Reviews

ANALYSIS, MATHEMATICS, AND LOGIC IN RUSSELL'S EARLY PHILOSOPHY

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Jolen Galaugher. Foreword by Michael Beaney. *Russell's Philosophy of Logical Analysis: 1897–1905*. (History of Analytic Philosophy series.) Basingstoke and New York: Palgrave Macmillan, 2013. Pp. xii, 218. ISBN 13: 978-1-137-30206-9. £60; US$110 (hb).

In this book Jolen Galaugher discusses a number of key turning-points in Russell's early philosophy: his break from Idealism in 1898–99; his acceptance of logicism in the period following the International Congress of Philosophy in Paris in August 1900 at which he was impressed by Peano and his students; and his arrival at his theory of descriptions in 1905 in the context of his attempts to grapple with various versions of what Russell called "the Contradiction". Her overall purpose is to examine the interplay between the view of the analysis of propositions that Russell accepts following his break with Idealism and his developing views within the philosophy of mathematics. The main merits of the book are the way in which it highlights unresolved issues in our understanding of Russell's early development and brings to bear relevant evidence in attempting to address those issues. However, as Galaugher acknowledges, the book's "arguments are complicated in places" (p. 2), and it is not an easy read. Like some other books in this series, it appears to be based on a recently completed PHD dissertation, and, like some dissertations, it combines a serious engagement with recent scholarship with a manner of presentation that presumes a shared understanding with the reader of a number of central issues and positions, and so does not clearly articulate and defend her understanding of those issues and positions to the uninitiated or sceptical reader. Hence, while the book will be useful to scholars already familiar with the sorts of issues it addresses, it would be hard to recommend it to someone coming to this material for the first time.

The book consists of five chapters. In the first, Galaugher outlines some

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central elements of Russell's position during his Idealist period and some of the steps involved in his dismantling that position. In the second, she argues more specifically, that in rejecting Idealism, Russell was not, as it is sometimes presented, simply following G. E. Moore—that his engagement with Leibniz's views led Russell to go beyond Moore in certain ways and to adopt his characteristic view that relations are "external". In the third, she discusses aspects of Russell's logicism and its development in the period following the Paris Congress. In the fourth, she focuses on Russell's logicist definition of the cardinal number and the notion of "class" that underpins it, arguing that in the face of the Contradiction, and his changing view of class and propositional function, it becomes questionable whether Russell and Frege should be said to endorse the same definition of cardinal number. In the fifth, she examines aspects of Russell's coming to reject the theory of denoting concepts in favour of the theory he presents in "On Denoting", arguing that by doing so Russell reaffirms his commitment to his post-Idealist view of a "decompositional" view of analysis as opposed to Frege's function/argument style of analysis.

Each of the aspects of Russell's views that Galaugher examines concerns issues on which Russell often changed his mind, sometimes within a quite short period of time. While much of the material relevant to understanding these changes of mind has now been published in the *Collected Papers of Bertrand Russell*, Russell scholars have not fully assimilated it. Galaugher not only calls attention to a number of features of Russell's early development that are not yet fully understood; she also makes use of the full range of materials now available in defending her account. Most notable in this regard is her use of Russell's correspondence with Couturat, which, while published, has not been translated from French, in which the correspondence was conducted. From Galaugher's translation of some of the correspondence, which took place in 1897–1913, it is clear that this is a valuable resource for understanding Russell's philosophical development and should be fully explored by scholars.

In what follows, I focus on two of the topics that Galaugher considers: the relative influence of Moore and Leibniz on Russell's rejection of Idealism; and the development and nature of Russell's logicism. In each case, I believe that Galaugher points to significant gaps in our current understanding of the issues involved in these shifts in Russell's philosophy. While I take issue with some of her claims, I do so in an effort to further the project—to which her book makes a significant contribution—of achieving as fine-grained an understanding as is possible of Russell's philosophical development.

MOORE, LEIBNIZ, AND RUSSELL'S REJECTION OF IDEALISM

During his Idealist period (1894–98) Russell uses a form of the dialectic method that incorporates the view that any science that falls short of the complete "metaphysical construction of the real" is subject to "contradictions … which unavoidably result from the incompleteness" of that science (*Papers* 2: 5). Hence, for the Idealist Russell, the proper method in philosophy is to begin with some limited science, find the inevitable contradiction within it, and then use that contradiction as the means to "pass outside to a new science, which may then be similarly treated" (*ibid.*). Once he rejects Idealism, Russell no longer holds that there are inherent contradictions in any limited science, and thus denies that a successful analysis of any limited science requires finding a contradiction in it. As he writes in a manuscript (completed in May 1900): "[I]t is time that contradictions should cease to be regarded as commending a theory" (*Papers* 3: 783).

