the form $X|X'$; it is itself harmless, but must not be used to obtain a function of S or of p.

: : : : : : : : : *folio ⟨3⟩*

Proof of $\text{Sim}|\text{Cls}‘a > \text{Sim}|a$

$$[\text{⍭}\,1 \to 1|R\,.\,\breve{\rho} \subset \text{Cls}‘\rho\,.\,w = \rho \cap \acute{x}\{\sim\imath(\breve{R}|x)|x\}\,.\supset.\,\breve{\rho}|w.]$$

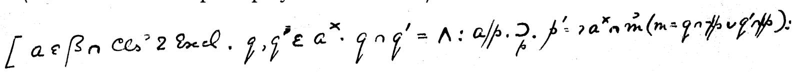
If we *never* allow $X|X'$, the proof fails always: we shan't have even $2^{\alpha_0} > \alpha_0$. Thus $X|X'$ must be sometimes admissable. Try Cantor's form of the proof [Lectures $*39.6$[2]]:

$$[a\ \varepsilon\ \beta \cap \text{Cls}‘2\ \text{Excl}\,.\,q, q'\ \varepsilon\ a^{\times}\,.\,q \cap q' = \Lambda : a|p\,.\supset_p.\,p' =$$
$$\imath a^{\times} \cap \acute{m}(m = q \cap -|p \vee q' \cap |p):^{3}$$
$$\ni\!\!-b = \acute{x}\{\exists a \cap \acute{p}(x = p')\} : \supset : b \subset a^{\times}\,.\,b\ \text{Sim}\ a :$$
$$\supset.\,\text{Sim}\ a^{\times} \geq \text{Sim}\ a \qquad (1)$$

This involves $\acute{p}\acute{p}'\{a|p\,.\,p' = q \cap -p \cup q' \cap p\}$. This relation is unobjectionable: but $\exists\beta \cap \text{Cls}‘2$ Excl remains to be proved.

[2] ⟨These references are presumably to Russell's 1901–02 Cambridge lectures; for discussion, see *Papers* 3: 382.⟩

[3] ⟨Russell's manuscript displays the line thus:⟩



$$a \,\varepsilon\, \beta \cap \text{Cls‘2 Excl} \,.\, S \,\varepsilon\, 1 \to 1 \,.\, \sigma = a \,.\, \breve{\sigma} \subset a^{\times} : a|p \,.\supset .\, p_1 \,.$$
$$= \imath(p \cap \imath\breve{S}|p) . p_2 = \imath(p \cap - \imath\breve{S}|p) :$$
$$q = \acute{x}\{\exists a \cap \acute{p}(x = p_2)\} : \supset : a^{\times}|q : a|p \,.\supset_p . q|p_2 \,.\sim(\breve{S}|p)|p_2 :$$
$$\supset . q \sim= \breve{S}|p \,.\supset .\sim\breve{\sigma}|q \qquad (2)$$

This step again is unobjectionable. Thus this proof is preferable to the other; but $\exists\beta \cap$ Cls‘2 Excl remains to be proved.

If a β consists of simple functions, and no two are each other's negatives, $\acute{u}\{\exists a \cap \acute{\phi}(u = \phi \,.\vee .\, u = -\phi)\}$ will do. More often, $\acute{u}\{\exists a \cap \acute{x}$ $(u = x . \vee . u = \iota x)\}$ will do; for $x \sim= y . \supset . \iota x \sim= \iota y$. But if $x = \iota x$ is ever true for a term of u, this will fail; or (what will be more frequent) if x and ιx are ever both terms of u. We require two mutually exclusive classes of β terms each:

$$1 \to 1|S . \beta|\sigma . \sigma \cap \breve{\sigma} = \Lambda \,.\supset . \acute{u}\{\exists\sigma \cap \acute{x}(u = \iota x \cup \breve{S}|x)\} \,\varepsilon\, \beta \cap \text{Cls‘2 Excl} \,.$$

The $\beta \cap$ Cls‘2 Excl required can be got from Cls‘a.

folio (4)

$$\begin{array}{ll} & \quad a \\ q \to & \\ & \ldots\ldots\ldots\ldots \quad \beta \text{ dots in} \\ & \ldots\ldots\ldots\ldots \quad \text{each row.} \\ q' \to & \end{array}$$

There are here β pairs of dots.

(1). To prove $\text{Sim}|a^{\times} \geq \text{Sim}|a$.
Take any two mutually exclusive terms q, q' of $a^{\times}$: let q be the top row, q' the bottom row. For any term of a which belongs to the bottom row, add all those in the top row: the result is a term of $a^{\times}$: thus

~~$S = \acute{u}\acute{p}\{q'|p \,.\, u = \iota p \cup q \cap - R|p\} \,.\supset . \sigma = q' \,.\, \breve{\sigma} \in$~~

[we start from $1 \to 1|R \,.\, \rho \cap \breve{\rho} = \Lambda \,.\, \beta|\rho$]
taking any term a, take the q-term for this couple, and the q'-terms for all the rest, so that we put

$$S = \acute{u}\acute{p}\{a|p : u = q \;\cap\; p \cup q' \cap -p\} \,.\supset . \sigma = a \,.\, \breve{\sigma} \subset a^{\times}$$

This function seems unobjectionable. But let us examine the proof of $1 \to 1|S$.

$$pSu \,.\, pSu' \,.\!\supset\!.\, u = q \cap p \cup q' \cap -p \,.\, u' = q \cap p \cup q' \cap -p \,.$$
$$\supset\!.\, u = u' \qquad (1)$$
$$pSu \,.\, p'Su \,.\!\supset\!.\, q \cap p \cup q' \cap -p = q \cap p' \cup q' \cap -p' \,.$$
$$\supset\!.\, -q' \cap p \cup q' \cap -p = -q' \cap p' \cup q' \cap -p' \,.$$
$$\supset\!.\, -q' \cap p = -q' \cap p' \,.\, q' \cap -p = q' \cap -p' \,.\!\supset\!.\, p = p' \; (2)$$

Thus there is no doubt of $1 \to 1|S$.

Proof of Sim|Cls‘$a = 2^{\mathrm{Sim}|a}$ *folio ⟨5⟩*

$$[a|x \,.\!\supset_x\!.\, a_x = \mathrm{Cls}‘a \cap \acute{b}(b|x) \,.\, a'_x = \mathrm{Cls}‘a \cap -a_x :$$
$$c = \acute{k}\{\exists a \cap \acute{x}(k = \iota a_x \cup \iota a'_x)\} : \supset\!.\, (\mathrm{Sim}|a \cap \mathrm{Cls}‘2\ \mathrm{Excl})|c \quad (1)$$
$$c^{\times}|x \,.\!\supset\!.\, d_m = \acute{x}(m|a_x) : \supset\!.\, \mathrm{Cls}‘a = \acute{w}\{\exists c^{\times} \cap \acute{m}(w = d_m)\} \,.$$
$$\supset\!.\, \mathrm{Sim}|\mathrm{Cls}‘a = \mathrm{Sim}|c^{\times}]$$

To make this proof right, put

$$S = \acute{y}\acute{x}[a|x \,.\, y = \mathrm{Cls}‘a \cap \acute{b}(b|x)] \,.\, S' = \acute{y}\acute{x}[a|x \,.\, y = \mathrm{Cls}‘a \cap \acute{b}(-b|x)] \,.$$
$$c = \acute{k}[\exists a \cap \acute{x}(k = \breve{S}|x \cup \breve{S}'|x)].$$
$$S'' = \acute{k}\acute{x}[a|x \,.\, k = \breve{S}|x \cup \breve{S}'|x].$$
$$\supset : 1 \to 1|S'' \,.\, \sigma'' = a : \breve{\sigma}''|k \,.\!\equiv\!.\, \exists a \cap \acute{x}(k = \breve{S}|x \cup \breve{S}'|x) \,.\!\supset\!.\, 2|k :$$
~~$xSy \,.\, xS'y \,.\!\supset\!.\, y = \mathrm{Cls}‘a \cap \acute{b}(b|x)$~~
$$\supset\!.\, (\mathrm{Sim}|a \cap \mathrm{Cls}‘2\ \mathrm{Excl})|\breve{\sigma}'' \qquad (1)$$
$$R = \acute{d}\acute{m}[c^{\times}|m \,.\, d = \acute{x}(m|\, a_x)] \,.\!\supset\!.\, \mathrm{Cls}‘a = \breve{\rho} \,.\, c^{\times} = \rho \qquad (2)$$
$$\vdash .\, (1) \,.\, (2) \,.\!\supset \vdash \ \mathrm{Prop}]$$

Thus Sim|Cls‘$a = 2^{\mathrm{Sim}|a}$ seems always true, and also $2^{\mathrm{Sim}|a} >$ Sim|a. It would follow that Sim|Cls‘$a >$ Sim|a.

