RUSSELL'S TWO LECTURES IN CHINA ON MATHEMATICAL LOGIC[1]

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In 1921 Bertrand Russell delivered two lectures on mathematical logic at Peking University. Manuscripts for the lectures have not been found, but two sets of Chinese notes, which were based on a simultaneous oral translation of Russell's lecturing, were published. The notes are translated into English based on the best readings of both sets. An introduction and notes with a glossary discuss the background and content of the lectures as well as the linguistic difficulties in translating logical terms.

Russell visited China during 1920–21. He decided to lecture to Chinese students on the topic of mathematical logic, probably because he received an invitation (dated 22 November 1920) to "give one or a series of lectures on Mathematical Philosophy" from Fù Tóng, a philosophy professor at the Government University of Peking (now Peking University), on behalf of the Study Circle for Mathematics and Science of the High Normal School (now Beijing Normal University).[2] Interestingly, in a letter to Wittgenstein on 11 February 1921, Russell commented that his Chinese students were "not advanced enough for mathematical logic".[3] This may be part of the reason why he prepared very elementary material for his lectures on mathematical logic in the following month.

[1] The authors wish to thank two anonymous referees for their comments and Russell Society members at their 2017 annual meeting, Central Connecticut State U.

[2] Another letter from Fu to Russell on 24 November 1920 shows that Russell seems to have accepted the invitation. See ZHOU, "A Critical Bibliography of Russell's Addresses and Lectures in China" (2016), p. 161.

[3] Our thanks to Landon Elkind for directing us to this letter dated 11 February 1921 (in MCGUINNESS, ed., *Wittgenstein in Cambridge* [2008], p. 124).

Main Hall of the Second Campus, Peking University, *c*.1919–49.

Russell delivered two lectures on mathematical logic at Peking University in March 1921. We know that the first lecture was delivered in the Main Hall of the Second Campus, Peking University, on 8 March, 7:30–9:30 pm (see above for a photograph of the Main Hall).[4] The exact date when the second lecture was delivered remains uncertain. But we know that Russell fell ill on 14 March and that he had to cancel his planned lectures due to his illness. So the date of the second lecture should be between 8 and 14 March. Relatedly, some secondary sources indicate that Russell was scheduled to give four lectures, but had to stop after the first because of his illness.[5] So it is also possible that these "two lectures" are simply parts of a single evening's presentation. Nonetheless, both sets of notes taken by Wú Fànhuán and Mù Yán, which are the basis of the current English translation of these lectures, numbered two lectures or two parts of a single presentation. Wu, in his notes, explicitly stated that Russell gave *two* lectures on mathematical logic.[6] The notes are consistent with respect to the content of the lectures because they are both based on a simultaneous oral Chinese translation of Russell's lecturing by Zhào Yuánrèn. Zhao took

4 See YUAN, SUN and REN, eds., *Zhōngguó Dào Zìyóu Zhī Lù: Luósù Zàihuá Jiǎngyǎnjí* [*China's Road to Freedom: a Collection of Russell's Lectures in China*] (2004), p. 309. The time period of this lecture corresponds to a suggested period in the invitation letter from Fu. This fact supports the surmise that Russell decided to lecture to Chinese students on mathematical logic because of Fu's invitation. Another letter, from *Fù* Zhòngsūn and Zhāng Bāngmíng to Russell on 23 March 1921, confirms that he lectured on mathematical logic and that they were still expecting him to give more lectures on this topic in the near future (see ZHOU, p. 159 n. 48).

5 See YUAN, SUN and REN, p. 309. For further details of Russell's illness and the cancelling of the lectures due to it, see ZHOU, pp. 145–6, 155 n. 28, 158–60 nn. 45, 48.

6 For discussion of the date and number of these lectures, see ZHOU, pp. 159–60.

a PHD in philosophy from Harvard University in 1918 under the supervision of Henry Sheffer (before that he studied mathematics and physics at Cornell University). He taught at Tsinghua College (now Tsinghua University) during Russell's visit.[7] Zhao did an excellent job of picking appropriate Chinese words for technical terms (the translator will discuss in notes the linguistic difficulties in finding equivalents in Chinese for such terms), although some of the translations follow previous scholars.[8] Wu's notes were first published in *Shùlǐ Luójī* [*Mathematical Logic*] (Peking University New Knowledge Press, 1921), and an early version of Mu's notes can be found in *Luósù Jí Bólākè Jiangyǎnlù* [*A Collection of Lectures by Russell and Black*] (Beijing Weiyi Daily Press, 1921). Both versions have been reprinted in China in recent years.[9] However, they have not been translated into English until now. Since Russell's manuscripts for these lectures have not been found (he probably used brief notes when he gave these lectures), translating their Chinese versions into English will be of assistance to scholars and readers in the English-speaking world. Those interested in the development of his views on logic between the first edition of *Principia Mathematica* (1910–13) and the second edition (1925–27) may find the lectures particularly interesting.

The material of the first lecture, the discussion of the Sheffer stroke as a new primitive connective in propositional logic, and the proposed single axiom from Jean Nicod, are familiar from lectures that Russell presented before going to China.[10] These became the basis for the system of the second edition of *PM* in 1925.[11] That Russell introduced these ideas first in this popular set of lectures may reflect his own thinking about a new formulation of *PM* in the light of Sheffer and Nicod's ideas as much as a view about how to introduce logic to novices.[12] The second lecture, which introduces ideas of the "Algebra of Classes" is a novelty. This material is the subject matter of *22 in *PM*,

7 Zhao (known in the West as Y. R. Chao) later taught for many years at UC Berkeley, where the Bancroft Library has a large collection of his papers. To gain further information about Zhao, the translator hopes for an opportunity to examine them.