For Galaugher, Russell's 1898 manuscript "An Analysis of Mathematical Reasoning" is "a transitional work" (p. 27). On the one hand, in "An Analysis of Mathematical Reasoning", as in his 1903 *Principles of Mathematics*, Russell "entirely reject[s]" views that attempt "to restrict the logical subject" and holds instead that "[e]very possible idea, everything that can be thought of, or represented by a word, may be a logical subject" (*Papers* 2: 168; *cf. PoM*, pp. 43–4). Further, as in the *Principles*, Russell adopts a terminology according to which "[w]hatever can be a logical subject I call a *term*" (*Papers* 2: 167; *cf. PoM*, p. 43) and distinguishes "being" and "existence", so that while every term has "being", only some terms "exist" (see *Papers* 2: 168, 170; *cf. PoM*, pp. 449–50). On the basis of such points, Galaugher claims that by "An Analysis of Mathematical Reasoning", "Russell had arrived at a position, similar to Moore's, on the nature and proper constituents of judgment" (p. 13).

On the other hand, in "An Analysis of Mathematical Reasoning", Russell holds that "[t]he foremost class of judgments, from every point of view, is the class in which a predicate is asserted of a subject" (*Papers* 2: 167), and, on the basis of doing so, he derives what he calls "the contradiction of relativity". Assuming that relational judgments are to be grounded in subject–predicate judgments, Russell argues that since judgments involving transitive asymmetric relations (such as *greater than* and *earlier than*) are not reducible to subject–predicate judgments, then there is an unavoidable "contradiction" infecting such judgments. In particular, he holds that since transitive asymmetric relations "pervade almost the whole of Mathematics" (2: 226) and since all the relations of this type fall prey to the "contradiction of relativity", this "pervading contradiction" helps "to define the realm of Mathematics" (2: 166).

As Nicholas Griffin has emphasized, an—and perhaps the—key event in Russell's rejection of Idealism occurs when he ceases to regard the irreducibility of propositions involving transitive asymmetric relations as establishing the "contradiction of relativity" and instead regards it as a *reductio* of the view that relational propositions are to be grounded in subject–predicate propositions. Griffin characterizes this change in view as a "Gestalt shift" on

Russell's part[1] and writes that "[i]t is hard to over-estimate the importance of this move"[2] in Russell's philosophical development. As Griffin recognizes, this shift has already occurred by January 1899, when Russell presents "The Classification of Relations", where he holds that every proposition, including every subject–predicate proposition ("if there be any" [*Papers* 2: 141]) is ultimately relational in form.

Questions arise, however, as to exactly when—between his completing "An Analysis of Mathematical Reasoning" and writing "The Classification of Relations"—this change occurred and what brought it about. Throughout his writings, Russell typically credits Moore with leading him to reject Idealism, writing, for example, in *My Philosophical Development* that "Moore led the way, but I followed closely in his footsteps" (p. 54). However, he also writes there that "I first realized the importance of the question of relations when I was working on Leibniz" (p. 61). In 1991 Griffin argued that the influence of Moore "was more powerful and probably earlier" than his study of Leibniz, thereby suggesting that Russell's interaction with Moore, not his reading Leibniz, occasioned the "Gestalt shift".[3] However, in recent years, Griffin has modified his position, suggesting tentatively in his "Russell and Leibniz on the Classification of Relations" (which was actually completed in 2008) and more definitely in his "What Did Russell Learn from Leibniz?" that, as a result of studying Leibniz's writings in preparation for a course on Leibniz that began in January 1899, Russell came to regard the doctrine of "internal relations" as a dispensable assumption, the rejection of which would enable him to avoid the "contradiction of relativity". Responding primarily to the sort of view expressed in Griffin's *Russell's Idealist Apprenticeship*,[4] Galaugher claims that "the significance of the work on Leibniz to Russell's development has not received sufficient attention" (p. 40), and, going beyond what Griffin has claimed in his recent writings, she argues that through his study and criticism of Leibniz, Russell came to accept certain views that go beyond what Moore had advanced in "The Nature of Judgment", the 1899 paper that is typically credited with inaugurating the Moore–Russell "revolt against Idealism".

In my view, resolving these issues (to the extent that they can be resolved) requires establishing not only an accurate and detailed timeline but also a clear understanding of how the views of Moore and Russell change over the relevant time period. Russell and Moore met in both May and June 1898,

1 Griffin, *Russell's Idealist Apprenticeship* (1991), p. 364.

2 Griffin, "What Did Russell Learn from Leibniz?" (2012), p. 3.

3 *Russell's Idealist Apprenticeship*, p. 342.

4 She also comments (p. 51f.) on Griffin's "Russell and Leibniz on the Classification of Relations" but not "What Did Russell Learn from Leibniz?", which may not have been published before her book went to press.

during the period in which Moore was beginning to draft a second version of his dissertation—the first version of 1897, which reflected a commitment to some form of Bradleian Idealism, having not been awarded a Fellowship—while Russell was working on "An Analysis of Mathematical Reasoning".[5] On 19 June, before his week-long stay with "the Russells", Moore writes to Desmond MacCarthy that he has written "about six pages [of the second version of his] dissertation and done less work than ever" (*ibid.*, p. 109). On 20 July 1898, Russell writes to Moore that he has finished Book I of "An Analysis of Mathematical Reasoning" and will have a typed copy of it sent to him (*ibid.*, pp. 109–10). On 4 August, Russell sends his paper "Are the Axioms of Geometry Empirical?" to Couturat. As in "An Analysis of Mathematical Reasoning", he accepts the "contradiction of relativity" and claims that it "infects all of Mathematics" (*Papers* 2: 328, see nn. 4–5). Thus, it would appear that by early August 1898, at any rate, Russell has not yet experienced the "Gestalt shift" central to his rejecting Idealism.