Necessary to investigate how Cantor’s proof differs from mine.

Now to prove $1 \to 1|S \,.\, \sigma = a \,.\, \breve{\sigma} \subset a^{\times} \,.\!\supset\!.\, \exists a^{\times} \cap -\breve{\sigma}$ *folio ⟨6⟩*

If $1 \to 1|S \,.\, \sigma = a \,.\, \breve{\sigma} \subset a^{\times}$, if $a|p$, one term of p belongs to $\imath\breve{S}|p$ and one doesn’t. Put

$$R_s = \acute{y}\acute{p}_{\ddagger}[a|p \,.\, \iota y = p \cap \imath(\breve{S}|p)] \,.\, R'_s = \acute{y}\acute{p}[a|p \,.\, \iota y = p \cap -\{\imath(\breve{S}|p)\}].$$

$$q = \dot{x}[\exists a \cap \dot{p}\{x = \imath(\breve{R}'_s|p)\}] .$$
$$\supset : a^{\times}|q : a|p . \supset_p . q|\{\imath(\breve{R}'_s|p)\} . \sim(\breve{S}|p)|\{\imath(\breve{R}'_s|p)\} :$$
$$\supset : -\breve{\sigma}|\{\imath(\breve{R}'_s|p)\}$$

p_1
. . . × . . . .
. . . × . . . .
p_2

i.e. we establish a correlation of p's and $a^{\times}$'s; if p is an a, we call p_2 the term of p which does not belong to $\imath\breve{S}|p$; the class of all p_2's is not a $\breve{\sigma}$.

When we prove Sim|Cls‘a > Sim|a, we have instead the notion of p itself not belonging to its correlate, and this gives an impermissible function. But in the above case, the function is all right.

But Nc|β. ⊃. $\exists\beta \cap$ Cls‘2 Excl is derived from Cls‘a. Let $\beta|a$: put $xSu . = . a|x . u =$ Cls‘$a \cap \dot{b}(b|x)$; *i.e.* u consists of all classes of a to which x belongs. Put $xS'u . = . a|x . u =$ Cls‘$a \cap \dot{b}(-b|x)$, *i.e.* u consists of all classes of a to which x does not belong. Then u varies as x varies, both for S and S'; *i.e.* $1 \to 1|S . 1 \to 1|S'$. Thus $\breve{S}|x \cup \breve{S}'|x$ always contains two terms; and x 0‘ $y . \supset . \breve{S}|x \sim= \breve{S}|y . \breve{S}'|x \sim= \breve{S}'|y$.

But do we necessarily have *folio ⟨7⟩*

$$x\ 0‘y . \supset . \breve{S}|x \sim= \breve{S}'|y\ ?$$

i.e. is it possible to have

$$x\ 0‘y . \text{Cls‘}a \cap \dot{b}(b|x) = \text{Cls‘}a \cap \dot{c}(-c|y)$$

If so, Cls‘$a|b . b|x . \equiv_b .$ Cls‘$a|b . \sim b|y$

Consider $b|x . \equiv_b . \sim b|y$.

This is to hold only if $b \subset a$; and it is only relevant when $a|x . a|y$. We want to know if two members x and y of a can be found such that every proposition implying a which is satisfied by x is not satisfied by y, and vice versa. This ⟨is⟩ impossible, because Cls‘$a|a . a|x . a|y$ excludes it.

Hence the members of $\breve{\sigma}''$ are mutually exclusive, where

$$x\ S''\ u . \equiv . a|x . u = \breve{S}|x \cup \breve{S}'|x .$$

Then $\breve{\sigma}'' \in \mathrm{Sim}|a \cap \mathrm{Cls}‘2\ \mathrm{Excl}.$

Perhaps proof of $\mathrm{Sim}|\mathrm{Cls}‘a = 2^{\mathrm{Sim}|a}$ may go wrong.

This depends on proving

$$R = \dot{d}\acute{m}[c^{\times}|m\,.\,d = \acute{x}\{m|(\iota\breve{S}|x)\}]\,.\supset.\,\breve{\rho} = \mathrm{Cls}‘a.$$

where $c = \acute{k}[\exists a \cap \acute{x}(k = \breve{S}|x \cup \breve{S}'|x)]$

Here $\breve{\rho}|d\,.\equiv.\,\exists\acute{m}[c^{\times}|m\,.\,d = \acute{x}\{m|(\iota\breve{S}|x)\}]$
$\equiv.\,\exists\acute{m}[c^{\times}|m\,.\,d = \acute{x}\{m|(\mathrm{Cls}‘a \cap \acute{b}(b|x))\}]$

$c^{\times}$ is formed by taking, in all possible ways, for every x, either those Cls‘a’s which contain it, or those which don’t; *i.e.* *folio ⟨8⟩*
$c^{\times}|m\,.\equiv:m|u\,.\equiv.\,\exists a \cap \acute{x}(u = \mathrm{Cls}‘a \cap \acute{b}(b|x)\,.\vee.\,u = \mathrm{Cls}‘a \cap \acute{b}(-b|x))$

Hence $\breve{\rho}|d\,.\equiv.\,\exists\acute{m}[m|u\,.\equiv.\,\exists a \cap \acute{x}\{u = \mathrm{Cls}‘a \cap \acute{b}(b|x)\,.\vee.$
$u = \mathrm{Cls}‘a \cap \acute{b}(-b|x)\} : d = \acute{x}\{m|\mathrm{Cls}‘a \cap \acute{b}(b|x)\}]$

Here $d|x.\equiv.\,m\{\mathrm{Cls}‘a \cap \acute{b}(b|x)\}.$
$\equiv.\,\exists a \cap \acute{y}\{\mathrm{Cls}‘a \cap \acute{b}(b|x) = \mathrm{Cls}‘a \cap \acute{b}(b|y).\vee.\,\mathrm{etc.}\}$
$\equiv.\,a|x$

~~Put~~ One term of c is $\breve{S}|x \cup \breve{S}'|x$; if $c^{\times}|m$, either ~~one ter~~ $\iota\breve{S}|x$ or $\iota\breve{S}'|x$ belongs to m, but not both. *I.e.*

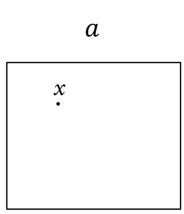
$$c^{\times} = \acute{m}[a|x\,.\supset_x : m|(\iota\breve{S}|x)\,.\vee.\,m|(\iota\breve{S}'|x) : \sim\{m|(\iota\breve{S}x)\,.\,m|(\iota\breve{S}'|x)\}]$$

i.e. $c^{\times} = \acute{m}[a|x\,.\supset_x : m|(\iota\breve{S}|x)\,.\equiv.\sim m|(\iota\breve{S}'|x)]$

Hence

$$\breve{\rho}|d\,.\equiv.\,\exists\acute{m}[a|x\,.\supset_x : m|(\iota\breve{S}|x)\,.\equiv.\sim m|(\iota\breve{S}'|x) :.\, d = \acute{x}\{m|(\iota\breve{S}|x)\}]$$

Take a square a: *folio ⟨9⟩*



$\mathrm{Cls}‘a \cap \acute{b}(b|x)$ = Areas containing x.
$\mathrm{Cls}‘a \cap \acute{b}(-b|x)$ = … not ……
Put $xSu\,.\equiv.\,a|x\,.\,u = \mathrm{Cls}‘a \cap \acute{b}(b|x)$
so that $\iota\breve{S}|x = \mathrm{Cls}‘a \cap \acute{b}(b|x)$
$xS'u\,.\equiv.\,a|x\,.\,u = \mathrm{Cls}‘a \cap \acute{b}(-b|x)$

whence $\iota\breve{S}'|x = \text{Cls}‘a \cap \acute{b}(-b|x)$.