8 See, for example, n. 35.

9 For publication information with respect to these lectures, see ZHOU, p. 160.

10 See *PLA* in *Papers* 8: 186 and *IMP*, p. 148.

11 *PM*2, Introduction and Appendix A. See LINSKY, *The Evolution of Principia Mathematica* (2011), pp. 124–6.

12 An anonymous referee reminded us that the relationship between the two editions of *PM* is contested by scholars. See LANDINI's review (2013) of LINSKY.

but it does not occur in his popularizations of logic such as "The Philosophy of Logical Atomism" or *Introduction to Mathematical Philosophy*. Presumably Russell was going to go on to discuss quantifiers, the "no-classes" theory of classes in *20 of *PM*, and in the later lectures the theory of types. Basing further discussion of logic on the algebra of classes may reflect an early version of the idea which distinguishes the second edition of *PM* from the first, namely the adoption of extensionality, by which the difference between propositional functions and classes is reduced. For Russell the reduction of mathematics to logic is a combination of a reduction of the theory of classes to logic, which is the material in sections *1–*20 in *PM*, and then the further reduction of mathematics to relations "in extension", which occupies the remainder of the three volumes. Russell may have thought that this latter reduction was the best subject-matter for an introduction to "Mathematical Logic" for an audience that was new to the subject. He may have already decided that the foundational logical material in the "Introduction" to *PM* and the first sections up to *20 might not be genuinely introductory but still subject to development and changes. If so, these lectures reveal a feature of Russell's "public lectures", which, while admirably accessible to a general audience, always stated Russell's current position on his topic, however well the technical difficulties he was struggling with might be hidden.

BERTRAND RUSSELL

Mathematical Logic [*Shùlǐ Luójī*/*Shùxué Luójī*][13]

I

Everyone [*Zhūjūn*],[14]

The difference between mathematical logic and ordinary mathematics [*pǔtōng shùxué*] lies in that they proceed in different directions: ordinary mathematics goes forward, while mathematical logic goes backwards.[15] But you should know that going backwards does

[13] These are the Chinese Pinyin terms for "**Mathematical Logic**" used by Wu and Mu, respectively. (Such variants are listed in the order "[Wu/Mu]", including in the Appendix II glossary.) The terms are in English in the lecture notes and in bold type here. Variant English words are limited to "**Mathematical Logic**" and "**informal principle**" (only in Wu's notes), "**proposition function**"—corrected here to **function of propositions** (see *IMP*, p. 147, and *PM2* I: xiv, 6; see also n. 35)—and "**equivalency**" in Wu's notes, and "**incompatibility**" (only in Mu's notes).

[14] This is found in Wu's notes (but not in Mu's notes).

[15] For the "directions" of logic and mathematics, see the opening words of *IMP*.

not mean getting worse. Rather, it means an approach to seeking grounds.[16]

When we encounter many mathematical propositions [*mìngtí*],[17] two different kinds of questions are going to arise: (1) What can we deduce from these propositions? This is what ordinary mathematics is supposed to study. (2) From what kind of propositions can we deduce these propositions? That is to say, we need to find simpler and fewer propositions from which we can deduce these relatively more complicated propositions. Taking steps like this, we can go backwards and find propositions that are even simpler than the simple propositions that we have found at the previous stage. This is what mathematical logic is supposed to study.

We can randomly pick a kind of deductive [*tuīlùn*][18] system, such as arithmetic [*suànshù*],[19] geometry [*jǐhé*],[20] Newtonian mechanics [*Níudùn lìxué*], etc., in which we can deduce the totality of a system from a few **axioms** [*gōnglǐ/zìlǐ*][21] and **postulates** [*gōngfǎ/jiǎdìng*][22] (I will clarify the distinction between axioms and postulates

[16] "*Zhuīběnsùyúan*"/"*zhuīqíuběnyuán*" literally means "tracing the source" in English. The philosophical term corresponding to "source" would be the Greek word, "*arche*", which means "origin", "beginning", or "source". But according to what Russell says in the next paragraph, this was meant to tell the audience that mathematical logic is a subject that seeks to find out logical grounds, namely the simplest and fewest propositions from which we can deduce relatively more complicated propositions.

[17] Russell takes "propositions" to mean a form of words or symbols which expresses either truth or falsehood (he also seems to use "propositions" and "symbols" interchangeably; see *IMP*, Ch. 15). "*Mìngtí*" originally means to assign a topic and is used as a technical term for a statement which expresses either truth or falsehood.

[18] Although the Chinese Pinyin "*tuīlùn*" is close in meaning to the English term "deduction", in contemporary literature Chinese logicians and philosophers usually use "*yǎnyì*" to capture the meaning of "deduction", and "*yǎnyì de*" to capture the meaning of "deductive".

[19] "*Suànshù*" literally means "methods for calculation". It can also mean "mathematics in general". For example, the Chinese mathematics book, *Jǐuzhāng Suànshù*, which is known as one of the earliest surviving mathematics texts from China, includes chapters on both arithmetic and geometry. But in the current context, the translator used "*suànshù*" to capture the meaning of "arithmetic".

