A NOTE ON *PRINCIPIA*'S ∗38 ON OPERATIONS

Gregory Landini
Philosophy / U. of Iowa
Iowa City, IA 52242–1408, USA
gregory-landini@uiowa.edu

Principia Mathematica ∗38 introduces what it calls "Relations and Classes Derived from a Double Descriptive Function". The notion of a relation-e (relation in extension) so derived is called an *operation*, and of course all dyadic relation-e theorems rely ultimately on the comprehension axiom schema for relations in intension given at ∗12.11. But in attempting to give a general pattern of definition, ∗38 uses the odd-looking "$x♀y$" which lends itself to the misconception that ♀ is itself an operation sign. The informal summary makes matters worse, writing "E! $(x♀y)$" which is ungrammatical. This paper argues that with P, R and S as relation-e variables and α, β, and μ as class variables, operations are comprehended by *wffs* such as "$P = x♀y$", "$\mu = \alpha♀\beta$" and "$P = R♀S$". Relying on triadic relations-e, I explain how the sign ♀ can be entirely avoided using comprehension. Along the way, puzzling cases such as $\underset{''}{♀}$ and $\underset{''}{\hat{♀}}$ are resolved.

1. INTRODUCTION: CURIOSITIES IN ∗38

In a recent issue of *Russell*, Nicholas Griffin offered a brief note discussing curiosities in ∗38 of *Principia Mathematica*. He points out that the section uses the sign ♀ and it "… occasionally appears in a metalanguage for the practical purpose of stating general principles of operator behaviour concisely."[1] For example, we find:

∗38.01 $\quad x♀ =_{df} \hat{u}\hat{y}(u = x♀y)$
∗38.02 $\quad ♀y =_{df} \hat{u}\hat{x}(u = x♀y)$

[1] Griffin, "A Sexist Joke in *Principia Mathematica*" (2020), p. 139.

$*38.03 \quad \alpha\, {}^{\text{"}}\!\!\female y =_{df} \female y\text{“}\alpha$

$*72.14 \quad \vdash x\female, \female x \in 1 \to \mathrm{Cls}$

$*182.01 \quad \hat{\female} =_{df} \hat{y}\hat{x}(y = x\female x).$[2]

The expression "$u = x\female y$" can be rewritten in many different ways such as "$P = x\female y$", "$\mu = \alpha\female\beta$" and "$P = R\female S$" where P, R and S are relation-e (relations in extension) variables and α, β, and μ are class variables. It is curious that we find the symbol ♀. In alchemy, it stands for copper and became associated with *female*. Since "♀" and the *wff* "$u = x\female y$" shift in so many ways, Griffin (*ibid.*) ponders whether *Principia* reflects the sexism of women being naturally fickle. I'm reminded of Virgil's "*Varium et mutabile semper femina*".[3] But using ♀ in an algebra was not without historical precedent, and Whitehead and especially Russell may have seen Leibniz use it in the following expression: "☽ $= 2\female\sqrt{(f\colon c)}$".[4] I can imagine cases where it would usefully stand in for $\pm$ as is found in the familiar quadratic *wff*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Indeed, *Principia* also uses the alchemy symbol ♂ for iron (associated with *male*). It also uses the alchemy sign ∇ for water, which appears at $*216.05$ so that $\nabla\text{‘}P$ is the derivative of a function. The sign ♂ is defined at $*314.03$ which concerns the development of Real numbers, not as classes of Rationals that form lower sections of Dedekind cuts, but as relations-e of relations-e. Like the Rationals (which are also relations-e of relations-e), this construction of Reals makes them immediately applicable for the measurement of physical magnitudes

2 Special thanks to the Editor and Arlene Duncan for dealing with this paper's difficult notation. Citations of *PM*'s starred numbers are not intended as direct quotations that keep its notation intact. Notation used in this paper and in such citations (apart from direct quotes) will depart from *PM* in the following ways: (1) brackets are always used for the scope of quantifiers; (2) both dots and brackets are used for punctuation, and dots are always used symmetrically and only where punctuation requires disambiguation; (3) the turnstile is used without dots since it means that what follows it is a thesis (axiom or theorem); (4) conjunction is an enlarged dot; (5) brackets are sometimes introduced for clarity (e.g., in $*113.02$ and $*116.01$).

3 *Aeneid* 4.569–70.

4 Gerhardt, ed., *Leibnizens Mathematische Schriften*, 7: 390. Thanks to Landon D. C. Elkind for pointing this out.

(spatial distance, temporal duration, height, weight, etc.). Magnitudes are construed as relations ("vectors") which can repeat (such as the relation implemented by a swinging pendulum clock) and which form series.[5] Though the use of the sign ♀ is certainly chameleonic, its use is no less important in *Principia* than ♂ and ∇.