Thus for every value of x, $\breve{S}|x \cup \breve{S}'|x$ is a class of two terms; and its relation to x is $1 \rightarrow 1$. This relation is

$$S'' = \acute{y}\acute{x}(a|x \,.\, y = \breve{S}|x \cup \breve{S}'|x)$$

Thus $\breve{\sigma}''$ contains Sim$|a$ terms, each a couple, and each couple excluding the others. Hence $\breve{\sigma}'' \, \varepsilon \, \text{Sim}|a \cap \text{Cls}‘2 \text{ Excl}$. What this proves is

$$\vdash : \text{Nc}|\beta \,.\supset.\, \exists\beta \cap \text{Cls}‘2 \text{ Excl}$$

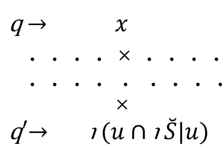
We now have to prove $\text{Sim}|\text{Cls}‘a = 2^{\text{Sim}|a}$. We have $2^{\text{Sim}|a}|\breve{\sigma}''^{\times}$
Hence we have to prove $\text{Cls}‘a|\text{Sim}|\breve{\sigma}''^{\times}$.

Take any term of $\breve{\sigma}''^{\times}$; if $\iota\breve{S}|x$ is a term of this, let $d|x$ hold; if not, not. *I.e.* $\breve{\sigma}''^{\times}|m \,.\, a|x \,.\supset: m|(\iota\breve{S}|x) \,.\supset.\, d|x : \sim m|(\iota\breve{S}|x) \,.\supset.\, \sim d|x$; in this way, every Cls‘a will be included sooner or later. Thus no objectionable function occurs here. Hence always

$$\vdash \; \text{Sim}|\text{Cls}‘a = 2^{\text{Sim}|a}$$

Thus it is $2^{\beta} > \beta$ that goes wrong.

To prove $2^{\beta} > \beta$. folio ⟨10⟩
Take two rows of β dots.



Call the class of couples c. Call the top row q, the bottom row q'. Take any $1 \rightarrow 1 \cap \acute{S}(\sigma = c \,.\, \breve{\sigma} \subset c^{\times})$.

If $c|u$, it will happen either that $u \cap \iota\breve{S}|u \subset q$, or that $u \cap \iota\breve{S}|u \subset q'$. Whichever happens, choose the other: *i.e.*

$$w|x \,.\equiv: \cup‘c|x : \cancel{u \cap \iota\breve{S}|u \subset q \,.\supset.\, u \cap x \subset q' : u \cap \iota\breve{S}|u \subset q' \,.\supset.}$$
$$c|u \,.\, u|x \,.\, u \cap \iota\breve{S}|u \subset q \,.\supset.\, q'|x :$$
$$c|u \,.\, u|x \,.\, u \cap \iota\breve{S}|u \subset q' \,.\supset.\, q|x :$$
$$\equiv:. \cup‘c|x :.\, c|u \,.\, u|x \,.\supset: u \cap \iota\breve{S}|u \subset q \,.\equiv.\, \sim q|x$$

We have to prove $\sim\breve{\sigma}|w$. This follows from $w \sim= \iota\breve{S}|u$. Now
$w = \iota\breve{S}v \,.\supset: w|x \,.\equiv.\, \iota\breve{S}v|x\,.$
$\supset: v \cap \iota\breve{S}v \subset q \,.\not\supset \equiv.\, v \cap w \subset q$

If w were the correlate of u, its term in u would be $\iota(u \cap \iota\breve{S}|u)$; but it isn't.

$$w = \iota\breve{S}|u \,.\supset.\, w \cap u = \iota\breve{S}|u \cap u$$

Define: $\iota\breve{S}|u \cap u \subset q \,.\supset.\, \sim q|x \,.\supset.\, \sim\{(\iota\breve{S}|u \cap u)|x\}$
What we want is $\iota(u \cap -\iota\breve{S}|u)$
i.e. $w = \dot{x}[\exists c \cap \dot{u}\{x = \iota(u \cap -\iota\breve{S}|u)\}]$

To prove $2^\beta > \beta$. *folio ⟨11⟩*

Take two rows of β dots. Call the class of couples c. Take a $1 \to 1$ $\cap\, \dot{S}(\sigma = c \,.\, \breve{\sigma} \subset c^{\times})$. Then choose w so that it picks out from each couple of c the term not belonging to the correlate of the couple, *i.e.* $w = \dot{x}[\exists c \cap \dot{u}\{x = \iota(u \cap -\iota\breve{S}|u)\}]$. Then
$c|v \,.\, w = \iota\breve{S}|v \,.\supset: w|x \,.\equiv_x.\, (\iota\breve{S}|v)|x$. But
$w|x \,.\equiv.\, \exists c \cap \dot{u}\{x = \iota(u \cap -\iota\breve{S}|u)\} : \supset :$ ~~$w|x \,.\, u|x \,.\supset.\, \sim(\iota\breve{S}|u)|x$~~
$\supset : u|x \,.\supset_u.\, w|x \equiv \sim(\iota\breve{S}|u)|x : \supset.\, \sim\exists\dot{u}(w = \iota\breve{S}|u)\,.$

Thus unless $u \cap -\iota\breve{S}|u$ is objectionable, there is no harm in this.

$$u \cap -\iota\breve{S}|u = \dot{x}\{u|x \,.\, \sim(\iota\breve{S}|u)|x\}$$

and $\breve{S}|u = \dot{v}(uSv)$. Thus $u \cap -\iota\breve{S}|u = \dot{x}[u|x \,.\, \sim\{\iota\dot{v}(uSv)\}|x]$
We get an objectionable complex if we start from x, thence define u as $\iota c \cap \dot{u}(u|x)$, and thence put $w = \cup\text{‘}c \cap \dot{x}[x \sim\varepsilon\, \iota\breve{S}|\{\iota c \cap \dot{u}(u|x)\}]\,.$

Frege's Proposition. $f^2 = \dot{x}\{\exists\dot{\phi}(x = f|\phi \,.\, \sim\phi|x)\}$ Df

$$f^2|x \equiv.\, \exists\dot{\phi}(x = f|\phi \,.\, \sim\phi|x)$$
$$\sim f^2|x \,.\equiv: x = f|\phi \,.\supset_\phi.\, \phi|x$$

$f^2|(f|f^2) \equiv.\, \exists\dot{\phi}\{f|f^2 = f|\phi \,.\, \sim\phi(f|f^2)\}$
$\sim f^2(f|f^2).\equiv: f|f^2 = f|\phi \,.\supset_\phi.\, \phi(f|f^2) : \supset.\, f^2(f|f^2)\,.$