[20] "*Jǐhé*" literally means "measurement of size". This Chinese translation of "geometry" derives from the translation by Xú Guāngqǐ and Matteo Ricci during the Ming dynasty (1368–1644).

[21] "*Gōnglǐ/zìlǐ*" literally means "a theory or statement that is self-evidently true".

[22] "*Jiǎdìng*" literally means "assumption". "*Gōngfǎ*", which literally means "accepted law", is not an appropriate translation here.

later).[23] In addition to the aforementioned systems, every kind of pure mathematics can be deduced from a finite number of axioms and postulates. All axioms and postulates are purely logical.[24]

Ordinary mathematics, except for a few special kinds, such as projective geometry [*tóuyǐng jǐhé*] and group theory [*qúnlùn*],[25] are associated with numbers [*shùmù*].[26] But there are many subjects which have nothing to do with numbers but can be studied rigorously by using mathematical methods, and mathematical logic is a required instrument for such studies. Therefore, although mathematical logic is part of mathematics, it is not concerned with numbers.

At this point, we cannot provide a clear definition [*dìngyì*][27] of mathematics: we will do this once we have covered more material. However, we can now give a simple definition as follows: a subject which seeks to deduce results by using symbols [*fúhào*] is called "mathematics".[28] Whether we use numbers or not does not matter; even if we use numbers, we just do so by chance. But we should know several characteristics of this subject: (1) mathematics is rigorous; (2) mathematics is certain and beyond doubt; and (3) for mathematicians, if they are responsive to one part [of the subject in question], they must be responsive to other parts. Generally speaking, if one is readily

[23] Russell did not specify this distinction in the two lectures. Nevertheless, since he planned more than two lectures, he could have planned on discussing this distinction in later lectures.

[24] With "axiom" Russell is referring to "Primitive Propositions (*Pp*)", which are basic logical principles from which others are derived, now presented as axioms or axiom schemata. "Postulates" are assumptions that are explicitly added as antecedents of conditional theorems. Both Russell's "multiplicative axiom" (*Mult ax*), now known as the "Axiom of Choice", and the "Axiom of Infinity" (*Ax Inf*), are in fact used as "hypotheses" in *PM*. The "Axiom of Reducibility" (*Red Ax*), however, is a "primitive proposition" and hence a real "axiom" in Russell's terminology used here.

[25] "*Qúnlùn*" is a literal translation of "group theory": "*qún*" literally means "group", and "*lùn*" literally means "theory".

[26] Nowadays, Chinese people simply use "*shù*" to capture the meaning of "number".

[27] "*Dìngyì*" literally means "eternal or correct meaning", which is close to what "definition" means.

[28] This definition of mathematics seems too general: it may pick out a subject which is not mathematics. E.g., Russell himself mentions Newtonian mechanics earlier on as a deductive system, one that presumably uses symbols, but which is not a part of mathematics. Perhaps by "symbols" ["*fúhào*"] Russell means here "defined symbols", or "symbols introduced as abbreviations". Even so, it is unclear how his definition of mathematics here can distinguish mathematics from other deductive systems which also presumably use symbols. But he may be providing only a preliminary or simplified definition of mathematics at this point.

able to use abstract symbols, he will be responsive to all parts of mathematics.

In fact, we know that many kinds of mathematics are independent of numbers. When we study these subjects, we need to use mathematical logic. Mathematical logic is essential to our study of these subjects, just as calculus is important to ordinary mathematics.

For those of you who studied mathematics, you must have known that many problems, which used to be regarded as philosophical problems, are now treated as mathematical problems. These problems, to which philosophers had no solutions for thousands of years, are now solvable by using mathematical methods. For example, people attempted to explain physical **entity** [*wù de shízaì/wùlǐ de shítǐ*],[29] but they did not succeed because the previous philosophical enquiries offered no solutions to problems concerning matter [*wùtǐ/wùzhì*],[30] space [*kōngjiān*], time [*shíjiān*],[31] etc. Now we know that we have to use mathematical methods to address these problems, and hence we treat these problems as problems to be addressed in the field of mathematical logic, in which we have achieved satisfying results.

As discussed earlier, every kind of pure mathematics can be deduced from a few purely logical axioms or postulates. Now let me add that in pure mathematics we do not deal with particular things to which we are actually able to point [*zhǐchū*]—all that we need is just symbols, such as x, y, z, etc., and that asking questions concerning the meaning of such symbols is not necessary. Moreover, things that can be verified [*zhèngmíng*] or falsified [*fǒurèn/fǒuzhèng*][32] by experiments are also not included in pure mathematics. For example, "Two straight lines [*zhíxiàn*] cannot enclose a space" is not a purely mathematical proposition. More than that, this proposition is problematic,

29 "*Wù de*" or "*wùlǐ de*" literally means "physical". Nowadays, when Chinese philosophers use terms like "*shízaì*" or "*shítǐ*", they intend to mean "substance". In both Wu and Mu's notes, "**entity**" is the English term corresponding to "*shízaì*" or "*shítǐ*".

30 "*Wùzhì*" is an appropriate translation of "matter" here. "*Wùtǐ*" or "*Wù*" literally means "physical entities". "*Zhì*" is perhaps related to "*zhìliàng*", which literally means "measurement of mass".

31 "*Kōngjiān*" and "*shíjiān*" are widely used to capture the meaning of "space" and "time" respectively.