Principia's *38 on operations has the title "Relations and Classes Derived from a Double Descriptive Function". The title is supposed to reflect *30 which concerns what it calls "Descriptive Functions". In a handwritten 1922 manuscript for Carnap, Russell informally explained operations, as follows:

> Here "♀" stands for any functional sign which can be put between two letters, e.g. $\alpha \cup \beta$, $R \mid S$, $\mu + \nu$, etc. Thus, $\alpha \underset{''}{+} 2$ e.g. will be the class of numbers resulting from adding 2 to each member of the class α.[6]

I wonder whether Carnap understood. It is quite important to realize the operation signs Russell means to mention are $\alpha\cup$ and $\cup\beta$, as well as $R \mid$ and $\mid S$, and $\mu +$ and $+\nu$. In fact, ♀ is not an operation sign. The operation signs are x♀ and ♀y. Thus, we expect:

$$\vdash u(x♀)y \equiv u = x♀y$$
$$\vdash u(♀y)x \equiv u = x♀y.$$

For example, with the roman letters "u" and "x" and "y" rewritten with lower-case Greek class variables, we obtain:

$$\vdash \mu(\alpha\cup)\beta \equiv \mu = \alpha\cup\beta$$
$$\vdash \mu(\cup\beta)\alpha \equiv \mu = \alpha\cup\beta.$$

Operation signs are relation-e signs, and they interface with the following definite description notation:

$$*30.01 \quad R`y =_{df} (\imath x)(xRy).$$

Principia explains that this should have its scope marker, and thus it

[5] WHITEHEAD discussed periodicity in his shilling shocker, *An Introduction to Mathematics* (1911), Ch. 12.
[6] LINSKY, *The Evolution of Principia Mathematica* (2011), p. 192.

should be:

$$[R\text{‘}x][f(R\text{‘}x)] =_{df} [(\imath y)(yRx)][f((\imath y)(yRx))]. \qquad (PM\ 1{:}\ 233)$$

There are many different sorts of instances of this definition for relation-e variables P, R and S and class variables such as α, β, and μ. For example, there are $R\text{‘}\alpha$ and $R\text{‘}S$. The following reveals the connection to operation signs $x♀$ and $♀y$ as relations-e:

$$x♀\text{‘}y =_{df} (\imath u)(u(x♀)y)$$
$$♀y\text{‘}x =_{df} (\imath u)(u(♀y)x).$$

This goes a long way toward explaining $*38$.

Clarifying the connection between $*30$ and $*38$ helps immensely. Nonetheless, the following admittedly informal passage introducing $*38$ remains a curiosity:

> A double descriptive function is a non-propositional function of two arguments, such as $\alpha\cap\beta$, $\alpha\cup\beta$, …, $R \mid S$, $\alpha \upharpoonleft R$…. In order to deal with all analogous cases at once, we shall in this number adopt the notation
>
> $x♀y$,
>
> where "♀" stands for any such sign as $\cap$, $\cup$, $\dot{\cap}$, $\dot{\cup}$, $|$, $\upharpoonleft$, $\restriction$, $\downharpoonright$, or any functional sign to be hereafter defined and satisfying the condition
>
> $(x, y) \,.\, E!\,(x♀y)$.
>
> The derived relations and classes with which we shall be concerned may be illustrated by taking the case of $\alpha\cap\beta$. The relation of $\alpha\cap\beta$ to β will be written $a\cap$, and the relation of $\alpha\cap\beta$ to α will be written $\cap\beta$. Thus we shall have
>
> $\vdash \alpha\cap\beta = \alpha\cap\text{‘}\beta = \cap\beta\text{‘}\alpha.$ (*PM* 1: 296)

As we see from the above passage, the use of $x♀y$ is illustrated in a case where we have the expression "$\alpha\cap\beta$". Here $\alpha\cap$ is the relation-e of $\alpha\cap\beta$ to β. It is properly expressed using $*30.01$ as follows:

$$\alpha\cap\text{‘}\beta =_{df} (\imath\mu)(\mu = \alpha\cap\beta).$$

And thus one expects to have:

$$(\alpha, \beta)\ E!\ \alpha\cap\text{‘}\beta$$

i.e., $(\alpha, \beta)\ \mathrm{E!}\,(\imath\mu)(\mu(\alpha\cap)\beta)$.

The expression "$(x, y)\mathrm{E!}\,(x♀y)$" is ungrammatical. Since "$x♀y$" is not a definite description, "$\mathrm{E!}\,(x♀y)$" is clearly ungrammatical. The sign "E!" must flank a definite description in accordance with

$$*14.02 \quad \mathrm{E!}\,(\imath y\phi y) \ =_{df} (\exists b)(\phi y \equiv_y y = b).$$

Thus, one expects "$\mathrm{E!}\,(\imath\mu)(\varphi\mu)$" or "$\mathrm{E!}\,(\imath R)(\phi R)$". The expression "$(x, y)\ \mathrm{E!}\,(x♀y)$" is certainly curious.[7]

Happily, *Principia* never uses the ungrammatical "$\mathrm{E!}\,(x♀y)$". What is expected is

$$(x, y)\ \mathrm{E!}\, x♀‘y.$$

i.e., $$(x, y)\ \mathrm{E!}\,(\imath u)(u(x♀)y).$$

In fact, this is what we get in the following theorem:

$$*38.12 \quad \vdash \mathrm{E!}\, x♀‘y \,.\, \mathrm{E!}\, ♀y‘x.$$

So the oddity of "$\mathrm{E!}\,(x♀y)$" is rather unimportant, as long as one is not misled by it.