Hence $\vdash f^2(f|f^2)$, $\vdash \exists\,\dot{\phi}\{f|f^2 = f|\phi \,.\, f^2 \sim= \phi\}\,.$

Hence no function is such that $f|x = f|y \,.\supset.\, x = y$. Now $1 \to 1|R \,.\supset$
$:.\, \rho|x \,.\supset:\, \breve{R}|x = \breve{R}|y \,.\supset.\, x = y$.
Hence $1 \to 1|R \,.\supset.\, \exists - \rho \,.\, \exists - \breve{\rho}$. If now we apply this to 1‘, putting $f = 1‘$, we have $1‘^2|x \,.\equiv.\, \exists\acute{\phi}\{x = 1‘|\phi . \sim\phi|x\}$, $1‘^2(1‘(1‘|1‘^2) \,.\equiv.\, \exists\, \acute{\phi}$ $[1‘|1‘^2 = 1‘|\phi \,.\sim\phi|(1‘|1‘^2)]$. Hence $1‘|x = 1‘|y \,.\sim \supset.\, x = y$. We had $\text{℩}(\iota x) = x$. Here $\iota x = \iota y \,.\, \text{℩}(\iota x) = x \,.\, \text{℩}(\iota y) = y$. If ℩ is included among functions, $\iota x = \iota y \,.\supset.\, \text{℩}(\iota x) = \text{℩}(\iota y)$. It may be that $1‘|x = \Lambda$ sometimes. We had $x\ 1‘\ x$, *i.e.* $(1‘|x)|x$, *i.e.* $x = x$.

If Frege's Proposition is to be false, we must deny functions of the form $\dot{x}\{X = Y \,.\, Y'|X\}$ *i.e.* of the form where y const $p \,.\, y$ const $p' \,.\, (x = y)$ const $q \,.\, (y|x)$ const q. In these cases, $\dot{x}(q)$ and $\dot{y}(q)$ are to be inadmissable.

Put $(\theta|z)\text{const}\, X \,.\supset_{\theta, z} : x \sim\!\text{const}\, \theta \,.\vee.\, x \sim\!\text{const}\, z :.$ *folio ⟨12⟩*
$(\theta|z)\text{const}\, X \,.\, (z = y)\text{const}\, X \,.\, x \text{ const}\, \theta \,.\supset_{\theta, z, y}.\, x \sim\!\text{const}\, y :.$
$(\theta|z)\text{const}\, X \,.\, (\phi = \theta)\text{const}\, X \,.\, x \text{ const}\, z \,.\supset_{\theta, \phi, z}.\, x \sim\!\text{const}\, \phi :.$
$\supset.\, X$ Focp x Pp

This destroys Frege's theorem; ~~also~~ $2^{\beta} > \beta$ ~~remains~~ also, but $\exists\beta \cap$ Cls‘2 Excl remains, and $\text{Sim}|\text{Cls}‘\alpha = 2^{\text{Sim}|\alpha}$.

It is obvious that $\dot{x}(\sim X|x)$ won't do if X takes all simple values; for then all ₣ classes are denied. Again $\dot{x}(\exists y).(x = Y'. \sim Y|x)$ won't do if Y can take all values which are classes. Thus $\dot{x}[(\exists\phi).(x = f|\phi \,.\sim\phi|x)]$ will deny $\phi|x$ for some value of x, unless $f|\phi = f|\phi'$. Hence if $f|\phi \sim= f|\phi'$, $\dot{x}[(\exists\phi).(x = f|\phi \,.\sim\phi|x)]$ must not be admitted. This would be $= \dot{x}[(\exists\phi).\{x = f|\phi \,.\sim \phi|(f|\phi)\}]$; and the latter is already excluded. But such complexes must be sometimes admissable. But when?

Consider $\alpha < \beta \,.=:\, \alpha \sim= \beta : \text{Nc}|\alpha \,.\, \text{Nc}|\beta : \alpha|u \,.\, \beta|v \,.\, 1 \to 1|R \,.\, \rho = u \,.\, \breve{\rho} \subset v \,.\supset_R.\, \exists v - \breve{\rho}$

It is obvious that the $v - \breve{\rho}$ will generally (always, unless $\alpha = 0$) have to be defined in terms of R. For $2^{\alpha_0} > \alpha_0$, we must find some other proof: I think Cantor has one.

Until $2^{\alpha} > \alpha$ we never require $\dot{x}(X|X')$ or $\dot{x}[(\exists y).(X = Y' \,.\sim Y|X)]$. I do not think these forms occur again anywhere, except in considering the greatest ordinal, and in the Df of P_c. [Lectures, 122·5]

We want first of all to admit all functions not of the forms folio ⟨13⟩

$$\dot{x}(X|X') \quad \dot{x}[X = Y \,.\supset_y. Y'|X] \quad \dot{x}[X = Y \,.\supset_y. X|Y']$$
$$\sout{\dot{y}[X = Y \,.\supset_y}.$$

or containing constituents of the forms of the above complexes. With the admission of the functions thus allowed, we can do everything except $2^\alpha > \alpha$, no greatest ordinal, and P_c. Functions of the form $\dot{x}(xRx)$ are obviously sometimes legitimate. But if we allow ε, they are not always legitimate; and $\mathrm{a}R|R$, which becomes $(\mathrm{a}|R)|R$, is not legitimate. Hence $\dot{x}(xRx)$ won't always do. $\breve{\mathrm{a}}R|R \,.\equiv.\, (\exists x).\{(R|R)|x\}$; $x \mathbin{\varepsilon} x \,.\equiv.\, x|x$. Thus $\breve{\mathrm{a}}R|R$ involves $R|R$.

We put $\mathrm{a} = \acute{R}\dot{x}[(\exists y).(R|y)|x]$
$\mathrm{a}|R = \dot{x}[(\exists y).(R|y)|x]$
$(\mathrm{a}|R)|x = [(\exists y).(R|y)|x] \quad (\mathrm{a}|R)|R = [(\exists y).(R|y)|R]$

It would seem $X|X'$ is legitimate when it is of the form $(R|X'')|X'$; but how ⟨to⟩ exclude $(\mathrm{a}|R)|R$? *i.e.* $[\{\acute{R}\dot{x}[(\exists y).(R|y)|x]\}|R]|R$? We must make it a rule if $\{\dot{x}(p)\}|x$ occurs in a complex, p is to be substituted for it. When this is done, $\dot{x}\{(R|X)|X'\}$ may be admitted.

We have $x \sim\text{const}\, \phi \,.\supset.\, (\phi|x)\frac{y}{x} = \phi|y$.

$$(\phi|x) \text{ Focp } x \,.\supset.\, \{\dot{x}(\phi|x)\}|y = (\phi|x)\frac{y}{x} =?\ (\phi\tfrac{y}{x})|y$$

Hence the proof of $\dot{x}(\phi|x) = \dot{y}(\phi|y)$ fails.