32 "*Zhèngmíng*" literally means "to demonstrate the truth of something by evidence or argument"; "*fǒurèn*", "to deny the truth of something"; and "*fǒuzhèng*", "to deny the truth of something by evidence or argument".

because we do not have an exact definition of straight lines. Early on it was widely held that light travels in a straight line. Given this, people attempted to define straight lines in terms of light [*guāngxiàn*]. Nevertheless, we now know that light is affected by gravity and becomes bent as a result. So the previous definition does not work. Additionally, in theory it is very difficult to "verify or falsify" such things as straight lines. With that being said, the kind of propositions discussed above cannot be included in pure mathematics.

When we study mathematical logic we exclude particular things that we can point to. Consider this example: "If it is raining, then I want an umbrella; now I don't want an umbrella, so it is not raining." In mathematics we can use p to represent "it is raining" and use q to represent "I want an umbrella". Therefore, we would just say, "If p, then q; now not q, so not p." This shows that in mathematics we only use symbols, regardless of what kind of things they represent. We also use variables [*biànliàng*]—meaningless letters, such as x, y, z, etc., represent variables—and we study them on the basis of a few postulates, regardless of whether they are true or not. So I sometimes define mathematics in a way that enemies of mathematics may be happy with. My definition is as follows: for the people who study mathematics, they do not know what they are talking about, nor do they know whether what they are saying is true.[33]

The totality of pure mathematics, as we have known it, can be deduced from a few axioms or postulates. It follows that methods of deduction are very important. For example, proposition q can be deduced from proposition p. If p is true, and given that if p is true then q is true, then we know that q is true. This sort of case is called "if p then q" or "p implies [*bāohán*][34] q". Furthermore, "If p then q" is a **function** [*hánshù/hánjiàn*] of p and q, which is called a "**function of propositions**" / "**propositional function**") [*mìngtí hánshù/mìngtí hánjiàn*].[35] Functions of propositions are very important to

33 This joke is first found in "Mathematics and the Metaphysicians" (1901; *ML*, p. 75; *Papers* 3: 366).

34 "*Bāohán*" literally means "to contain". Here, it is used to capture the meaning of the verb "imply".

35 "*Hánshù*", as a Chinese translation of "**function**", was invented by Lǐ Shànlán (1810–1882), a well-known Chinese mathematician. He specified the meaning of "*hánshù*" as follows: if a variable contains another variable, then the former variable is the function of the latter variable (*cf. Daìshùxué* [*The Study of Algebra*]). To see how this translation goes, we should know that "*hán*" in Chinese means "box" or

mathematical logic. In functions of propositions variables are one or several propositions, just as in mathematical functions variables are one or several numbers. If the truth of a function of propositions depends on whether the propositions that it contains are true or not, then this proposition function is called a "**truth function**" [*zhēnlǐ hánshù/zhēnlǐ hánjiàn*].[36] To illustrate this concept, we can take a "non-truth function" as an example: "I believe p" is a function that contains p, but the truth or falsity of "I believe p" does not depend on the truth or falsity of p. This is because even if p is false, I can still believe it. Therefore, "I believe p" is not a truth function.

Every truth function can be derived from a single truth function—**incompatibility** [*bùxiāngróng*],[37] which means "two things cannot be both true". For example, we can symbolize the incompatibility of p and q as $p \mid q$. This expression means "p and q cannot be both true". That is to say, "either p is false or q is false". We can derive other functions from this function:

(1) $p \mid p$. This expression means "either p is false or p is false". That is to say, "p is false". We symbolize "p is false" or "not-p" as $\sim p$. It follows that $\sim p = not\, p\ (p\ is\ false) = p \mid p$ Df (**Df = Definition**).

(2) $(p \mid p) \mid (q \mid q)$. This expression means "either p is true or q is true". That is to say, "p and q cannot be both false". In symbols: $p \vee q = (p \mid p) \mid (q \mid q)$ Df, where "$p \vee q$" is the **logical sum** [*luójī de hé*] of p and q.

(3) $(p \mid q) \mid (p \mid q) = \sim (p \mid q) = not\, (p \mid q)$. This expression means

"envelop". "*Shù*", as mentioned before, means "number". So putting these two words together, we have "*hánshù*", which means "box/envelop with number inside". This Chinese translation of "**function**" is particularly appropriate for the notion of function used in ordinary mathematics, where variables (arguments) are numbers. "*Hánjiàn*" is variant of "*hánshù*", which originally means "envelop with letter or something inside". "*Hánjiàn*" seems to be a more appropriate Chinese translation of "**function**" in "**function of propositions**" than "*hánshù*", since in a function of propositions the arguments are propositions rather than numbers. A combination of "*mìngtí*" and "*hánshù*"/"*hánjiàn*" is therefore used to capture the meaning of "function of propositions". (See *IMP*, p. 147; *PM*2 1: xiv, 6.) The use of "proposition function" (in English in Mu's notes) is treated here as Mu's mistake and corrected to "**function of propositions**". In Wu's notes, the term used here is "**propositional function**", which is problematic in the current context because propositional functions can have arguments that are not propositions.

36 "*Zhēnlǐ*" literally means "a theory or fact that is accepted as true".

37 "*Bùxiāngróng*" literally means "(two things) cannot exist together".