Nor, from this chronology, is it clear that, in his discussions with Russell in May and June, Moore has yet adopted the views regarding propositions and their constituents that he comes to formulate in "The Nature of Judgment". However, on 14 August, Moore writes to MacCarthy that he has "some 60 new pages written" and that "I had not seen where my principles would lead me" (Preti, p. 108). On 11 September, in a letter to Russell discussing changes he has made to his dissertation, Moore writes:

> I carefully state that a proposition is not to be understood as any thought or words, but the concepts + their relation *of* which we think. It is only propositions in this sense, which can be true, and from which inference can be made.… There would need, I think, to be several kinds of ultimate relations between concepts—each of course necessary.… (*Early Philosophical Writings*, pp. xxxiv–xxxv)

In claiming that a proposition consists of "concepts + their relation", Moore indicates that every proposition is relational in form, a claim he also makes in "The Nature of Judgment", where he indicates that a proposition "would certainly seem to involve at least two terms and a relation between them" (p. 65, see also p. 64).

Galaugher does not, I think, appreciate how central this relational view of propositions is to "The Nature of Judgment" and to Moore's influence on Russell. For the view Moore presents in "The Nature of Judgment", that each proposition is relational, is opposed to Russell's view in "An Analysis of Mathematical Reasoning" that "[t]he foremost class of judgments, from every point of view, is the class in which a predicate is asserted of a subject"; and in that

5 See Preti, "'He Was in Those Days Beautiful and Slim'" (2008), p. 108.

case, there is at least one fundamental respect in which Galaugher is wrong to claim that in "An Analysis of Mathematical Reasoning", "Russell had arrived at a position, similar to Moore's"—presumably, Moore's view in "The Nature of Judgment"—"on the nature and proper constituents of judgment".

Further, in support of her claim that through his study and criticism of Leibniz, Russell came to accept certain views that go beyond what Moore had advanced in "The Nature of Judgment", Galaugher writes that by "The Classification of Relations", Russell's view "has become so radical that propositions asserting identities are not propositions, identities are not relations, and there are no subject–predicate propositions, strictly speaking, but only ones which assert a relation between subject and predicate, taken as terms", and has thereby "pressed Moore's new realist views to a radical conclusion"(p. 63). However, each of these views is a relatively straightforward consequence of Moore's relational view of propositions. Since a true identity proposition would have to consist of one term in relation to itself, it fails to meet Moore's condition in "The Nature of Judgment" that each proposition consists of "at least two terms and a relation between them". More generally, by that condition, there can be no relation (such as identity would have to be) that relates a term to itself; for then there would be a (true) proposition containing only one term in relation to itself. (Accordingly, in "The Classification of Relations", Russell indicates that identity understood as "mere self-sameness" does not meet "the formal condition of relations, namely plurality of terms" [*Papers* 2: 140].) Again, if there are any subject–predicate propositions, then on Moore's view in "The Nature of Judgment", they will ultimately have to be understood as relational (involving, as Russell indicates in "The Classification of Relations", the relation of *predication*). Galaugher notes that in his 1900 paper "Necessity", Moore writes that in a case of a "supposed analytic proposition … expressed in the form, *A* is *A*, … we have no proposition",[6] but she indicates (p. 51) that in doing so, Moore "has adopted the view … which Russell advocated" in "The Classification of Relations". In fact, in "Necessity", Moore claims that the reason that there is no proposition in such a case is that "we have certainly not two terms" (Moore, *ibid.*), having earlier claimed in that paragraph, as he had in "The Nature of Judgment", that "[a]ny proposition, it would seem must contain at least two different terms and their relation" (p. 88). What is radical here is Moore's view in "The Nature of Judgment" that each proposition consists of at least two terms and a relation between them; and there is no reason to suppose that Moore needed Russell to recognize the consequences of that view.[7]

[6] Moore, "Necessity" (1900), p. 89.