Is it simply an error, typographical or otherwise, of omitting an inverted apostrophe? Is it an outright error? If so, was it Russell's or Whitehead's? Quine remarks that Whitehead invented the notation for operations and was quite pleased by it.[8] Based on such comments, Grattan-Guinness reported that Whitehead originated the general strategy of using operation expressions.[9] Sheffer's notes taken from Russell's 1910 Cambridge mathematical logic course corroborate this. He explicitly wrote: "Notation invented by Dr. Whitehead."[10] So it is clear that Whitehead is likely to have written *38. But once again it is curious that in Whitehead's 23 August 1910 letter to Russell, we find

[7] Observe that it cannot be that "$\mathrm{E!}\,(\alpha\cap\beta)$" was supposed to have been "$\exists!\,(\alpha\cap\beta)$" which, by definition *24.03, would say that the intersection is not empty. The class $\alpha\cap\beta$ might well be empty without impacting the operations $\alpha\cap$ and $\cap\beta$ that are involved.

[8] *The Time of My Life* (1985), p. 84.

[9] *The Search for Mathematical Roots, 1870–1940* (2000), p. 394.

[10] See Levine and Linsky, eds., *Bertrand Russell's Lectures 1910–1914* (in progress).

the following:

> *Note* sentence (about operations) inserted in page 312 and heading of page 313.[11] I like the sentence and don't care for heading—but don't like "Double Descriptive Functions" as a heading. Suggests an investigation which is not there. Also on page 311 note that E! $(x♀y)$ is not defined anywhere. Does this matter? Its meaning is obvious. Do as you like about it. I am inclined to leave it. In fact have done so, after a few futile attempts.[12]

Whitehead himself calls attention to the oddity of "E! $(x♀y)$", and he mentions explicitly that it is not defined anywhere. Whitehead says he is inclined to leave it on grounds that its meaning is "obvious". Its meaning, as we have seen, is not at all obvious. In fact it is ungrammatical. But his comments strongly suggest that it was not a typo of omitting an inverted apostrophe. If Whitehead was vetting a passage and found such a typo, he, or Russell, would have easily fixed it as "E! $x♀‘y$". Only a little space and the inverted apostrophe need be added, and this couldn't have interfered seriously with respacing and resetting a page that was already typeset. Surely, if Whitehead regarded it as a typo he wouldn't have advocated leaving it. The better explanation is that Whitehead intended to say that though he hasn't found a formal definition for "E! $(x♀y)$", he thinks that it doesn't matter and that it is useful for the purposes of making introductory comments for $*38$.

This leaves open the question of whether "E! $(x♀y)$" could have been given a separate definition. It could not. Observe that in *Principia*, we don't find a definition such as:

$$\text{E!}\,\hat{x}\phi x =_{df} (\exists f)(f!\,x \equiv_x \phi x).$$

Such a definition would clash with the definition $*14.01$ which defines "E! $(\imath x)(\varphi x)$". There is no expression "E! (y)" in *Principia* and thus no way to regard "E! $(\imath x)(\varphi x)$" as if it put an expression "$(\imath x)(\phi x)$" in the position of "y". The expression is taken as a whole. For the same

[11] These are page numbers of the then as yet unpublished first edition of Volume 1 of *PM*. The published running head on p. 313 (2nd ed., p. 297) is "Operations". There is no definition in the 2nd edition either.

[12] Griffin and Gandon, eds., *The Whitehead–Russell Correspondence* (in progress). The original letters may be viewed in the Russell Archives.

reason, one cannot give a fresh definition for "E! $(x♀y)$". The point is corroborated when we find comments after:

$*37.05 \quad \text{E!!}\, R``\beta =_{df} (y)(y \epsilon \beta . \supset . \text{E!}\, R`y).$

Whitehead and Russell explain that they don't write "E!! $R``\beta$" because there is no expression "E!! $\mu$" with the expression "$R``\beta$" in the position of "μ". The expression "E!! $R``\beta$" is taken as a whole.

No fresh definition of "E! $(x♀y)$" is viable. But Whitehead's letter says that he made "a few futile attempts". It is, thus, not clear to what end his attempts were made. I suspect that he meant that he had made some futile attempts at avoiding ♀ altogether. We shall see that one can avoid the use of ♀ but that the explanation requires the introduction of triadic relations-e, something that Whitehead and Russell wanted to postpone until Volume 4 on Geometry. This best explains Whitehead's remark about futility.

Returning to "$u = x♀y$", we can see that things begin to be clear once it is rewritten with appropriate class or relation-e variables. In the case of $\alpha\cap\beta$, we have $\mu = \alpha\cap\beta$. Thus we get:

$$\alpha\cap =_{df} \hat{\mu}\hat{\nu}(\mu = \alpha\cap\nu)$$
$$\cap\beta =_{df} \hat{\mu}\hat{\nu}(\mu = \nu\cap\beta).$$

Using definition $*30.01$, we have

$$\alpha\cap`\beta =_{df} (\imath u)(\mu\,(\alpha\cap)\beta)$$
$$\cap\beta`\alpha =_{df} (\imath u)(\mu\,(\cap\beta)\alpha)$$
$$\vdash \alpha\cap\beta = \alpha\cap`\beta = \cap\beta`\alpha.$$

Alternatively put, this last is:

$$\vdash \alpha\cap\beta = (\imath\mu)(\mu(\alpha\cap)\beta) = (\imath\mu)(\mu(\cap\beta)\alpha).$$

But one must be on the lookout for a great many different cases where definitions require rewriting with class variables and relation-e variables.