"both p and q are true". In symbols: $p \cdot q = (p \mid q) \mid (p \mid q)$ Df, where "$p \cdot q$" is the **logical product** [*luójī de jí*] of p and q. (We have now observed the uses of symbols. Without symbols, when we say in everyday life, "Today is Tuesday, tomorrow is Wednesday", in logic we have to say, "Either today is not Tuesday or tomorrow is not Wednesday and either today is not Tuesday or tomorrow is not Wednesday are incompatible." This is very inconvenient and not easy to follow.)

(4) $p \mid (q \mid q)$. This expression means "p and not-q are incompatible", i.e., "either p is false or not-q is false". In other words, it means that "either p is false or q is true". So $p \mid (q \mid q) =$ not-p or q = if p, then $q = p$ implies q. In symbols: $p \supset q = p \mid (q \mid q)$ Df.

(5) $p \mid (q \mid r) = p$ implies q and r. What this expression means is very clear, so I will not say more about it.

When we study mathematical logic we must use a few principles [*yuánlǐ*][38] to deduce others. We now have six **deductive principle**s, among which the first five principles are **formal principle**s [*xíngshì de yuánlǐ*] and the last principle is an **informal principle** [*fēixíngshì de yuánlǐ*].[39] Here are the principles:

(1) $(p \vee p) \supset p$, i.e., "p or p" implies p.
(2) $q \supset (p \vee q)$, i.e., if q, then "p or q".
(3) $(p \vee q) \supset (q \vee p)$, i.e., if "$p$ or q", then "q or p".
(4) $p \vee (q \vee r) \supset q \vee (p \vee r)$.
(5) $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$, i.e., if q implies r, then "p or q" implies "p or r".[40]

M. Nicod has reduced these five formal principles to one as follows:

There are five propositions: p, q, r, s, and t.
Put $P = p \mid (q \mid r)$ (p implies q and r),
$\pi = t\,(t \mid t)$ (t implies itself),
$R = (p \mid s) \mid (p \mid s)$ (the conjunction of p and s),
$Q = (s \mid q) \mid R$ (p and s implies s and q).
Therefore, $P \mid (\pi \mid Q)$ (P implies π and Q).

38 "*Yuánlǐ*" literally means "a fundamental or original theory or fact".
39 "*Xíngshì*" literally means "form". "*Fēi*" literally means "no" or "negative".
40 A horseshoe, as the main connective, between the brackets is missing in Wu's notes, while in Mu's notes the main connective is "U", which is perhaps a typo. These are five of the "primitive propositions" of *PM*, *2. The sixth is the rule of Modus Ponens.

In addition to the preceding five formal principles, M. Nicod has yielded another informal principle:

(6) If we know that p is true and p implies q, then we can say, q is true.

By using the above symbols, we can express logical laws [*fǎzé/yuánzé*],[41] such as:

Law of contradiction [*máodùn lǜ/máodùn dìnglǜ*].[42] It says that p and not-p cannot exist simultaneously, i.e., $\sim(p \cdot \sim p) = p \mid (p \mid p)$.
Law of excluded middle [*pái zhōngxiàng lǜ/ wú zhōngxiàng dìnglǜ*].[43] It means either p or not-p, i.e., $p \vee \sim p = (p \mid p) \mid [(p \mid p) \mid (p \mid p)] = (p \mid p) \mid p$.
Syllogism [*sānduànlùn fǎ*].[44] In symbols: $[(p \supset q) \cdot (q \supset r)] \supset (p \supset r)$.[45]

2[46]

Everyone,

In some cases, two truth functions are **equivalent** [*děngzhí de*], i.e. that if one is true, then the other is true; and that if one is false, then the other is false. We often use ≡ to denote the **equivalence** relation

41 "*Fǎzé*" literally means "rules". In comparison with "*guīzé*", which also literally means "*rules*", "*fǎzé*" is often used in a more objective or stronger sense than "*guīzé*". "*Yuánzé*" literally means "fundamental or original rules".

42 "*Lǜ*" literally means "rules or laws"; "*Dìnglǜ*", "eternal rules"; "*Máodùn*", "spear and shield", with "contradiction" being its extended meaning.

43 "*Pái*" is the same as "*páichì*", which literally means "to exclude or keep (something or someone) out". "*Wú*" literally means "vacancy or emptiness". "*Zhōngxiàng*" literally means "items in the middle".

44 "*Sānduànlùn*", which literally means "a reasoning expressed in three parts", is a widely accepted Chinese (Pinyin) translation of "syllogism". "*Fǎ*" is often used to capture the meaning of "law or method".

45 In both versions of notes, brackets are missing here, or perhaps dots around the ⊃.

46 According to Wu's notes, in the second lecture, Zhao introduced a few symbols that Russell would use in this lecture to replace parentheses, brackets, and braces: one dot . replaces ( ), two dots : replace [ ], and three dots ∴ replace { }. For example, $4 - \{[(2+3) \times (3+5) - 12] \div 5\}$ will be expressed as $4 - \therefore : .2+3. \times .3+5. - 12 : \div 5 \therefore$. But Zhao's account is not an accurate characterization of Russell's use of dots in *PM*. (See *PM* itself, 1: 10.) The incorrect dots in Wu's notes might be a result of Zhao's account. Mu's notes simply keep brackets as brackets. The translator (Zhou) will follow Russell's usage of dots except for the case in which both versions of notes keep brackets as brackets (see n. 48 for the exception).