[7] As late as his 1913 manuscript, *Theory of Knowledge*, Russell writes that "[i]t is … doubtful whether" there are any complexes "in which there is only one term and one

Accepting Moore's view in "The Nature of Judgment" that each proposition is relational is sufficient to undermine Russell's commitment to the "contradiction of relativity". For recognizing that "contradiction" depends on holding both that subject–predicate propositions are not ultimately relational in form, and that relational propositions not reducible to subject–predicate propositions are contradictory; but on Moore's view in "The Nature of Judgment", all propositions (including subject–predicate propositions, if there are any) are ultimately relational in form, and there is no inherent contradiction in that being so. However, it is plausible that the influence of Moore worked in tandem with Russell's reading of Leibniz. As Griffin has detailed,[8] Russell read a translation of the *Nouveaux essais* in June 1898, Duncan's translated selections *The Philosophical Works* in August, and Latta's translated selections *Monadology and Other Writings* in October. Assuming that Russell first learned of Moore's "The Nature of Judgment" view of propositions from the September letter and perhaps only really assimilated that view when he read Moore's second dissertation in November, then while it seems that Moore's influence on Russell was not, as Griffin writes in *Russell's Idealist Apprenticeship*, "probably earlier" than his study of Leibniz, it is not clear either, as Griffin suggests in his more recent writings, that it was his reading Leibniz, rather than the influence of Moore, that provoked Russell's "Gestalt shift".

From Moore, Russell was exposed to the view that all propositions are relational in form; and while this may have been enough for Russell to reject the "contradiction of relativity", his study of Leibniz would have contributed to—and, at a minimum, would have reinforced—his view that assuming the subject–predicate form to be fundamental leads to extreme conclusions that can be avoided if that assumption is rejected. Moore adopted the view that every proposition is relational in form for general philosophical reasons; but he was not (apparently) concerned with the more technical issues regarding relations—including their relevance for mathematics and the irreducibility of certain sorts of relations to the subject–predicate form—with which Russell was occupied. Leibniz, on the other hand, was concerned with those sorts of technical issues; and while Russell could have undergone his "Gestalt shift" and avoided the "contradiction of relativity" simply by following Moore whether or not he had ever had read Leibniz, reading Leibniz could have only strengthened his view that holding, with Moore, that every proposition is relational in form would enable him to avoid a number of difficulties in his philosophy of mathematics. If so, then in his *Autobiography* Russell accurately characterizes the influence of both Moore and Leibniz in his "Gestalt switch". For there he

predicate, where the predicate occurs as relations occur in other complexes" (*Papers* 7: 80–1).

8 "What Did Russell Learn from Leibniz?", p. 3.

writes: "In the study and criticism of Leibniz I found occasion to exemplify the new views on logic to which, largely under Moore's guidance, I had been led" (*Auto.* 1: 135).

RUSSELL'S LOGICISM

In Chapters 3 and 4, Galaugher discusses a number of issues regarding Russell's logicism in the *Principles*. These include whether, and if so in what sense, Russell advocates an "if-thenist" version of logicism for geometry as well as issues regarding Russell's acceptance of the so-called "Frege–Russell" definitions of the cardinal numbers. With regard to Russell's alleged "if-thenism", she argues not only, following Sébastien Gandon, that Russell accepts a "topic-specific logicism" incompatible with "if-thenism" as it is typically portrayed, but also that one should distinguish two different versions of "if-thenism", each of which Russell advocates in different passages, one of which emerges later in the composition of the *Principles* than the other. Galaugher is right, I believe, to hold that issues concerning the composition of the *Principles* are central to its interpretation and is, in particular, right to distinguish two different versions of "if-thenism". However, I take issue with some aspects of the history of the composition of the *Principles* that she presents and suggest that taking that history into account may undermine some of Gandon's claims regarding Russell's "topic-specific logicism".

In his retrospective writings, Russell describes attending the Paris Congress in August 1900 as "the most important event" in "[t]he most important year in my intellectual life"[9] and writes that "[i]ntellectually, the month of September 1900 was the highest point of my life" (*Auto.* 1: 145). In October, he wrote a draft of his paper "The Logic of Relations"; and in November–December, he wrote final drafts of Parts III–VI of the *Principles*. However, as Galaugher recognizes, when Russell began this draft of the *Principles* at the end of 1900, he did not advocate logicism in the form he articulated it in the *Principles* as published, so that questions arise as to how and when Russell came to advocate logicism in that form.

First, Russell did not introduce his logicist definitions of the cardinal numbers—as classes of "similar" (or, in Fregean terminology, equinumerous) classes—until sometime in 1901. As Michael Byrd has detailed,[10] in the draft material he wrote in 1900 (as in his pre-Peano, post-Idealist writings), Russell regards each cardinal number as philosophically "indefinable". Further, as Gregory Moore has discussed (*Papers* 3: xxvi–xxvii), sometime between February and July 1901, in the course of revising "The Logic of Relations",

9 Russell, "My Mental Development" (1944), *Papers* 11: 12.
10 Byrd, "Part V of *The Principles of Mathematics*" (1994), pp. 56–64.