What makes $*38$ difficult is that the expression "$u = x♀y$" is itself a stand-in and must be rewritten, and yet the definitions at $*38$ use all roman letters. One cannot blame the typesetters. In fact, there is no right answer to the question of what letters to use. There are many instances. For example:

$\mu = \alpha ♀ \beta$	e.g., $\mu = \alpha \cap \beta$
$R = x ♀ y$	e.g., $R = x \downarrow y$
$P = \alpha ♀ R$	e.g., $P = \alpha \upharpoonleft R$
$P = x ♀ y$	e.g., $P = R \mid S$.

Note that in the first case above we find the lower-case Greek "μ" is for a class variable and not the roman "u". These use the lower-case Greek μ, α, β, σ, etc., for class variables, or with R, S, P, T, etc., for relation-e variables. We don't have to go into details of the no-classes and no-relation-e theories to see this. But note that bindable variables for classes and relations-e are defined:

$*20.07$	$(\alpha) f\alpha =_{df} (\phi)(f\{\hat{z}\phi!\, z\})$
$*20.071$	$(\exists\alpha) f\alpha =_{df} (\exists\phi)(f\{\hat{z}\phi!\, z\})$
$*21.07$	$(R) fR =_{df} (\phi)(f\{\hat{x}\hat{y}\phi!\,(x,y)\})$
$*21.071$	$(\exists R) fR =_{df} (\exists\phi)(f\{\hat{x}\hat{y}\phi!\,(x,y)\})$.

These definitions are for classes of *individuals* (of whatever simple type), and new definitions have to be written for relative types of classes of classes, classes of relations-e and relations-e of classes and relations-e of relations-e. There are:

cls$*20.07$	$(\alpha) f\alpha =_{df} (\phi)(f\{\hat{\mu}\,\phi!\,\mu\})$
rel$*20.07$	$(\alpha) f\alpha =_{df} (\phi)(f\{\hat{R}\phi!\, R\})$
cls$*21.07$	$(R) fR =_{df} (\phi)(f\{\hat{\alpha}\,\hat{\beta}\,\phi!\,(\alpha,\beta)\})$
rel$*21.07$	$(R) fR =_{df} (\phi)(f\{\hat{P}\,\hat{T}\phi!\,(P,T)\})$
clsrel$*21.07$	$(R) fR =_{df} (\phi)(f\{\hat{\alpha}\,\hat{P}\phi!\,(\alpha,P)\})$
relcls$*21.07$	$(R) fR =_{df} (\phi)(f\{\hat{P}\hat{\beta}\,\phi!\,(P,\beta)\})$.

As we see, such rewriting is quite important. *Principia* doesn't do it, and assumes that it is understood.

Properly understanding *Principia* requires readers to know how to rewrite its many definitions. Some definitions are particularly concerning. Consider the following:

$$
\begin{array}{ll}
*52.01 & 1 =_{df} \hat{\alpha}\,(\exists x)(\alpha = \iota`x). \\
cls*52.01 & 1 =_{df} \hat{\alpha}\,(\exists \mu)(\alpha = \iota`\mu) \\
rel*52.01 & 1 =_{df} \hat{R}(\exists \mu)(\alpha = \iota`R).
\end{array}
$$

Though we can often recognize the differences in context, formally speaking the above are not proper definitions because it is illicit to have different *definientia* for the same *definiendum*. Happily, *Principia* does provide notations that solve the problem, but not until *63–*65 which introduce notations of *relative types* of classes and relations-e. For example:

$$
\begin{array}{ll}
*65.01 & \alpha_y =_{df} \alpha \cap \mathbf{t}y \text{[13]} \\
cls*65.01 & \alpha_\sigma =_{df} \alpha \cap \mathbf{t}\sigma \\
*65.02 & \alpha_{(y)} =_{df} \alpha \cap \mathbf{tt}y \\
*63.01 & \mathbf{t}y =_{df} \iota`y \cup - \iota`y \\
*63.02 & \mathbf{t}_0\alpha =_{df} \alpha \cup - \alpha.
\end{array}
$$

As we can see, $\mathbf{t}y = \hat{z}(z = y \vee z \neq y)$. It is the universal class of individuals of the same simple type as y. If we restore *simple type* indices, it is this

$$\mathbf{t}y^t = \hat{z}^t(z^t = y^t \vee z^t \neq y^t).$$

One must never conflate $\mathbf{t}y$, which is the notation for a universal class of individuals, with y^t, which restores a simple type index to an individual variable. Restoring simple type indices to *52.01 yields

$$1 =_{df} \hat{\alpha}\,(\exists x^t)(\alpha = \iota`x^t).$$

This is for a class of classes of individuals (of whatever simple type t may be). There are also:

$$
\begin{array}{ll}
indiv*52.01 & 1_{(y)} =_{df} \hat{\alpha}\,(\exists x)(\alpha_y = \iota`x). \\
cls*52.\ 01 & 1_\sigma =_{df} \hat{\alpha}\,(\exists \mu)(\mathbf{t}_0\sigma = \iota`\mu).
\end{array}
$$

[13] I have omitted the inverted apostrophe in $\mathbf{t}`y$ and boldfaced the "t" instead, writing $\mathbf{t}y$. This is to emphasis that $\mathbf{t}`y$ is not a definite description and that therefore *30.01 does not apply to it.