Suppose we put *always* folio ⟨14⟩

$$\{\dot{x}(X)\}|y = X\frac{y}{x} \quad \text{Pp}$$

x var X would be better.
↑
x const $X \,.=.\, \sim(y).X\frac{y}{x} = X$ Df
$(\phi|x)\frac{y}{x} = (\phi\frac{y}{x})|y$ Pp

and suppose for X we put $\phi|x$. Then

$$x \sim\text{const}\, \phi \,.\supset.\{\dot{x}(\phi|x)\}|y =_y \phi|y \,.\supset.\, \dot{x}(\phi|x) = \phi \,.$$
$$\supset.\, \dot{x}(\phi|x) = \dot{y}(\phi|y).$$

But x const $\phi \,.\supset.\, \{\dot{x}(\phi|x)\}|y = (\phi\frac{y}{x})|y$ whence not $\dot{x}(\phi|x) = \dot{y}(\phi|y)$. Take e.g. $\dot{x}(x|x)$. We have

$$\{\dot{x}(x|x)\}|y = y|y = \{\dot{y}(y|y)\}|y = \{\dot{z}(z|z)\}|y$$

Thus $\dot{x}(\phi|x) = \dot{y}(\phi|y)$ is still true: for

$$\{\dot{x}(\phi|x)\}|z = (\phi \tfrac{z}{x})|z = \{\dot{y}(\phi|y)\}|z$$

With regard to $\dot{x}\{(R|X)|X'\}$, we may assume

$$\vdash :. \sim(\exists p) . [\{\dot{x}(p)\}|x] \text{ const } q : (x) . q = (R|X)|X' : \\ x \sim\text{const } R . x \text{ const } X : \supset . q \text{ Focp } x \qquad \text{Pp}$$

This will admit P_c and suicide $[\dot{x}(x \text{ kills } x)]$. But it leave⟨s⟩ $2^\alpha > \alpha$ doubtful. It does not admit $(\imath(\breve{R}|x))|x$, though it does admit $(\breve{R}|x)|x$. But if $\imath(\breve{R}|x)$ can be brought into the form $\breve{S}|x$, then it becomes admissable.

Thus in correlating u and Cls‘u, take a $1 \to \text{Nc} \cap \dot{R}(\rho = u . \breve{\rho} \subset u)$. ~~Assume further~~ Then $x \sim= y . \supset . \breve{R}|x \sim= \breve{R}|y$. Hence correlation of x and $\breve{R}|x$ is $1 \to 1$. Thus $S = \dot{u}\dot{x}(\rho|x . u = \breve{R}|x)$ gives the required correlation. Now consider $w = \dot{x}(x \sim Rx)$. This $\sim \varepsilon\, \breve{\sigma}$. For $(\breve{R}|x)|x . \equiv . -w|x : \supset . \breve{R}|x \sim= w$. Here there seem to be no illegitimate functions.

We have $xSu . xSu' . \supset . u = \breve{R}|x . u' = \breve{R}|x . \supset . u = u'$

$$xSu . x'Su . \supset . \exists\breve{R}|x . \exists\breve{R}|x' . u = \breve{R}|x . u = \breve{R}|x' . \supset . \breve{R}|x = \breve{R}|x' . \exists\breve{R}|x$$
$$\supset : xRy . \equiv_y . x'Ry : \exists\breve{R}|x :$$
$$\supset : xRy . \equiv_y . xRy . x'Ry : \exists\breve{R}|x :$$
$$\supset : xRy . \supset_y . x = x' : \exists\breve{R}|x : \supset : \exists\breve{R}|x . \supset . x = x' : \exists\breve{R}|x :$$
$$\supset . x = x'$$

We have to prove $\exists 1 \to \text{Nc} \cap \dot{R}(\rho = u . \breve{\rho} \subset u)$ *folio ⟨15⟩*

1‘ $\dot{\cap}$ $(u \uparrow u)$ will do what we want. In this case, Λ is omitted. If $u =$ Cls, $\breve{R}|x = \iota x$ when Cls$|u$. Hence $\imath(\breve{R}|x) = x$. This gives a $1 \to 1$ of all classes to all classes; but we want one of all classes to all classes of classes. We have $\text{Cls} = \dot{u}\{(x) . \text{Indiv}|(u|x)\}$

$$\text{Cls}' = \text{Fo} \cap \dot{u}(u \subset \text{Indiv}) \quad \text{Cls}^2 = \text{Cls} \cap \dot{u}(u \subset \text{Cls})$$[4]

The contention is: a $1 \to \text{Nc}$ relating classes to classes will never have

4 ⟨In the definition of Cls^2, Russell struck out primes after the two occurrences of “Cls”.⟩

all Cls[2]'s ~~in its~~ of the form $\breve{R}|x$. What makes this case odd is that $\breve{R}|x$ is itself a Cls, *i.e.* $(x).\rho|(\breve{R}|x)$

Correlating all entities with classes, the contention is

$$1 \rightarrow \mathrm{Nc}|R : (x).\rho|x : \supset .\exists \mathrm{Cls} \cap \dot{w}[\sim(\exists x).(w = \breve{R}|x)]$$

⟨I⟩n other words, $\breve{R}|x$ will never take all values that are classes. If Ɑ were admitted, $\breve{\text{Ɑ}}|R$ would be such a function, roughly; here the omitted function is $\dot{R}\{\sim(\breve{\text{Ɑ}}|R)|R\}$. But $\breve{\text{Ɑ}}$ is only $1 \rightarrow \mathrm{Nc}$ for relations.

N.B. If we admit the relation Ɑ, $\breve{\text{Ɑ}}$ must not have its present meaning. We have $\text{Ɑ}|R = \dot{x}[(\exists y).(R|y)|x]$ *i.e.* $\text{Ɑ} = \dot{R}\dot{x}[(\exists y).(R|y)|x]$. Thus $\breve{\text{Ɑ}} = \dot{x}\dot{R}[(\exists y).(R|y)|x]$. $\breve{\text{Ɑ}}|x = \dot{R}[(\exists y).(R|y)|x]$ *i.e.* $\breve{\text{Ɑ}}|x$ is the class of relations holding between x and any other terms.

Put $xRy \,.=.$ ~~$y = \text{℩}x$~~ $x = \text{℩}y$. This is $1 \rightarrow \mathrm{Nc}$.
Then $\breve{R}|x = \dot{y}(x = \text{℩}y)$. If x is a 1, this $= \iota(\iota x)$; if not, it $= \iota(\iota x) \cup \iota x$.

It would seem $\dot{x}(xRx)$ is not in general admissable, though it may be so if R is constant and does not contain x. The proof of $\breve{R}|x \sim= w$ may fail in odd cases, e.g. when $\breve{R} = x$.

<u>$\dot{x}(xRx)$</u> *folio ⟨16⟩*

$$\text{Put} \quad R = \dot{y}\dot{z}(y \times z). \quad \text{Then } xRx = (x \times x)$$

If now $\dot{x}(x \times x)$ is admissable, so is $\dot{x}(xRx)$.

In proving Sim|Cls‘u > Sim|u, we put

$$1 \rightarrow \mathrm{Nc}|R \,.\, \breve{p} \subset \rho \,.\, S = \dot{u}\dot{x}(\rho|x \,.\, u = \breve{R}|x) \,.\, w = \dot{x}(x \sim Rx) \,.\supset .\sim\breve{\sigma}|w \,.$$

In order that this proof may be sound, we require that, if $R = \dot{y}\dot{z}(y \times z)$, $\dot{x}(x \times x)$ should be admissable. If this condition fails, we may not have $w|x = (x \sim Rx)$, and hence the proof fails. Thus what is demonstrated is

$$1 \rightarrow \mathrm{Nc}|R \,.\, \breve{p} \subset \rho \,.\, S = \dot{u}\dot{x}(\rho|x \,.\, u = \breve{R}|x) \,.\, w = \dot{x}(x \sim R\, x) : \\ (\exists p).\{R = \dot{y}\dot{z}(p).(p\tfrac{x}{z})\tfrac{x}{y}\ \mathrm{Focp}\ x\} : \supset .\sim\breve{\sigma}|w \,.$$

Thus Sim|Cls‘u > Sim|u only follows in cases where

$$\rho = u \,.\, \breve{\rho} \subset u \,.\, 1 \to \mathrm{Nc}|R \,.\supset.\, (\exists p).\{R = \acute{y}\acute{z}(p).\, (p\tfrac{x}{z})\tfrac{x}{y} \text{ Focp } x\}$$

We have now to consider $\{\imath(\breve{R}|x)\}|x$ where ~~$1 \to \mathrm{Nc}|1$~~ $\mathrm{Nc} \to 1|R$. Suppose $R = \acute{z}\acute{y}(y \times z)$. Then $\breve{R} = \acute{y}\acute{z}(y \times z)$, $\breve{R}|x = \acute{z}(x \times z)$, $\{\imath(\breve{R}|x)\}|x = \{\imath\acute{z}(x \times z)\}|x$.