[*děngzhí de guānxì*].[47] Hence: $(p \equiv q) = (p \supset q) \cdot (q \supset p)$.[48]

The equivalence (**equivalency**?) relation possesses three properties:

(1) **reflexiveness** [*fǎnshè*], i.e., $p \equiv q$.
(2) **symmetry** [*duìchèn*], i.e., $p \equiv q . \supset . q \equiv p$.
(3) **transitiveness** [*yíxiàng*],[49] i.e., $p \equiv q \cdot q \equiv r . \supset . p \equiv r$.

Among the properties listed above, (2) and (3) are independent. For example, given $a > b$, $b > c$, we get $a > c$. Hence this relation is transitive. But it is not symmetrical, because it is not right to derive $b > a$ from $a > b$. Take another example: assume that a is not like b, it follows that b is not like a. So this relation is symmetrical. But it is not transitive, because assume that a is not like b and b is not like c, it does not follow that a is not like c. Moreover, (1) is associated with (2) and (3): if (2) and (3) exist, then (1) exists as well.

"Equivalence" in mathematical logic is comparable to "**equality**" [*xiāngděng*][50] in ordinary mathematics. But in mathematical logic we have the following expressions:

$$p . \equiv . p \vee p$$
$$p . \equiv . p \cdot p$$

In ordinary mathematics we know that the following two expressions are not right:

$$x = x + x$$
$$x = x \cdot x$$

Therefore, unlike ordinary (numerical) algebra [*xúncháng daìshù*], **logical algebra** [*luójī de daìshù*] is **non-numerical**.[51]

47 "*Děngzhí de*" is the adjectival form of "*Děngzhí*", which literally means "equivalent values". "*Guānxì*" literally means "connection or relation".

48 In both Wu and Mu's notes, dots are not used in this case. Moreover, an identity sign occurs here as the main connective. The translator (Zhou) understands it as a symbol for definition and so would expect "Df" at the end.

49 "*Fǎnshè*" literally means "reflection or the throwing back by a surface of light, sound, etc." "*Duìchèn*" literally means "exact correspondence between two things". "*Yíxiàng*" literally means "moving of items".

50 "*Xiāngděng*" is the same as "*děngzhí*"; both mean that some given values are equivalent.

51 "*Daìshù*", which literally means "replacement of numbers", was first used by Lǐ Shànlán as a Chinese translation of "algebra" (see n. 34).

In logical algebra we have several laws, which are comparable to laws in ordinary algebra:

(1) the **commutative law** [*hùhuàn lǜ/hùhuàn dìnglǜ*]: $p \vee q . \equiv . q \vee p, p \cdot q . \equiv . q \cdot p$, which is the same as the commutative law in ordinary algebra, i.e., $p + q = q + p, p \cdot q = q \cdot p$;
(2) the **associative law** [*liánhé lǜ/liánhé dìnglǜ*]: $p \vee (q \vee r) . \equiv . (p \vee q) \vee r$, $p \cdot (q \cdot r) . \equiv . (p \cdot q) \cdot r$, which is the same as the associative law in ordinary algebra;
(3) the **distributive law** [*jiāohuàn lǜ/jiāohuàn dìnglǜ*].[52] This law includes two expressions. One of the two expressions is the same as the one in ordinary algebra (see A below), while the other does not apply to ordinary algebra (see B below): A. $p \cdot q \vee r . \equiv : p \cdot q . \vee . p \cdot r$, which corresponds to $x(y + z) = xy + xz$ in ordinary algebra; B. $p . \vee . q \cdot r : \equiv . p \vee q \cdot p \vee r$, whose corresponding expression in ordinary algebra, $x + (y \cdot z) = (x + y) \cdot (x + z)$, is not right.

We turn now to the **logic of classes** [*zǔ de luójī*].[53] In this lecture, we use α, β, γ to represent classes.[54] I begin by introducing several kinds of symbols.

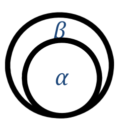


Diagram 1

As shown in Diagram 1, α is contained in β. We use the symbol, $\subset$, to represent this relation. Hence: $\alpha \subset \beta$. That is to say, every **member** [*zǔyuán*] of α is a member of β.

The multiplication/[logical] product of classes [*zǔ de chéngfǎ*]:

52 "*Hùhuàn*" literally means "two things switch their positions". "*Liánhé*" literally means "the act or state of uniting". "*Jiāohuàn*" is close in meaning to "*hùhuàn*" and so is not a good translation of "distributive". In contemporary literature, Chinese logicians often use "*fēnpèi lǜ*" to mean "the distributive law". "*Fēnpèi*" literally means "the act of sharing or distributing".

53 "*Zǔ*" literally means "group".

54 Lower-case Greek letters in *PM* stand for classes. Free lower-case Greek letters can be replaced by a class expression "$\hat{x}$(øx)". In the remainder of this lecture, the reader will find nothing of *PM*'s contextual definitions of class and relation-in-extension expressions.

$\alpha \cap \beta$ means the common part of both α and β.[55] See the overlapping area in Diagram 2 below.