One cannot apply $*65.02$ to $1_{(y)}$. Definitions of $*63$–$*65$ properly apply only to class and relation-e variables.

Roman letter "*u*" (for individuals of some or other simple type) as used in $*38$ is particularly misleading because it is hard to imagine a good case.[14] When ♀ stands in for the sign $\cap$ we have a class $\alpha\cap\beta$, and the roman "*u*" must give way to the Greek "μ". Thus:

$$\textit{clscls}{*}38.01 \quad \alpha♀ =_{df} \hat{\mu}\hat{\nu}(\mu = \alpha ♀ \nu)$$
$$\textit{clscls}{*}38.02 \quad ♀\beta =_{df} \hat{\mu}\hat{\nu}(\mu = \nu ♀ \beta).$$

The following are then expected:

$$\alpha\cap =_{df} \hat{\mu}\hat{\nu}(\mu = \alpha\cap\nu)$$
$$\cap\beta =_{df} \hat{\mu}\hat{\nu}(\mu = \nu\cap\beta).$$

When ♀ stands in for the sign $\downarrow$, $x{\downarrow}y$ is for a relation-e. We can't use roman "*u*" but need "*R*". Thus,

$$\textit{relindiv}{*}38.01 \quad x♀ =_{df} \hat{R}\hat{y}(R = x♀y)$$
$$\textit{relindiv}{*}38.02 \quad ♀y =_{df} \hat{R}\hat{x}(R = x♀y).$$

The following are expected:

$$x\downarrow =_{df} \hat{R}\hat{y}(R = x\downarrow y)$$
$$\downarrow y =_{df} \hat{R}\hat{x}(R = x\downarrow y)$$

$$x\downarrow \text{‘} y =_{df} (\imath R)(R(x\downarrow)y)$$
$$\downarrow y\text{‘}x =_{df} (\imath R)(R(\downarrow y)x)$$

$$\vdash (x, y)\mathrm{E!}\,(\imath R)(R = x\downarrow y).$$

When ♀ stands in for $\alpha{\downarrow}\beta$ we again expect:

$$\textit{relcls}{*}38.01 \quad \alpha♀ =_{df} \hat{R}\hat{\beta}(R = \alpha♀\beta)$$
$$\textit{relcls}{*}38.02 \quad ♀\beta =_{df} \hat{R}\hat{\alpha}(R = \alpha♀\beta).$$

[14] It may be noted that no help is found in the 1922 manuscript (in LINSKY) for Carnap in which Russell explained many of the important definitions found in *Principia*. His entry for $*38$ also uses the roman "*u*".

There are also cases where ♀ stands in for the sign $|$, and we have $R \mid S$ of relative product. For the relative product of two relations R and S, we have: $x\ (R\,|S)\ y \equiv_{x,y} \exists z(xRz\ .\ zSy)$. Thus,

$$\textit{relrel}{*}38.01 \qquad R♀ =_{df} \hat{T}\hat{S}(T = R♀S)$$
$$\textit{relrel}{*}38.01 \qquad ♀S =_{df} \hat{T}\hat{R}(T = R♀S).$$

The following are expected:

$$R\,| =_{df} \hat{T}\hat{S}(T = R \mid S)$$
$$|\,S =_{df} \hat{T}\hat{R}(T = R \mid S)$$
$$R\,|‘S =_{df} f(\imath T)(T(R\,|\,)S)$$
$$|\,S‘R =_{df} (\imath T)(T(|\,S)R)$$
$$\vdash \mathrm{E!}\,(\imath T)(T = R \mid S).$$

There are mixed cases where ♀ stands in for the sign $\upharpoonleft$ and we have the relation-e that is $\alpha \upharpoonleft T$, which is a relation-e whose domain is restricted to members of the class α. Accordingly, we have:

$$\textit{relrelcls}{*}38.01 \qquad \alpha♀ =_{df} \hat{T}\widehat{R}(T = \alpha♀R)$$
$$\textit{relrelcls}{*}38.01 \qquad ♀R =_{df} \hat{T}\widehat{\alpha}(T = \alpha♀R).$$

It is now the following that are expected:

$$\alpha \upharpoonleft\ =_{df} \hat{T}R(T = \alpha \upharpoonleft R)$$

$$\alpha \upharpoonleft ‘R =_{df} (\imath T)(T(\alpha \upharpoonleft)R)$$

$$\vdash (\alpha, R)\mathrm{E!}\,\alpha \upharpoonleft ‘R$$
$$\text{i.e.,} \quad \vdash (\alpha, R)\mathrm{E!}\,(\imath T)(T = \alpha \upharpoonleft R).$$

This is just the tip of an iceberg in *Principia* when it comes to definitions. But you see the point.