Suppose $\acute{z}(x \times z) = \iota\acute{z}(x + z)$; then $\{\imath(\breve{R}|x)\}|x = (x + x)$.

This must be the test: whether $\acute{x}(x + x)$ will do.

Consider <u>$\phi(f|\phi)$</u>. Is this ever functional ⟨with⟩ respect to ϕ? *folio ⟨17⟩*

$$\phi|(f|\phi) \,.\equiv.\, (\exists x).\{x = f|\phi \,.\, \phi|x\}$$

(α). If f is not a function, $\acute{\phi}\{\phi|(f|\phi)\} = \acute{\phi}\{\phi|$~~A~~$((p).p)\}$ which is functional.

(β). If f is a function, put $f = \acute{\psi}\{\psi \times \psi)\}$. Then if $(\psi \times \psi)$ is functional, $f|\phi = (\phi|\phi)$. Hence $\phi|(f|\phi) = \phi|(\phi \times \phi)$

There seems no way of proving that this is ever functional.

Consider e.g. $\rho|\breve{\rho}$. This is probably not functional.

Consider $(f|x)|(f'|x)$

Put ~~$f|x = f = \acute{x}(X)$,~~ $f' = \acute{x}(X')$ Then $(f|x)|(f'|x) = (f|x)|X'$.

If $f = \acute{x}\acute{y}(y \times x)$, $(f|x)|X' = (x \times X')$. This may or may not be functional.

To prove by Cantor's method that Sim|Cls‘u > Sim|u, we need to prove

$$u|y \,.\!\supset\!\equiv_y.\, (\exists z).\,[u|z \,.\, \{(y \times z)\}] : \supset. \{(x \times x)\} \text{ Focp } x$$

i.e. if, when $(y \times z)$ is satisfied by some z when and only when $u|y$, and is then satisfied by a z which is a u, then $(x \times x)$ is functional.

When these conditions are satisfied, $(x \times x)$ is only satisifed when $u|x$. Hence if all complexes contained in $u|x$ are functional, the above condition is fulfilled. Now all non-functional complexes have x in the functional place, and none of them are null. Hence all are satisfied by functional values of x. Hence if $u \subset$ Indiv, the proposition holds; otherwise, it may not. } Wrong

The condition $u \subset$ Indiv is too narrow for our purposes, since existence-theorems in Arithmetic are proved by classes of numbers. *folio ⟨18⟩*

If we assume that $x|x$, $\iota x|x$, $(x|y)|x$, and so on, are the only objectionable complexes, then any argument satisfying them must be of no definite order. For a function of a definite order is only satisfied by an argument of lower order. Thus $(x|y)|x$ requires that x should be a relation which can have itself as one of its terms. This will still not enable us to be content with numbers, for Sim$|\phi$ is of no definite order. But given any class u, ~~$\dot{x}(\exists u \cap \{y(x = y)\}$~~ $\dot{p}\{\exists u \cap \dot{y}[p = (y = y)]\}$ is a function only satisfied by propositions, *i.e.* by individuals, having the same number of terms as u provided $\dot{p}\dot{y}[u|y \,.\, p = (y = y)]$ is admissable, and provided $u|y \,.\, u|y' \,.\, y \; 0‘ \, y' \,.\, \supset_{y,y'} . \, (y = y) \sim= (y' = y')$. But this last can hardly be always true. If it is, as seems to be the case, the number of propositions is as great as the number of objects altogether. Now that Cantor's proposition is disproved, there is no harm in this.

The above argument, that if $u \subset$ Indiv Cantor holds, is fallacious. x may occur in a functional place, and yet the complex hold when Indiv$|x$. E.g. $\sim(x|x)$ holds always when Indiv$|x$. We want the proposition to fail for <u>Indiv</u> itself. Thus Indiv $\cap \, \dot{x}(\sim x|x)$ is not functional, being in fact Λ. But it is enough if the complex can be made functional, and Indiv$|x \,.\, \sim x|x \,. \equiv .$ Indiv$|x$ which is functional.

Perhaps a non-functional complex, if it holds for individuals at all, holds for all of them.

If ~~ϕ~~ $X \sim$Focp x, $X . X' \sim$Focp x; unless $X . X' . \equiv . X''$ and X'' Focp x. It would be convenient to take as our Pp *folio ⟨19⟩*

$$\vdash :. \, (x) . \, \text{Indiv}|X : (x) . \, \text{Indiv}|X' . \, (x) . \, X \equiv X' : \supset : \sout{(x)} X = X' \qquad \text{Pp}$$

Then we do not have two different Pp's about $\phi|x$. We have

$$\vdash :. \, \text{Fo}|\phi \,.\, \text{Fo}|\psi \,.\supset : \; (x) . \, \phi|x = \psi|x \,.\supset . \, \phi = \psi \qquad \text{Pp}$$

Thence $\vdash : \; (x) . \, \{\dot{x}(X)\}|x = \{\dot{x}(X')\}|x \,.\supset . \, \dot{x}(X) = \dot{x}(X')$

Thence $\vdash :. \, X \text{ Focp } x \,.\, X' \text{ Focp } x \,.\supset : \; (x) . \, X = X' \,.\supset . \, \dot{x}(X) = \dot{x}(X')$

and $\vdash :. \, (x) . \, \text{Indiv}|X : (x) . \, \text{Indiv}|X' : (x) . \, X \equiv X' :$
$X \text{ Focp } x \,.\, X' \text{ Focp } x : \supset . \, \dot{x}(X) = \dot{x}(X')$.

Apparently we don't have generally $X =_x X' . \supset . \dot{x}(X) = \dot{x}(X')$. The reason is that $\dot{x}(X)$ has to do with the *meaning* of X, whereas $X = X'$ has to do with the denotation. But this is inconvenient, and should be remedied by a Pp, if possible.

We may put

$\vdash :. X \text{ Focp } x . X' \sim\text{Focp } x : (x). X = X' : \supset . \dot{x}(X') = \dot{x}(X)$ Pp

~~$\vdash :. X \text{ Focp } x . \supset$~~

$\vdash :: (\phi) :. \sim(x) : \phi|x = X :. \supset . \dot{x}(X) = \Lambda$ Pp

Thus always $(x). X = X' . \supset . \dot{x}(X) = \dot{x}(X')$ which should be Pp

We might take $(\imath x). X$ as indefinable, meaning

(1) if $(\exists x). X : X\frac{y}{x} . X\frac{z}{x} . \supset . y = z$, the x satisfying X

(2) if not, ~~$\sim(\exists x) X$~~, $(p). p$. But $\dot{x}(p) = (p). p$ won't do.

Consider $(\exists \phi). \{x = f|\phi . \sim\phi|x\}$ *folio (20)*

This is non-functional, and may hold only for individuals.

Put $X \text{ Focp } x . = . (\exists \phi). \{x \sim\text{var } \phi : (x). \phi|x = X\}$ Df

Then $\vdash :. (x). X = X' . \supset : X \text{ Focp } x . \equiv . X' \text{ Focp } x$

Also $x \sim\text{var } \phi . \phi|x =_x X . \supset . \phi|y =_y X\frac{y}{z}$

We might put $\vdash : x \sim\text{var } \phi . \text{Fo}|\phi . \phi|x =_x X . \supset . \dot{x}(X) = \phi$ Pp

This Pp seems more powerful than the others. It gives

$\vdash : x \sim\text{var } \phi . \text{Fo}|\phi . \supset . \dot{x}(\phi|x) = \phi$ provided $x = x$ can be proved.

