Documents

INTRODUCTION TO G. E. MOORE'S UNPUBLISHED REVIEW OF *THE PRINCIPLES OF MATHEMATICS*

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Several interesting themes emerge from G. E. Moore's previously unpublished review of *The Principles of Mathematics*. These include a worry concerning whether mathematical notions are identical to purely logical ones, even if coextensive logical ones exist. Another involves a conception of infinity based on endless series neglected in the *Principles* but arguably involved in Zeno's paradox of Achilles and the Tortoise. Moore also questions the scope of Russell's notion of material implication, and other aspects of Russell's claim that mathematics reduces to logic.

We here publish for the first time a lengthy review G. E. Moore wrote of Russell's *The Principles of Mathematics*. The review was intended for the German journal *Archiv für systematische Philosophie*,[1] and was likely composed in the late summer and/or early autumn of 1905.[2] Moore mentions the review in his autobiographical contribution to his Library of Living Philosophers volume, and indeed suggests that he spent a significant amount of

[1] MOORE published a review in the *Archiv* the previous year (1904) entitled "Philosophy in the United Kingdom for 1902", and seems to have intended it to be the first in a series of reviews covering British philosophy, including Russell's work. However, no further installments were published.

[2] In a letter to Russell dated 23 October 1905 (RA1 710.052987), Moore mentions having completed it. The review also cites RUSSELL's *OD*, published that month.

time during his 1904–11 fellowship in Edinburgh studying Russell's book but that he encountered some difficulty doing so.[3] It is not entirely clear why it was never published. Moore's self-perceived difficulty provides one possible explanation: perhaps he was not sufficiently happy with the result. Other explanations may involve his declining friendship and working relationship with Russell,[4] or negative feedback from the journal, from Russell, or elsewhere.[5] This has unfortunately postponed until now the opportunity to examine a direct interaction between these two seminal figures in the early history of analytic philosophy. Russell states his philosophical indebtedness to Moore in the Preface of the *Principles* (p. xviii). Moore's reciprocal admiration for Russell's work is evident even in the brief discussion in his autobiography, and the importance he saw in it is manifest in the opening line of the review. His estimation of the *Principles* as the most important philosophical book published in the UK in 1903 silently places it above his own *Principia Ethica*, published the same year.

Moore's prose in the review has his usual clear and straightforward style, and the review hardly requires additional commentary to be comprehensible. Yet it may be worth highlighting a few places where it might be of interest to contemporary researchers. Moore's questioning of Russell's claim to have established that the propositions of pure mathematics can be derived from logic when Russell excludes certain (apparently) mathematical truths in non-conditional form, can be seen as anticipating a criticism, now called "if-thenism", made by a number of later commentators. According to these critics, Russell did not so much succeed in deriving mathematics from logic, but rather

[3] Moore, "An Autobiography" (1942). The full passage reads: "At the beginning of the period I spent at Edinburgh what I was chiefly occupied with was trying to understand Russell's *Principles of Mathematics*, a thing which I found very difficult since the book was full of conceptions which were quite new to me. Many parts of it I never did succeed in understanding, but the earlier fundamental parts about logic I think I did in the end succeed in understanding pretty thoroughly. I was helped in understanding by the fact that, as I mentioned before, I did not merely think about and read over and over again what seemed to me to be of cardinal importance, but actually wrote a long review of the book."

[4] Their personal and working relationship seems to have been suffering from tensions from 1899 onward; for discussion, see Preti, "'He Was in Those Days Beautiful and Slim'" (2008), and Levy, *G. E. Moore and the Cambridge Apostles* (1979).

[5] Russell was at least aware the unpublished review existed, making note of it in his reply to Moore dated 25 October 1905 (RA1 710.053032).

derived only conditional claims with mathematical axioms as antecedents and mathematical theorems as consequents, thereby greatly reducing the achievement of his form of "logicism". As Moore hints, if the main reason Russell has for not counting, e.g., the claim that "the three angles of every triangle are equal to two right angles" as a proposition of pure mathematics, is that it cannot be established purely logically, the claim that all pure mathematics reduces to logic threatens to become trivial and uninteresting. However, unlike some later thinkers pushing this worry, Moore rightly connects Russell's contention that mathematical truths take the form of universal hypotheticals with his views about how *pure* mathematics gets applied in concrete situations, which helps explain what might otherwise appear to be an oddity in Russell's position.[6] It is perhaps worth mentioning that at one point (p. 144 below), Moore gets Russell's position wrong. Moore asserts that Russell holds all the propositions of the science of logic to be universal hypotheticals, when in fact Russell himself gives examples of non-mathematical truths of the science of logic not taking this form, such as "implication is a relation" (*PoM*, §10).