2. DOING WITHOUT ♀

We are still left with the sign ♀. Happily, it can be eliminated and an even greater clarity emerges. But ultimately this will require the introduction of triadic relation-e signs. Operations of $*38$ are simply dyadic relations-e that can be introduced by comprehension of triadic relations in terms of *wffs* $\phi(u, x, y)$. Thus we can use the schematic form $\phi(u, x, y)$ instead of the form "$u = x♀y$" and thereby eliminate "♀" altogether. Whitehead and Russell must certainly have understood this, but, as noted earlier, their official plan was to postpone the introduction of notations for triadic (and higher *adicity*) relations-e until Volume 4 on Geometry. This is explicit in the following:

> … relations between more than two terms will be distinguished as *multiple* relations, or (when the number of their terms is specified) as triple, quadruple, … relations, or as triadic, tetradic, … relations. Such relations will not concern us until we come to Geometry. For the present, the only relations we are concerned with are *dual* relations. (*PM* 1: 26)

Of course by "relation" here, what is meant is relation-e. The existence of relations-in-intention is understood. In his 1924 "Logical Atomism", Russell explained about geometry:

> How far it is necessary to go up the series of three-term, four-term, five-term … relations I don't know. But it is certainly necessary to go beyond two-term relations. In projective geometry, for example, the order of points on a line or of planes through a line requires a four-term relation.
> (*LK*, p. 332; **26** in *Papers* 9, p. 169)

Indeed, the point had already been made in Chapters 45–8 of *The Principles of Mathematics*.[15] In his 1914 *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*, Russell made the point again, writing:

> We have already seen how the supposed universality of the subject-predicate form made it impossible to give a right analysis of serial order, and therefore made space and time unintelligible. But in this case it was only necessary to admit relations of two terms. (*OKEW*$_3$, p. 61)

[15] The epistemic question remains of interest. Is there a relational adicity that is accessible to the human mind such that the understanding of all relational structures never needs to appeal to relations of higher adicity? It seems that nothing in logic assures this. There may well be relational structures that are epistemically inaccessible.

Obviously, one shouldn't think that *Principia* hoped to eliminate all but dyadic relations-e. And just as obviously, the work is committed to an ontology of relations in intension.

Principia has $f!(x_1, \ldots, x_n)$ as a *wff*, and its use indicates a relation (in intension) of *adicity* n, for any finite n whatsoever. Where $f!$ is not free in the *wff* ϕ, *Principia*'s comprehension-axiom schemas, given in Volume 1, assure the existence of properties and *dyadic* relations in intension as follows:

$*12.1 \quad \vdash (\exists f)(\phi x \equiv_x f!x)$

$*12.11 \quad \vdash (\exists f)(\phi xy \equiv_{x,y} f!(x, y))$

Where $f!$ is not free in the *wff* ϕ, Whitehead and Russell surely realized that mathematical logic needs

$*12.n \quad \vdash (\exists f)(\phi(x_1, \ldots, x_n) \equiv_{x_1,\ldots,x_n} (x, y) f!(x_1, \ldots, x_n)).$

For example, in *Principia* we find the following telling passage:

> In dealing with relations between more than two terms, similar assumptions would be needed for three, four, … variables. But these assumptions are not indispensable for our purpose, and are therefore not made in this work. (*PM* 1: 167)

The comment means to speak of relation-e, and the reference to "this work" means Volumes 1–3. One must not neglect that there was to be a Volume 4 where relations in intention of higher *adicity* would be needed with accompanying notations for such relations-e. In *Principia* one can always discern whether one is dealing with a dyadic relation-e sign R used in a *wff* such as xRy, as opposed to a dyadic relation in intension sign $f!$ in the *wff* $f!(x, y)$. A dyadic relation-in-intension sign always has the exclamation $f!$, $\phi!$, etc. Indeed, signs such as f and φ without the exclamation are schematic for *wffs*. We know from the letters x and y being on the left and right of "xRy" that the free occurrence of the relation-e variable "R" is a stand-in for $\hat{z}\hat{w}\phi(z, w)$, so that "xRy" abbreviates the expression "$x\{\hat{z}\hat{w}\phi(z, w)\}y$".

A question arises, however, as to how best to render notations for relations-e that are of higher *adicity*. There is a letter from Whitehead in which he expresses this concern to Russell. He explains that the notations of relations-e at ∗21, which put variables on left and right sides, are too limiting. The expression "xRy" indicates that "R" is a dyadic relation-e sign by putting the variables on either side. What then do the authors write for triadic relations-e? Whitehead's letter of 27 April 1905 to Russell praises Veblen's work in geometry and raises the question. Whitehead wrote:

> He [Veblen] proves that Descriptive Geometry is the study of the properties of a *single three-term* relation, and the points are the field of this relation. Of course he does not quite know that this is his point of view; but it is the gist of it, and it throws a flood of light on the whole subject.
>
> Now this advance makes it urgent that we produce a notation suitable for three-term relations. In fact since four-term relations occur (harmonic relations etc.) we want a notation suitable for relations with any number of terms.... I should propose to keep xRy as a simplification in this instance of the general form, *but otherwise use the new symbolism*.[16]

One solution to providing a notation for relations-e of higher *adicity* is quite simple. Write

$$< a_1, \ldots, a_n > \epsilon\ \phi! =_{df} \phi!\,(a_1, \ldots, a_n).$$

Now free occurrence of R stand in for $\hat{x}\hat{y}\phi!\,(x, y)$, and we find

$$*21.02 \quad a\{\phi!\,(\hat{x}, \hat{y})\}b =_{df} \phi!\,(a, b).$$

This can be preserved in the dyadic case with

$$a\{\phi!\}b =_{df} < a, b > \epsilon\ \phi!$$

In this way triadic relations-e are expressed.[17]

[16] In Gandon and Griffin (in progress).