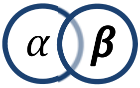


Diagram 2

The addition/[logical] sum of classes [*zǔ de jiāfǎ*]: $\alpha \cup \beta$ means the [logical] sum of α and β.[56]

Put $x \in \alpha \ = \ x\ is\ a\ member\ of\ \alpha$ Df. Given all these definitions, we can get:

$$\alpha \subset \beta \,.\!\equiv\ :x \in \alpha \,.\!\supset_x .\ x \in \beta^{57}$$
$$x \in \alpha \cap \beta \,.\!\equiv\ .\ x \in \alpha \ \cdot\ x \in \beta^{58}$$
$$x \in \alpha \cup \beta \,.\!\equiv\ :x \in \alpha \,.\!\vee\!.\, x \in \beta^{59}$$

We can see from the above expressions that the symbols used in class logic, such as $\subset$, $\cap$, and $\cup$, have some connections with the symbols that we introduced earlier, such as $\supset$, $\cdot$, and $\vee$.

Negative class [*fùzǔ*]. A class composed by members that are not members of another class is called the "negative class" of that class. Negative classes, which we use – to represent, are comparable to negative propositions [*fǒudìng mìngtí*][60] in propositional logic [*mìngtí luójī*]. Hence:

$$-\alpha \ = \ not\text{-}\alpha^{61}$$
$$x \in -\alpha \ = \ \sim(x \in \alpha)\ \mathrm{Df}$$
$$\alpha - \beta \ = \ \alpha \cap -\beta\ \mathrm{Df}$$
$$x \sim\ \in \alpha \ = \ \sim(x \in \alpha)\ \mathrm{Df}$$

Classes can be divided into three kinds:

[55] "Multiplication" is a literal translation of the Chinese term, "*chéngfǎ*". For Russell's notion of the "product" of classes, see *PM* 1: Summary of *22.

[56] "Addition" is a literal translation of the Chinese term, "*jiāfǎ*". For Russell's notion of the "sum" of classes, see *PM* 1: Summary of *22.

[57] Surprisingly, in both versions of lecture notes, the main connective in this expression is a horseshoe (see the images in Appendix 1). It should be the symbol for equivalence. Moreover, in this expression *x* should be bound universally. See *PM*, *22.1.

[58] See *PM*, *22.33.

[59] See *PM*, *22.34.

[60] Both "*fù*" and "*fǒudìng*" means "negative".

[61] A proper definition of "not-*α*" is this: $-\alpha \ = \ \hat{x}(x\!\sim\ \in \alpha)$ Df (see *PM*, *22.04).

(1) A class which has no members is called a "**null class**" [*língzǔ*], represented by Λ. E.g., the class which is composed by all even prime numbers except 2.
(2) A class which contains some thing(s) but not all things in the universe as its member(s) is called an "**existing class**" [*cúnzàizǔ*], represented by $\exists$.[62] E.g., the class which is composed by all prime numbers.
(3) A class which contains all things in the universe as its members is called a "**universal class**" [*yǔzhòuzǔ*],[63] represented by V. E.g., the class which is composed by all things that are identical to themselves.

Now I want to introduce the definition of **identity** [*xiāngděng*], which is comparable to the concept of equivalence that we discussed previously: $\alpha = \beta \,.\!\equiv.\, \alpha \subset \beta \cdot \beta \subset \alpha$ Df. From this definition we know that $\Lambda = -V$, and $V = -\Lambda$. Put $\exists!\ \alpha \,.\!=.\, (\exists x) \,.\, x \in \alpha$ Df.[64] This means that α contains at least one member, i.e., $\sim (\alpha = \Lambda)$. So we can get:

$$(\alpha \cap \beta = \Lambda) = \textit{no members of } \alpha \textit{ are members of } \beta,$$
$$(\exists!\ \alpha \cap \beta) = \textit{some members of } \alpha \textit{ are members of } \beta,$$
$$(\exists!\ \alpha - \beta) = \textit{some members of } \alpha \textit{ are not members of } \beta,$$
and [*hé*][65]
$$\alpha \subset \beta = \textit{all members of } \alpha \textit{ are members of } \beta.$$

People once thought that in the logic of classes nothing is beyond the four categorical [*zuìdà*] propositions.[66] Now we know that there are

62 The comment that says that not all things in the universe are members of an existing class is surprising. Russell himself accepted $\exists!$V. See *PM*, *24.52.

63 "*Líng*" literally means "zero". "*Cúnzài*" literally means "existence". "*Yǔzhòu*" literally means "universe".

64 See *PM*, *24.03.

65 In Wu's notes, there is a strange sentence between the third expression (i.e., the one that says that some members of α are not members of β) and the fourth expression (i.e., the one that says that all members of α are members of β): "The sum [*hé*] of these three expressions is". To compare, it is worth mentioning that a similar sentence in Mu's notes says that "these three *hé*", where "*hé*" seems to mean "and" rather than "sum". So Russell might simply mean that there is a fourth expression, in addition to the preceding three, as one of the four categorical propositions. The difference between these notes shows a different grasp of Russell's original speech and Zhao's oral translation.

66 "*Zuìdà*" literally means "maximum". The Chinese translation here is close to the meaning of "categorical". Russell's four categorical propositions seems to correspond to the four traditional Aristotelian categoricals: A: All *S* are *P*; E: No *S* are *P*;

many things that are outside them.

We end by discussing several syllogisms:

$$\alpha \subset \beta \ \cdot \ \beta \subset \gamma \,.\supset.\, \alpha \subset \gamma,$$

which is comparable to $p \supset q \ \cdot \ q \supset r \,.\supset.\, p \supset r$. Both $\subset$ in the logic of classes and $\supset$ in propositional logic are transitive.