Russell added a comment to the effect that "the cardinal number of a class *u*" can be defined as "the class of classes similar to *u*". Moreover, as late as May 1902, Russell indicates (for reasons having to do with the paradoxes) that while defining the cardinal numbers as classes of similar classes is acceptable "for formal purposes", those numbers remain "philosophically indefinable".[11] And it is only sometime after June 1902 that Russell deletes this passage and affirms his logicist definitions of number both "formally" and "philosophically" (*PoM*, p. 136).[12]

With regard to "if-thenism", Russell indicates that he was led to some form of this view by his concern with providing an account according to which geometry—including different, apparently incompatible, systems of geometry—is part of "pure mathematics". As he writes in his 1937 "Introduction" to the *Principles*:

> It was clear that Euclidean and non-Euclidean systems alike must be included in pure mathematics, and must not be regarded as mutually inconsistent; we must, therefore, only assert that the axioms imply the propositions, not that the axioms are true and therefore the propositions are true. (*PoM*, p. vii)

Galaugher (p. 89) rightly distinguishes two different views. On the first, which Galaugher identifies as "if-thenism" and which I'll call "ungeneralized if-thenism", what are asserted are indicative conditionals (Russell's "implications"), whose antecedents and consequents contain geometrical vocabulary, such as "point", "line", and "distance". On the second form, which I'll call "logically generalized if-thenism", what are asserted are generalized conditionals, in which all geometrical vocabulary has been replaced by variables, so that a given geometric system is, in effect, regarded as a logical structure.

Russell illustrates a central difference between these two views by considering the so-called "Peano postulates" for arithmetic:

(1) 0 is a number.
(2) If *a* is a number, the successor of *a* is a number.
(3) If two numbers have the same successor, the two numbers are identical.
(4) 0 is not the successor of any number.
(5) If *s* be a class to which belongs 0 and also the successor of every number belonging to *s*, then every number belongs to *s*.

[11] See Byrd, "Part II of *The Principles of Mathematics*" (1987), p. 69.

[12] Galaugher is aware of these points but does not make them as clearly as she might. Compare pp. 85, 116 and 117 on her dating of Russell's introduction of the logicist definitions of the cardinals.

As stated, these postulates contain the arithmetical vocabulary "0", "number", and "successor". However, as Russell discusses, those postulates can be generalized from to characterize a logical structure he calls a "progression", which he defines as follows: "A 'progression' is a one-one relation such that there is just one term belonging to the domain but not to the converse domain, and the domain is identical with the posterity of this one term" (*IMP*, p. 82; see also *PoM*, p. 240, and *PM* 2: ∗122). For Russell, that is, Peano's axioms exemplify the structure that is a progression because the *successor* relation is a one–one relation meeting his stated conditions; 0 is the member of the domain of that relation that is not a member of the converse domain; and *number* is the domain of that relation. Further, given that each element of Russell's definition of "progression", such as "one–one relation" and "posterity", can be explicated in purely logical terms, then Russell has provided a purely logical characterization of the "progression" structure that is exemplified in Peano's postulates. Likewise, in Chapter XLIX of the *Principles*, Russell gives examples as to how the axioms of different systems of geometry can by generalized from to provide purely logical definitions of the structures—or, as he calls them "spaces"—that are exemplified in those different systems of geometry. And, given the logical definition of the structure exemplified by the axioms of a given system of geometry along with the corresponding logical characterization of the structure exemplified in a theorem of that system of geometry, one may then form a generalized conditional stating, in effect, that whatever meets the structural conditions determined by those axioms meets the structural condition determined by that theorem.

At the outset of "Recent Work on the Principles of Mathematics", written apparently in January 1901, Russell suggests the view he states in Chapter I of the *Principles*, according to which the propositions of "pure mathematics" are generalized conditionals ("formal implications") that contain only variables and "logical constants", and so "contain no indefinables except logical constants" (*PoM*, p. 8).[13] This characterization of the propositions of pure mathematics is in accord with "logically generalized if-thenism" but not with "ungeneralized if-thenism", since the conditionals of "ungeneralized if-thenism" contain non-logical geometrical vocabulary. However, in a manuscript

[13] However, as Galaugher discusses (pp. 102–4), Russell achieves the understanding of "formal implication" that he presents in the *Principles* only sometime after May 1901. Note also that in passages added relatively late in the composition of the *Principles*, Russell supplements his characterization of the propositions of "pure mathematics" to include "existence theorems", establishing that the various structures that have been logically defined have instances (so that the generalized conditionals of pure mathematics are not vacuously true). See, for example, §474, inserted in January 1903 (see BYRD, "Part VII of *The Principles of Mathematics*" [1999], p. 174), and §300, added in proofs between May 1902 and May 1903 (see BYRD, "Part V", p. 81).

entitled "Recent Italian Work on the Foundations of Mathematics", Russell writes:

> What distinguishes a special branch of mathematics is a certain collection of primitive or indefinable ideas, and a certain collection of primitive or indemonstrable propositions concerning these ideas. When once these have been assigned, symbolic logic appropriates the subject, and effects whatever deductions are legitimate. (*Papers* 3: 353)

Thus, unlike the view he presents in the *Principles*, but in accord with "ungeneralized if-thenism", Russell indicates here that while logic is sufficient for all mathematical deductions, it does not provide the content for all propositions of pure mathematics—a view that is far weaker than the logicism of the *Principles*, according to which there are no "primitive or indefinable ideas" of "pure mathematics" other than the "logical constants".