Put also $\vdash :. \text{Fo}|\phi . \supset_\phi . \sim(x). \phi|x = X : \supset . \dot{x}(X) = \dot{x}\{(\phi). \phi|x\}$ Pp

Then ~~put $X \text{ Focp } x . = . (\exists \phi). \{\text{Fo}|\phi . x \sim\text{var } \phi . \phi|x =_x X\}$ Df~~

~~and~~ bring in the old assumptions as to what complexes are functional.

To prove a complex non-functional, we shall have to prove that it is not equivalent to $\phi|x$ for any ϕ. This can sometimes be done. It is when this occurs that Cantor fails. We wish to prove

(1) that it occurs for some Cls‘Indiv.

(2) that it does not occur for certain simple classes.

Consider $(\exists x).\{p = \sim(x|x)\}$ *folio ⟨21⟩*

Put $f = \acute{p}[(\exists x).\{p = \sim(x|x)\}]$ Df
Then $f|p = (\exists x).(p = \sim x|x)$

Consider $(\exists\phi).\{x = \exists\phi\,.\sim\phi|x\}$

Put
$$f = \acute{x}[(\exists\phi).\{x = \exists\phi\,.\sim\phi|x\}]$$
$$f|x = (\exists\phi).\{x = \exists\phi\,.\sim\phi|x\}$$
$$f|(f|x) = (\exists\phi).\{f|x = \exists\phi\,.\sim\phi|(f|x)\}$$

Now $f|x = \exists\acute{\phi}\{x = \exists\phi\,.\sim\phi|x\}$
Hence, if $\phi \sim= \psi\,.\supset.\,\exists\phi \sim= \exists\psi$, we have
$$f|(f|x) \equiv \sim[\acute{\phi}\{x = \exists\phi\,.\sim\phi|x\}]|(f|x)$$
$$\equiv \sim[x = \exists(f|x)\,.\sim(f|x)|x]$$
$$\equiv [x\ 0\text{‘}\ \exists(f|x)\,.\vee.\,(f|x)|x]$$

As for identity of apparently different propositions, observe $\phi = \psi\,.\supset.\,\phi|x = \psi|x$. Thus $x = x\,.=.\sim x = \sim x$ and so on. If $X \equiv_x X'$, and X Focp x, $X =_x X'$. Thus apparently different propositions may be functional, and there may be fewer propositions than functions. Hence we might assume ~~$(x).\text{Indiv}|X\,.\supset.\,X$ Focp x Pp~~

$$\vdash : X \supset_x \text{Indiv}|x\,.\supset.\,X \text{ Focp } x \qquad \text{Pp}$$

0. $\vdash : X$ Focp $x\,.\equiv.\sim X$ Focp x *folio ⟨22⟩*
1. To prove $\vdash : X$ Focp $x\,.\,X'$ Focp $x\,.\supset.\,X \vee X'$ Focp x
   $[X = \phi|x\,.\,X' = \phi'|x\,.\supset.\,X \vee X' = (\phi \vee \phi')|x]$
2. $\vdash : X$ Focp $x\,.\,X \vee X' \sim$Focp $x\,.\supset.\,X' \sim$Focp x
3. $\vdash : X$ Focp $x\,.\,X'$ Focp $x\,.\supset.\,(X\,.\sim X')$ Focp x
   $[X = \phi|x\,.\,X' = \phi'|x\,.\supset.\,(X\,.\sim X') = (\phi \cap -\phi')|x]$
4. $\vdash : X$ Focp $x\,.\,X \vee X'$ Focp $x\,.\supset.\,X'$ Focp x [3]
5. $\vdash : X \sim$Focp $x\,.\,X \vee X'$ Focp $x\,.\supset.\,X' \sim$Focp x [4]
6. $\vdash : X$ Focp $x\,.\,X'$ Focp $\langle x\rangle\,.\supset.\,X\,.\,X'$ Focp x
7. $\vdash : X \sim$Focp $x\,.\,X\,.\,X'$ Focp $x\,.\supset.\,X' \sim$Focp x and so on.

We ought to be able to prove $\sim(\exists\phi)\,.\,(\phi|x =_x X)\,.\supset.\,X$ is satisfied by arguments outside such and such a class.

The best suggestion is, that a non-functional complex is always satisfied by arguments of infinite order, *i.e.* by arguments which are themselves satisfied by arguments of all finite orders. Thus $\mathrm{Sim}|\phi$ is satisfied only by ψ's of definite finite orders, and is therefore not of infinite order; but Sim is of infinite order.

We have $\vdash :. (\phi) : (\exists x) . (X = \phi) : \supset . X|x \sim \mathrm{Focp}\, x$ *folio ⟨23⟩*
More specially $\vdash :. \mathrm{Cls}|\phi . \supset_\phi . (\exists x) . (X = \phi) : \supset . X|x \sim \mathrm{Focp}\, x$
Similarly $\vdash :. \mathrm{Rel}|\phi . \supset_\phi . (\exists x) . (X = \phi) : \supset . (X|x)|x \sim \mathrm{Focp}\, x$
$\vdash :. \mathrm{Triple}\big|\phi . \supset_\phi . (\exists x) . (X = \phi) : \supset . \{(X|x)|x\}\big|x \sim \mathrm{Focp}\, x$

We have $\mathrm{Order} = \acute{\phi}\{f|\mathrm{Cls} : f|\theta . \supset_\theta . f(\theta \cap \mathrm{Cls}`\theta) : \supset_f . f|\phi\}$ Df

There are functions not of any order, e.g. $\mathrm{Sim}|\phi$, but having values which are always of some definite order [$\mathrm{Sim}|\phi$ is not such].

(Won't work.)

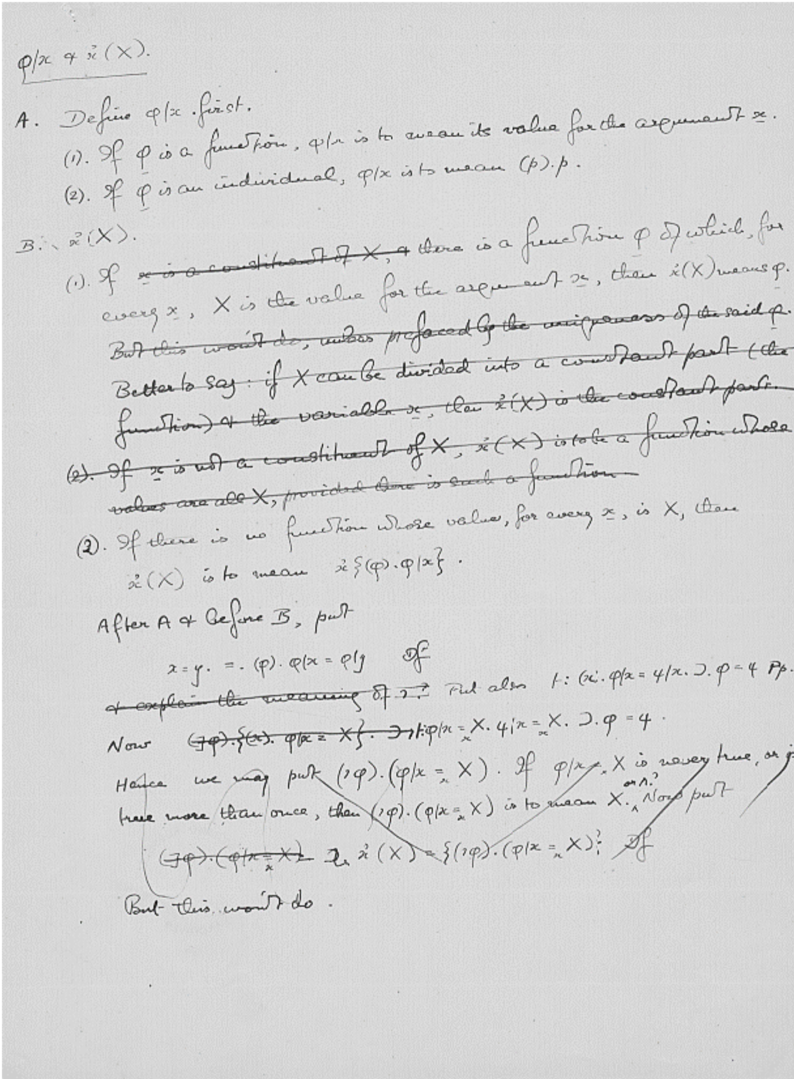
$\phi|x$ and $\acute{x}(X)$. *folio ⟨24⟩*