In the course of this discussion, Moore gives an argument (pp. 142–4) which is uniquely his, and which may be the most interesting part of the review for historians interested in the development of Moore's and Russell's philosophies. Moore makes note of Russell's admission that certain analyses he offers are meant only to meet mathematical standards of definition, not philosophical ones. Russell's definitions of the various cardinal numbers as classes of similar classes are not meant to capture what we ordinarily think when we consider, e.g., that $1 + 1 = 2$. Moore then argues that Russell's definition of 1 from logical primitives, while it may yield something equivalent to the usual notion of 1 (applying to all and only the same collections), still might not yield the very same property. The results Russell proves logically then might not be the very propositions we expected, but instead similar propositions using equivalent, but distinct, notions. To establish the

6 Later thinkers who push the "if-thenist" worry include PUTNAM, "The Thesis That Mathematics Is Logic" (1967); MUSGRAVE, "Logicism Revisited" (1977); COFFA, "Russell and Kant" (1981); and BOOLOS, "The Advantages of Honest Toil over Theft" (1994). There have been many responses, but for those stressing the importance of the pure/applied distinction, see GRIFFIN, "New Work on Russell's Early Philosophy" (1982); GALAUGHER, *Russell's Philosophy of Logical Analysis 1897–1905* (2013); and KLEMENT, "Russell's Logicism" (2018).

intended mathematical propositions themselves, one would need to be able to establish that the notions are identical and not merely equivalent, or at least show that the equivalences themselves are logical truths, which Moore worries Russell has not done. Moore's suggestion that there might be an entire system of purely logical properties, coextensive but not identical with those of mathematics, is a rather startling one, but from within Moore's own philosophy it is perhaps not an unnatural one. It is reminiscent of his famous "open question argument": if it is an open question whether not a class has the number 1 if and only if that class is a member of the class of all unit classes, then, arguably, having the number 1 and being in that class cannot be identical properties, even if they are coextensive. It is a difficult task to speculate what Russell's response might have been, but this argument has already sparked some debate as to whether or not it shows a deep disagreement between Russell and Moore during this period over the very nature and goals of analysis.[7]

Moore also calls into question (p. 146ff.) what Russell means by claiming that mathematics is deducible from logic, noting that he cannot simply mean that the truths of logic *imply* those of mathematics in Russell's own sense of *material implication*. In that sense, all truths imply all other truths. Moore further claims that material implication is not what we ordinarily mean by implication, and that Russell and others are committed to a different notion of implication. A full century of research into various forms of conditional logics would seem to support Moore's contention, although not as much his suggestion that this further notion is simple and analyzable. It is somewhat disappointing, however, that Moore does not go far in probing to what extent Russell's stronger notion of formal implication (*PoM*, §40) might be serviceable in this regard. Similarly, when it comes specifically to the deducibility of logic from mathematics, Moore does not consider the very straightforward interpretation that this means nothing more nor less than the existence of *deductions* using only logical axioms and inference rules for the various claims of mathematics.

In addition to these topics, the review contains praise (p. 152) for Russell's new theory of denoting in the newly published "On Denoting", and a surprisingly strong statement of disagreement (*ibid.*)

7 For contrasting standpoints, see Levine, "The Place of Vagueness in Russell's Mathematical Development" (2016), and Gandon, "Sidgwick's Legacy?" (2017).

with Russell's earlier theory of denoting concepts, though Moore demurs from elaborating. There is also a very nice statement (p. 146)—perhaps clearer than any similar statement made by Russell himself—of their shared anti-psychologism in logic, according to which the subject matter of logic is not anything to do with human reasoning, thought or psychology.