[17] Note that *Principia*'s no-relations-e theory is clearly not emulating relations-e as classes of ordered pairs. This is made explicit at *PM* 1: 26. Nothing in my notations suggests otherwise. Modern set-theorists assume a metaphysics of sets and imagine that relation-in-extension are sets of ordered pairs. They adopt the Weiner–Kuratowski definition which can have no analog in *Principia*'s simple types without ruling out its very important non-homogeneous relations (and their accompanying relations-e).

Once notations for triadic (and higher) relations-e are in place, it is easy to see that the use of ♀ in the definitions of *38 of operations is avoidable in favour of a straightforward appeal to the comprehension of relations in intension. Comprehension *12.11 together with the definitions *20.07.071 of the no-classes theory, yields:

$$\vdash (\alpha)(\exists f)(\phi(\mu, \alpha, \nu) \equiv_{\mu,\nu} f!(\mu, \nu))$$
$$\vdash (\beta)(\exists g)(\phi(\mu, \nu, \beta) \equiv_{\mu,\nu} g!(\mu, \nu)).$$

The following are instances of the above:

$$\vdash (\alpha)(\exists f)(\mu = \alpha{\cap}\nu \equiv_{\mu,\nu} f!(\mu, \nu))$$
$$\vdash (\beta)(\exists g)(\mu = \nu{\cap}\beta \equiv_{\mu,\nu} g!(\mu, \nu)).$$

After existential instantiation, $f!$ is $\alpha\cap$ and $g!$ is $\cap\beta$. Here we only needed dyadic comprehension *12.11 of relations in intension. But to see the point another way, observe that if we appeal to comprehension of triadic relations in intension, together with definitions *20.07.071, we get the general theorem:

$$\vdash (R)(R = \hat{\mu}\hat{\nu}\hat{\sigma}\,\phi(\mu, \nu, \sigma) \,.{\supset}.\, (\alpha).\exists S)(S = \hat{\mu}\hat{\sigma}(< \mu, \alpha, \sigma > \epsilon\ R))\ \boldsymbol{.}$$
$$(\beta)(\exists T)(T = \hat{\mu}\hat{\nu}(< \mu, \nu, \beta > \epsilon\ R)).$$

This theorem assures that for any triadic relation-e that is $\hat{\mu}\hat{\nu}\hat{\sigma}\,\phi(\mu, \nu, \sigma)$ we can form various dyadic operations. For the relations-e that are the operations $\alpha\cap$ and $\cap\beta$, we have the *wff* "$\mu = \alpha{\cap}\beta$" that is an instance of the schema $\phi(\mu, \alpha, \beta)$. Thus, we have the following cases:

$$(R)(R = \hat{\mu}\hat{\nu}\hat{\sigma}(\mu = \nu{\cap}\sigma) \,.{\supset}.\, (\alpha)\ (\exists S)(S = \hat{\mu}\hat{\sigma}(< \mu, \alpha, \sigma > \epsilon\ R))\ \boldsymbol{.}$$
$$(\beta)(\exists T)(T = \hat{\mu}\hat{\nu}(< \mu, \nu, \beta > \epsilon\ R)))$$
$$\alpha\cap\ =_{df} (\imath S)(S = \hat{\mu}\hat{\sigma}(\mu = \alpha{\cap}\sigma))$$
$$\cap\beta\ =_{df} (\imath S)(S = \hat{\mu}\hat{\nu}(\mu = \nu{\cap}\beta)).$$

In the case where we have relations-e that are the operations $x\downarrow$ and $\downarrow y$, we have the theorem schema:

$$\vdash (R)(R = \hat{P}\hat{x}\hat{y}\,\phi(P,x,y) \,.\supset.\, (y)(\exists S)(S = \hat{P}\hat{y}(<P,x,y> \epsilon\, R)) \,\centerdot$$
$$(x)(\exists T)(T = \hat{P}\hat{x}(<P,x,y> \epsilon\, R)).$$

Here is the *wff* "$T = x \downarrow y$" that is an instance of the schema $\phi(P,x,y)$. Thus:

$$\vdash (R)(R = \hat{T}\,\hat{x}\hat{y}\big(T = x \downarrow y\big) \,.\supset.\, (x)(\exists P)(P = \hat{T}\hat{y}(<T,x,y> \epsilon\, R)) \,\centerdot$$
$$(y)(\exists Q)(Q = \hat{T}\hat{x}(<T,x,y> \epsilon\, R))).$$
$$x\!\downarrow\, =_{df} (\imath P)(P = \hat{T}\hat{x}(T = x\!\downarrow\! y))$$
$$\downarrow\! y \, =_{df} (\imath Q)(Q = \hat{T}\hat{x}(T = x\!\downarrow\! y)).$$

As we can see, in this way "♀" may be eliminated altogether.