A. Define $\phi|x$ first.

(1). If ϕ is a function, $\phi|x$ is to mean its value for the argument x.
(2). If ϕ is an individual, $\phi|x$ is to mean $(p) . p$.

B. $\acute{x}(X)$.

(1). If ~~x is a constituent of X, and~~ there is a function of ϕ of which, for every x, X is the value for the argument x, then $\acute{x}(X)$ means ϕ. ~~But this won't do, unless prefaced by the uniqueness of the said ϕ. Better to say: if x can be divided into a constant part (the function) and the variable x, then $\acute{x}(X)$ is the constant part.~~
~~(2). If x is not a constituent of X, $\acute{x}(X)$ is to be a function whose values are all X, provided there is such a function.~~
(2). If there is no function whose value, for every x, is X, then $\acute{x}(X)$ is to mean $\acute{x}\{(\phi) . \phi|x\}$.



$\phi|x$ & $\hat{x}(X)$.

A. Define $\phi|x$ first.
(1). If ϕ is a function, $\phi|x$ is to mean its value for the argument x.
(2). If ϕ is an individual, $\phi|x$ is to mean $(p).p$.

B. $\hat{x}(X)$.
(1). If ~~x is a constituent of X, &~~ there is a function ϕ of which, for every x, X is the value for the argument x, then $\hat{x}(X)$ means ϕ.
~~But this won't do, unless prefaced by the uniqueness of the said ϕ.~~
~~Better to say: if X can be divided into a constant part (the function) & the variable x, then $\hat{x}(X)$ is the constant part.~~
~~(2). If x is not a constituent of X, $\hat{x}(X)$ is to be a function whose values are all X, provided there is such a function~~
(2). If there is no function whose value, for every x, is X, then $\hat{x}(X)$ is to mean $\hat{x}\{(\phi).\phi|x\}$.

After A & before B, put
$x = y . = . (\phi) . \phi|x = \phi|y$ Df
~~& explain the meaning of~~ Put also $\vdash : (x).\phi|x = \psi|x . \supset . \phi = \psi$ Pp.
Now ~~$(\exists\phi):\{(x). \phi|x = X\}. \supset$~~ $\vdash : \phi|x =_x X . \psi|x =_x X . \supset . \phi = \psi$.
Hence we may put $(\iota\phi).(\phi|x =_x X)$. If $\phi|x =_x X$ is never true, or is true more than once, then $(\iota\phi).(\phi|x =_x X)$ is to mean X. or Λ? Now put
~~$(\exists\phi).(\phi|x =_x X)$~~ $\supset\ \hat{x}(X) = \{(\iota\phi).(\phi|x =_x X)\}$ Df
But this won't do.

Figure 1. Some changes of mind about functions (folio ⟨24⟩).

After *A* and before *B*, put

$$x = y \,.\!=\!.\, (\phi) \,.\, \phi|x = \phi|y \qquad \text{Df}$$

~~and explain the meaning of $\imath$?~~ Put also $\vdash : (x).\phi|x = \psi|x .\supset. \phi = \psi$ Pp.

Now ~~$(\exists\phi).\{(x).\phi|x = X\}.\supset \imath$~~ $\vdash : \phi|x =_x X . \psi|x =_x X . \supset . \phi = \psi$.

~~Hence we may put $(\imath\phi).(\phi|x =_x X)$. If $\phi|x =_x X$ is never true, or is true more than once, then $(\imath\phi).(\phi|x =_x X)$ is to mean X. or Λ? Now put~~

$$\sout{(\exists\phi).(\phi|x =_x X).\supset.\acute{x}(X) = \{(\imath\phi).(\phi|x =_x X)\} \quad \text{Df}}$$

But this won't do.

<u>$X|X'$.</u> *folio (25)*

$\vdash :$ ~~If~~ $Y \sim \text{Focp}\ y .\supset. (\acute{y}Y)|X \sim \text{Focp}\ x$
$\vdash : X = \acute{z}(p) . p\ \text{Focp}\ z .\supset. X|X' = p\dfrac{X'}{z}$
Put $X' = \phi|x$. Then $X|X' = p\dfrac{\phi|x}{z}$

Now p must contain x and z. If $x|Z$ occurs, then if $\text{Indiv}|x$ we can substitute $(p).p$. But if $z|x$ occurs, we get $(\phi|x)|x$. If now $\text{Rel}|\phi$, $(\phi|x)|x$ is admissable if ϕ is derived from a complex, and if not, $(\phi|x)|x = (p).p$. Thus $(\phi|x)|x$ under the circumstances is always admissable.

It is simpler to put

$$\vdash : (x).\text{Indiv}|X .\supset. X\ \text{Focp}\ x \qquad \text{Pp}$$

But this Pp is probably capable of proof or disproof.

3. "MEANING AND DENOTATION" (230.030950)

Meaning and denotation.

A function is something *denoted*, not *meant*, by the form of words in which it is spoken of. Broadly, every single word which is not a proper name denotes a function; but there are some exceptions—e.g. *the* and the copulative *is*. "To be a rational animal" and "to be a man" differ in meaning, but the function involved is the same. Thus there must be an object which both denote—a property which may be expressed by either phrase.

But when a proper name is part of the complex which is the value of the function, it often happens that the proper name is part of the *meaning*, not of the denotation; yet change of the name changes the value of the function, and the name is the variable of the function. Take e.g. "the centre of gravity of the universe". The meaning is complex, but the denotation is utterly simple. Substituting x for *universe*, the denotation is always simple, but the meaning is complex, and contains x as constituent. Thus the variable of a function need only be part of the meaning, not of the denotation.

"The centre of gravity of x", if x is a function satisfied by material points and by nothing else, denotes a point of space; if x is anything else, it denotes Λ.

Consider: x enters into the *meaning*, not necessarily the *denotation*, *folio 2*
of $\phi|x$; the question is: does the meaning or the denotation of x form a constituent of $f|x$? E.g. "Edward VII is a man" is a value of $\dot{x}(x$ is a man); is "The present King of England is a man" the *same* value, or a different one? We should have to say, I think, that it is the *same* value. Thus x denoted enters in to $\phi|x$.

In the above case, the denotation as well as meaning of $\phi|x$ was complex. Take a case where this is not so, e.g., "The Centre of Gravity of the Crystal Palace" and the "the Centre of Gravity of the glass building which, after the great Exhibition, was moved from Hyde Park to Sydenham". Are there two values, or one, of $\dot{x}$(the centre of gravity of x)? I think we must say they are one. Thus always it is x denoted which is relevant in $\phi|x$, although it is in the meaning of $\phi|x$, not necessarily the denotation, that x is a constituent. When x has meaning and denotation, can the denotation, as opposed to the meaning, be part of the meaning, as opposed to the denotation, of a complex? This is a difficult question.