These topics make up roughly the first half of the review, and most of the remainder (pp. 152–64) is taken up by a lengthy discussion of issues related to infinity and continuity. Although this may not be evident in a contemporary context, Russell's discussion of then-new techniques for solving what had hitherto been regarded as "paradoxes" or "contradictions" of infinity would at the time have been seen as especially important. Moore makes note of two distinct conceptions of infinity discussed by Russell. The first is the notion—now often called "standard infinity" or "Frege infinity"—that applies to a class which does not have any of the inductive natural numbers $0, 1, 2, 3, \ldots$ for its cardinality. The second, which Russell calls "reflectiveness" but is now usually called "Dedekind infinity", is that which applies to a class which can be put in $1-1$ correspondence with a proper part of itself. Moore follows Russell in claiming that these two notions are equivalent, i.e., apply to all and only the same classes. This is now known to be an oversimplification, as in most forms of set theory, the equivalence is only true assuming the axiom of choice (or an equivalent assumption such as Russell's later multiplicative axiom), at least in the weak form of the axiom of countable choice. This is likely something he himself realized before Moore wrote the review, though it is unknown whether or not it was ever communicated to him.[8]

As he does so often, Moore questions whether or not either of these notions of infinity capture our pre-theoretic conception. He goes on to sketch yet another concept of infinity. He defines an *endless* series

[8] Russell had expressed doubts about results now known to be dependent on the axiom of choice as early as 1900 or 1901 (see *Papers* 3: 410, 596), and explicitly formulated his own equivalent multiplicative axiom in his 1904 manuscripts (*Papers* 4: 171–5). Explicit acknowledgement of the importance of this for the equivalence of the two notions of infinity came only after ZERMELO published his 1904 paper "Beweis, dass jede Menge wohlgeordnet werden kann", something that received much discussion in Russell's correspondence with Jourdain over the following year; see GRATTAN-GUINNESS, *Dear Russell—Dear Jourdain* (1977), pp. 46–9. In *Principia Mathematica*, theoretically possible cardinals that are Frege infinite, but Dedekind finite, are discussed and there called "mediate cardinals" (∗124).

as one "which has no beginning, or no end, or which has neither" (p. 154), and defines an infinite series as one that "*either* is itself endless *or* contains an endless series as part of itself" (p. 155). This definition applies to series, but as Moore notes (*ibid.*), one can obtain from it a concept applicable to those classes that are the fields of such series. In contemporary parlance, Moore's definition of infinity is essentially that of a class that has a subclass that can be partially ordered in a way that does not have maximal elements. Moore seems to think that this notion better captures at least one common notion of infinity used in pre-theoretic discourse, and since it is at least intensionally, if not extensionally, distinct from the other notions, one may rationally entertain doubts about whether or not they are equivalent. Moore seems to think it likely that it will turn out to be equivalent as well. In fact, Moore's notion is also not equivalent with the other notions unless the axiom of countable choice is assumed. All classes that are Dedekind infinite are Moore infinite, and all classes that are Moore infinite are Frege infinite, but the axiom is needed to complete the "circle" and obtain that all Frege infinite classes are also Dedekind or Moore infinite.[9] Perhaps Moore's lack of confidence with these issues—not feeling himself to have the technical chops to determine whether these equivalences hold—is one of the reasons he held back the review.[10]

Moore goes on to summarize how various conceptions of infinity can be used to pose Zeno's paradox of Achilles and the Tortoise (pp. 156–62), not only summarizing Russell's discussion but going on to restate what he takes to be a more natural formulation of the paradox involving his own notion of infinity stated in terms of endless series. So stated, the paradox involves the oddity that Achilles must traverse all of an endless series of locations before catching up with the Tortoise at a certain instant. As Moore sorts things out, however, it turns out not to be impossible to traverse every point of an endless series of locations with an endless series of instants, even if all those instants precede a given instant. A series may have an endless part without itself being endless, as, for example, with the series of rational numbers from 0 to 1 inclusive. This series has an end, namely 1, but

[9] For a discussion of these three, along with 23 other senses of "finite" and "infinite", and their mutual interrelations, see De la Cruz, Dzhafarov, and Hall, "Definitions of Finiteness Based on Order Properties" (2006).

[10] Thanks to Jim Levine for suggesting something along these lines to me.

there is an endless series, ½, ⅔, ¾, ⅘, etc., within it. Moore does a nice job summarizing both how the paradox seems puzzling when stated this way, and how it can be solved from within the new mathematics of series.

On the whole, Moore's review sheds new light on his philosophy, and perhaps on Russell's, and their interactions. Those interested in the topics of philosophical analysis, implication, infinity and other topics will no doubt find Moore's perspective valuable. It is not known whether or not Russell himself ever had a chance to read it, but either way it is a shame there is no official reply.

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