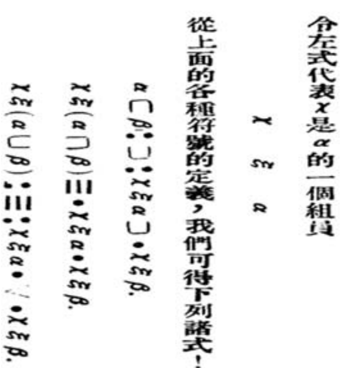
$$\alpha \subset \beta \ \cdot \ \exists!\ \alpha \cap \gamma \,.\supset.\, \exists!\ \beta \cap \gamma$$

$$\alpha \subset \beta \ \cdot \ \exists!\ \alpha - \gamma \,.\supset.\, \exists!\ \beta - \gamma$$

$$\alpha \subset \beta \ \cdot \ \beta \cap \gamma = \Lambda \,.\supset.\, \alpha \cap \gamma = \Lambda$$

APPENDIX I

RE PAGE 65. LEFT: IMAGE OF WU'S LECTURE NOTES IN 1921
RIGHT: OF MU'S LECTURE NOTES IN 1921 (2004 REPRINT)


令左式代表x是α的一個組員

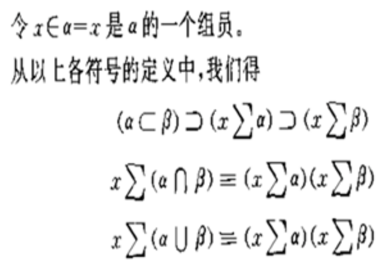
$x \varepsilon \alpha$

從上面的各種符號的定義，我們可得下列諸式！

$\alpha \subset \beta .\supset:. x \varepsilon \alpha .\supset. x \varepsilon \beta.$

$x \varepsilon (\alpha \cap \beta) .\equiv. x \varepsilon \alpha . x \varepsilon \beta.$

$x \varepsilon (\alpha \cup \beta) .\equiv:. x \varepsilon \alpha .\vee. x \varepsilon \beta.$


令 $x \in \alpha = x$ 是 α 的一个组员。

从以上各符号的定义中，我们得

$$(\alpha \subset \beta) \supset (x \sum \alpha) \supset (x \sum \beta)$$

$$x \sum (\alpha \cap \beta) \equiv (x \sum \alpha)(x \sum \beta)$$

$$x \sum (\alpha \cup \beta) \equiv (x \sum \alpha)(x \sum \beta)$$

APPENDIX II: GLOSSARY OF CHINESE TERMS[67]

B
bāohán 包含
biànliàng 变量
bùxiāngróng 不相容
C
cúnzàizǔ 存在组
D
děngzhí de 存在的
děngzhí de guānxì 等值的关系
dìngyì 定义
duìchèn 对称
F
fǎnshè 反射
fǎzé 法则
feīxíngshì de yuánlǐ 非形式的原理
fǒudìng mìngtí 否定命题
fǒurèn 否认
fǒuzhèng 否证
fúhào 符号
fùzǔ 负组
G
gōngfǎ 公法
gōnglǐ 公理
guāngxiàn 光线
H
hánshù 函数
hánjiàn 函件

I: Some *S* are *P*; and O: Some *S* are not *P*. In *PM* 1: 291, Russell remarked that the following argument cannot be captured by any Aristotelian syllogism: All horses are animals; therefore, the head of a horse is the head of an animal.

[67] All Chinese characters in the glossary are simplified Chinese characters.

hé 和
hùhuàn dìnglǜ 互换定律
hùhuàn lǜ 互换律
J
jiǎdìng 假定
jiāohuàn dìnglǜ 交换定律
jiāohuàn lǜ 交换律
jǐhé 几何
K
kōngjiān 空间
L
liánhé dìnglǜ 联合定律
liánhé lǜ 联合律
língzǔ 零组
luójī de daìshù 逻辑的代数
luójī de hé 逻辑的和
luójī de jí 逻辑的积
Luósù 罗素
M
máodùn dìnglǜ 矛盾定律
máodùn lǜ 矛盾律
mìngtí hánjiàn 命题函件
mìngtí hánshù 命题函数
mìngtí luójī 命题逻辑
N
Níudùn lìxué 牛顿力学
P
pái zhōngxiàng lǜ 排中项律
pǔtōng shùxué 普通数学
Q
qúnlùn 群论
S
sānduànlùn fǎ 三段论法
shíjiān 时间
shùlǐ luójī 数理逻辑
shùmù 数目
shùxué Luójī 数学逻辑
suànshù 算术
T
tóuyǐng jǐhé 投影几何
tuīlùn 推论
W
wù de shízaì 物的实在
wùlǐ de shítǐ 物的实体
wú zhōngxiàng dìnglǜ 无中项定律
wùtǐ 物体
wùzhì 物质
X
xiāngděng 相等
xíngshì de yuánlǐ 形式的原理
xúncháng daìshù 寻常代数
Y
yíxiàng 移项
yuánlǐ 原理
yuánzé 原则
yǔzhòuzǔ 宇宙组
Z
zhèngmíng 证明
zhēnlǐ hánjiàn 真理函件
zhēnlǐ hánshù 真理函数
zhǐchū 指出
zhíxiàn 直线
Zhūjūn 诸君
zìlǐ 自理
zǔ de chéngfǎ 组的乘法
zǔ de jiāfǎ 组的加法
zǔ de luójī 组的逻辑
zuìdà 最大
zǔyuán 组员

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