Russell appears to refer to "Recent Italian Work" in September 1900 when he writes to Couturat that he intends to write an article for *Mind* on "Peano and his disciples" and on 24 October when he writes to Alys that "I am going to begin my article on Peano" (see *Papers* 3: 351). However, in *Papers* 3, "Recent Italian Work" (which was never published in *Mind*) is dated as 1901, because, it seems, of a letter from Russell to Stout in December 1900 in which he writes that "I shall be delighted to send you my article on Peano and Co.", adding that "I can send you the article in time for the April number" (*Papers* 3: 351). However, this need not be evidence that Russell wrote "Recent Italian Work" following the December letter; rather, it may be evidence that Russell intended to revise the paper he drafted in October before submitting it to *Mind*.[14] If so, then Galaugher has been misled by the dating of "Recent Italian Work", for she appears to cite that paper (see pp. 89, 92, and 93) as reflecting Russell's views as of January 1901, and she holds (see p. 94) that Russell "arrived at his logicism" and "replaced non-logical constants with variables" in the period between January and May 1901. Instead, as I argue now, there is evidence that Russell made that change in the course of drafting the *Principles* in December 1900, thereby changing his view from "ungeneralized if-thenism" to "logically generalized if-thenism".

As detailed by Byrd, ("Part VI") the *Principles*, Part VI—entitled "Space" in which Russell discusses geometry—was drafted in December 1900, and in the

[14] While Russell discusses geometry in "Recent Italian Work" (*Papers* 3: 361–2), in his December letter to Stout, he writes that "by sticking to Symbolic Logic, I can, I think, make an article suitable for *Mind*" (3: 351). This suggests that "Recent Italian Work" was not written after the December letter, but rather that Russell planned to revise "Recent Italian Work" before submitting the article to *Mind* but never revised (or submitted) the paper.

December manuscript the chapters were numbered internally, from Chapters I to IX, rather than from XLIV to LII as they appear in the *Principles*. In the first chapter of the manuscript of Part VI, Russell writes:

> Geometry has become … a branch of pure mathematics, that is to say, a subject in which the assertions are that such and such consequences follow from such and such premisses, not that entities such as the premisses describe actually exist. That is to say, if Euclid's axioms be called A, and P be any proposition implied by A, … now-a-days, the geometer would only assert that A implies P, leaving A and P themselves doubtful. And he would have other sets of axioms, A_1, A_2 … implying P_1, P_2 … respectively: the *implications* would belong to Geometry, but not A_1 or P_1 or any of the other actual axioms and propositions. Thus Geometry no longer throws any direct light on the nature of actual space.… Moreover it is now proved (what is fatal to the Kantian philosophy) that every Geometry is rigidly deductive, and does not employ any forms of reasoning but such as apply to Arithmetic and all other deductive sciences. (*PoM*, pp. 373–4)[15]

Here, Russell characterizes geometry as "a branch of pure mathematics" simply because it asserts conditionals in which axioms of a given geometry appear as antecedents, and consequences of those axioms appear as consequents. Russell's approach here enables him to regard Euclidean as well as non-Euclidean geometries as belonging to "pure mathematics", and it further enables him to argue, against Kant, that geometry employs no reasoning that is not "rigidly deductive". However, there is no suggestion here of eliminating the geometrical vocabulary from the conditional propositions of geometry; nor is there any claim that since geometry is "a branch of pure mathematics", it is thereby a branch of logic. That is, in this passage, Russell only endorses "ungeneralized if-thenism" and hence does not present geometry as "a branch of pure mathematics" by the definition of "pure mathematics" given in the *Principles*, Chapter I.

However, elsewhere in the published Part VI—most prominently, in Chapter XLIX, the sixth chapter of Part VI—Russell presents "logically generalized if-thenism". In the first section of that chapter, Russell writes:

> [W]hen logic is extended, as it should be, so as to include the general theory of relations, there are, I believe, no primitive ideas in mathematics except such as belong to the domain of Logic. In the previous chapters of this Part, I have spoken, as most authors do, of certain indefinables in Geometry. But this was a concession, and must now be rectified. (*PoM*, pp. 429–30)

15 My claim that this passage was written in December 1900 (and other such claims I make below) is based on comparing the text of *PoM*, Part VI with Byrd ("Part VI", pp. 55–61).

And in the remaining sections of the chapter, he presents definitions of various geometries of the sort required to carry out "logically generalized if-thenism" for those geometries. Thus here, as opposed to what he writes in Chapter LXIV, Russell endorses the sort of logicism he defends in the *Principles*; and he recognizes that in order to sustain that logicism, he will have to produce definitions eliminating what he had previously taken to be indefinable geometrical vocabulary. The manuscript of this chapter of the *Principles* no longer exists; hence, we do not know whether Russell wrote the passage I have just quoted in December 1900 or inserted it at some later stage.

However, from sections of the manuscript that do exist, it is clear that in that chapter, as he originally drafted it, and in accord with the logicism of the *Principles* as published, Russell defines geometrical structures using purely logical vocabulary. In particular, in the final section of Part VI, in "briefly recapitulat[ing] the results of this Part", Russell writes in December 1900:

> We found that the abstract logical method, based upon the logic of relations, which had served hitherto, was still adequate, and enabled us to define all the classes of entities which mathematicians call spaces, and to deduce from the definitions all the propositions of the corresponding Geometries. We found that … no new indefinables occur in Geometry. (*PoM*, p. 461)

Thus, Russell indicates that by extending logic to include "the general theory of relations", he is able to define what had previously been regarded as "indefinables" of geometry; and it seems clear that this is what he undertook in the sixth chapter of Part VI.[16] That is, it seems that Russell undertook the critical step of logically generalizing the axioms of a given geometry—the step from which "logically generalized if-thenism" follows—in the course of drafting Part VI of the *Principles* in December.

That he did so is further suggested by the letter he wrote to Helen Thomas on 31 December 1900, the day, according to Russell (*Auto.* 1: 145), on which he completed his manuscript of the *Principles*. Part VI was the last part of the *Principles* that Russell drafted in December 1900 before writing that letter. If so, then on the day that he wrote that letter, Russell had also written the section of the *Principles* from which I have just quoted. In that letter, Russell writes: "In October I invented a new subject, which turned out to be all mathematics, for the first time treated in its essence" (*SLBR* 1: 208). The subject that Russell "invented" in October was the "general theory of relations"; and it seems that he came to hold, in the course of writing Part VI, that once the "general theory of relations" is recognized as part of logic, then logic "turn[s]

[16] See *PoM*, p. 458, as collated with Byrd ("Part VI", p. 61) for a reference to "the definitions of the various spaces" in "Chapter VI".

out to be all mathematics". In particular, in accord with what he writes in the final section of Part VI, he came to hold that by using the "logic of relations" not only to deduce geometrical theorems from geometrical axioms (as he does on "ungeneralized if-thenism") but also to define geometrical structures (as he does on "logically generalized if-thenism"), thereby avoiding the assumption that there are any specifically geometrical indefinables, he can hold that logic "turn[s] out to be all mathematics" in the strong sense in which he comes to present logicism in the *Principles* as published.

If it is the case that in the course of drafting Part VI in December 1900, Russell changed his view from the (non-logicist) "ungeneralized if-thenism" to the (logicist) "logically generalized if-thenism", then there is reason to doubt whether Russell accepted the "topic-specific logicism" that Galaugher, following Gandon, attributes to Russell. As Gandon discusses, in the *Principles*, Part VI, Russell discusses two different axiomatizations of projective geometry. In Chapter XLV, he presents Pieri's account, according to which there are two indefinables—*point* and a relation (which Gandon calls an incidence relation) that obtains between two points if and only if they are both in a given specified line, so that, for example, the relation R_{ab} will obtain between two points if and only if they both occur in the line *ab* (see *PoM*, p. 383). In Chapter XLVI, he presents Pasch's account, according to which there are again two indefinables—this time, *point* and the *betweenness* relation. Given that these different axiomizations yield the same geometrical structure, then it would seem that if Russell adhered simply to what I have called "logically generalized if-thenism", there would be no philosophically motivated reason for him to prefer one of these axiomatizations over the other. In fact, however, Russell indicates in Part VI of the *Principles* that he prefers Pieri's axiomatization to Pasch's, characterizing Pieri as the mathematician who took the step that renders projective geometry "complete" (*PoM*, p. 421). For Gandon, and following him Galaugher (see pp. 81–4), this establishes that Russell "was not just interested in formalizing geometry",[17] as a commitment to "logically generalized if-thenism" would suggest; rather he was concerned "to delineate *the* real essence of geometrical thought" (*ibid.*).

However, if Russell's views of geometry changed during the period in which he composed the *Principles*, Part VI—in particular, if, as I have suggested, he began writing it assuming "ungeneralized if-thenism" but came to accept "logically generalized if-thenism" by the time he completed it—then it may be that while Russell's preference for Pieri's axiomatization over Pasch's reflects his views when he began composing Part VI, he could no longer justify that preference—at least from the perspective of the logicism he accepts by the time he completes it. In particular, if he began writing Part VI assuming

[17] GANDON, *Russell's Unknown Logicism* (2012), p. 47.

that there are geometrical indefinables, then his preference for Pieri's axiomatization reflects his view that Pieri, as opposed to Pasch, gives the correct account of the "indefinables" of projective geometry. However, if he came to hold by the time he completed Part VI that there are no geometrical indefinables, but are rather only logically defined structures or "spaces", then he would no longer be in a position to hold that one way of defining a given structure is more ultimately correct than other, equivalent, ways of defining the same structure. If so, then there is a conflict within Part VI; but it would not be the only conflict within the *Principles* that is a product of the history of its composition.[18]

As Gandon (Ch. 2) discusses, Russell's preference for Pieri's account of projective geometry reflects an aspect of his work on geometry that goes back to his Idealist period. In his 1897 dissertation, *An Essay on the Foundations of Geometry*, and in accord with the "pure" or "synthetic" approach to geometry, Russell presents projective geometry as purely "qualitative", free from any "quantitative" elements, although he does not present mathematically adequate axioms for projective geometry so construed. While he argues in his dissertation that the "antinomy of the point" requires a dialectical transition from projective geometry to quantitative science, during his post-Idealist pre-Peano period, Russell abandons his Idealist view that there is any inevitable contradiction or antinomy in projective geometry, but retains his view of projective geometry as independent of any quantitative or, indeed, ordinal notions. Thus, as Galaugher notes (p. 82), he writes in 1899 that "Projective Geometry is not essentially concerned with order or series" (*Papers* 2: 379); and, again in 1899, he writes that "[t]he pure projective theory knows nothing of distance, nor, consequently of order among points, lines, or planes" taken as indefinables (*Papers* 3: 497). It is in this context that Russell enthusiastically receives Pieri's work, which he studied in August 1900, immediately following the Paris Congress, and prefers it to Pasch's. Thus, in "Recent Italian Work" (which, I have argued, was written in October 1900), Russell writes that "Pieri's work on projective Geometry ... is far the best I know" (*Papers* 3: 362) and that in projective geometry "[t]here is ... to begin with, no order of points on a straight line, and no such notion as *between*" (3: 361). However, as I have discussed above, in "Recent Italian Work", Russell had not yet adopted

[18] Thus, for example, BYRD ("Part II") discusses inconsistencies in *PoM* regarding *one* (in particular, whether it applies to absolutely every entity) that results from late additions to the text following Russell's study of Frege; and I have argued ("From Absolute Idealism to *The Principles of Mathematics*" [1998], pp. 110–17, 126–7, n. 57) that there are inconsistent characterizations in *PoM* of the "principle of abstraction" that result from Russell's originally accepting the "axiom of abstraction" when he drafted *PoM* in November–December 1900, but later regarding it as a provable "principle" when he made later changes to the text.

"logically generalized if-thenism", holding, instead, that each "special branch of mathematics" has "a certain collection of primitive or indefinable ideas". And it is given the assumption that there are geometrical indefinables that Russell favours Pieri's axiomatization; for Pieri's axiomatization is in accord with Russell's pre-existing view that the indefinables of projective geometry should include no ordinal notions.

However, on "logically generalized if-thenism", there are no geometrical "indefinables". Rather, there are logically defined structures, or "spaces", which the axioms of a given system of geometry exemplify; and from this point of view, if there are logically equivalent ways to define that structure, there is no philosophical reason for preferring one of them to the others. As Russell writes in the *Principles*, Chapter XLIX, in which he produces purely logical definitions of various "spaces":

> It is important to observe that the definition of a space, as of most other entities of a certain complexity, is arbitrary within certain limits. For if there be any property which implies and is implied by one or more of the properties used in the definition, we may make a substitution of the new property in place of the one or more in question.... In such cases, we can only be guided by motives of simplicity. (*PoM*, p. 432)

Thus, for Russell, where different "properties" can be used to define the same space, there is no reason, aside from "simplicity", to prefer one definition of that space to another. Again, we do not know exactly when this passage was written. However, it reflects what Russell regards as the consequence of accepting "logically generalized if-thenism". On this view, given that Pasch's axiomatization of projective geometry is logically equivalent to Pieri's, either one of them may be generalized from to characterize the logical structure of a projective "space", and the only reason for using one of them, rather than the other, is simplicity.

The *Principles*, Part VI, is the last of Russell's writings where he engages in the task that he undertook in his 1897 dissertation of examining the relation between projective and metrical geometry. It is not, however, the last of his writings in which he presents "logically generalized if-thenism". Thus, for example, at the outset of his 1927 book *The Analysis of Matter*, in discussing one of the questions "which we may ask concerning physics or, indeed, any science"—in particular, the question "What is its logical structure, considered as a deductive system?"—Russell presents "logically generalized if-thenism" as a "procedure" that "has become familiar" in geometry (*AMa*, pp. 1–2). In particular, as in the *Principles*, Chapter XLIX, he writes in *The Analysis of Matter* that when we turn the geometrical constants of the axioms of a given geometry into variables in order to characterize the "logical structure" of that geometry, there are in general "many different sets of initial hypotheses" we may use to characterize that structure and that the "choice between [them] is logically irrelevant and can be guided only by aesthetic considerations" (*ibid.*, p. 2). When (following the Paris Congress but prior to adopting logicism) Russell regards logic as providing the rules for all correct deductions but also holds that each "special branch of mathematics" has its own indefinables, he can engage in what he regards as the characteristically philosophical task of identifying the indefinables of a given "special branch of mathematics", such as projective geometry. However, once he adopts "logically generalized if-thenism" and the form of logicism that goes with it, he no longer holds that there are any such indefinables of a given "special branch of mathematics" and regards that philosophical task as "logically irrelevant".

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