The notions of $+\beta$ and *addition* $+_c 1$ are defined in *Principia's* ∗110 by appeal to operations formed from *wffs*. There is the definition of the *arithmetical class-sum* of two classes:

∗110.01 $\quad \alpha + \beta =_{df} \downarrow(\Lambda \cap \beta)``\iota``\alpha \cup (\Lambda \cap \alpha)\downarrow``\iota``\beta).$

Unlike the union $\alpha \cup \beta$, the above does not require that α and β be of the same relative type. Moreover, it acts as though α and β had no common members. For example, let $\alpha = \{a, b\}$ and $\beta = \{a, c\}$. Thus:

$$\alpha \cup \beta = \{a, b, c\}$$
$$\alpha + \beta = \{\Lambda\!\downarrow\!\iota`a, \Lambda\!\downarrow\!\iota`b, \iota`a\downarrow\!\Lambda, \iota`c\downarrow\!\Lambda\}.$$

Note that $\alpha + \beta$ is a class of relations-e. Using this definition of "+" *Principia* goes on to:

∗110.02 $\quad \mu +_c \nu =_{df} \hat{\xi}(\exists \alpha, \beta)(\mu = \mathrm{N_oc}`\alpha \,\centerdot\, \nu = \mathrm{N_oc}`\beta \,\centerdot\, \xi \,\mathrm{sm}\, \alpha + \beta).$

The operation $+_c 1$ is the relation-e introduced from the *wff* "$\mu = \alpha +_c 1$" as follows:

$$+_c 1 =_{df} \hat{\mu}\hat{\alpha}(\mu = \alpha +_c 1).$$[18]

[18] Grattan-Guinness, p. 402, mistakenly suggests that the operation $+_c \nu$ was used to define $\mu +_c \nu$.

In this case, it is $\mu(+_c 1)\alpha$ that has the pattern $\mu ♀ \alpha$. Thus we get:

$$+_c 1`\alpha =_{df} (\imath\mu)(\mu(+_c 1)\alpha)$$
$$\vdash (\alpha)E!(\imath\mu)(\mu = \alpha +_c 1)$$

At *120 of Volume 2, the ancestral $(+_c 1)_*$ of the operation $+_c 1$ is used in order to define the class of inductive cardinals each of which has the relative type ξ. And we find:

$$*120.011 \quad N_\xi C \text{ induct} =_{df} \hat{\alpha}\{\alpha(+_c 1)_* 0_\xi\}$$

Once we understand the pattern, we can always eliminate. Where the *wff* $\mu = \nu +_c \sigma$ is an instance of the schema $\phi(\mu, \nu, \sigma)$, we get:

$$(R)(R = \hat{\mu}\,\hat{\nu}\,\hat{\sigma}(\mu = \nu +_c \sigma) \,.\supset.\, (\exists T)(T = \hat{\mu}\,\hat{\nu}(< \mu, \nu, 1 > \epsilon R).$$
$$+_c 1 =_{df} (T)(T = \hat{\mu}\,\hat{\nu}(\mu = \nu +_c 1)).$$

Once again we have eliminated the use of ♀ altogether.

I cannot emphasize enough the importance *Principia*'s relations-e that are operations and their tie to the (impredicative) comprehension of relations in intension. Wittgenstein, it may be noted, rejected impredicative comprehension in his *Tractatus* and accepted recursive definition (and the "etc." of repeating an operation) as indefinables that are *shown*.[19] Using its impredicative comprehension to capture the *ancestral* relation R_* in *90, *Principia* is emphatic that one must define the "etc." (i.e., the notion of a consecutive repetition of an operation), and it does this in *91 on the "powers of a relation". We find:

> The study of R_* will occupy *90. The relation R_* holds between x and y if $x(I \upharpoonright C`R)y$ or $xRy$ or $xR^2y$ or etc. The study of this "etc." occupies *91. … If $S$ is a power of $R$, so is $S \mid R$. Now $S \mid R$ is $|R`S$, according to the definition in *38. (*PM* 1: 545)

Of course, *Principia* has

$$*34.02 \quad R^2 =_{df} R \mid R$$
$$*34.02 \quad R^3 =_{df} R^2 \mid R$$

[19] See Landini, "Tractarian Logicism" (2021).

and so on. But such definitions are not the "etc." notion R_{po} whereby $xR_{po}y$ says that y either equals x or comes at some point after x in a series produced by repeating the operation $|R$. This "etc." (the general notion of repetition of an operation), it should be noted, is not yet the notion introduced at ∗301.03 of *Principia*, Volume 3, on the "numerically defined powers of a relation", where we get the definition of R^{σ} with σ bindable as an object-language class variable. When R is relation-e $+_c 1$, *Principia* can put

$$(\exists\mu)(\mu \,\varepsilon\, \text{NC induct} \,.\, \mu = 4 \,.\, (+_c 3)^{\mu}\text{‘}0 = (+_c 6)^2\text{‘}0).$$

This says that by consecutively repeating the operation $+_c 3$ exactly four times starting from zero, we arrive that the same result as consecutively repeating the operation $+_c 6$ exactly two times starting from zero. By being able to quantify over the position of σ in R^{σ}, *Principia* also captures the arithmetic of positive and negative natural numbers and the theory of Rationals as relations-e on relation-e. It plays a role in the treatment of measurement of magnitudes—where a magnitude, e.g., +1 gramme, a second, a centimeter (comparative weights and distances in space and time), involves the repetition of a relation. (See *PM* 3: 339.) Moreover, the very definition of the operation R^{σ} with σ bindable offers an illustration of how *Principia* avoids accepting *recursive* definition as primitive. Its legitimation of *recursive definition* (historically called "inductive definition") relies on the impredicative comprehension of relations.

3. CURIOUS CASES: JUST DO IT

As we saw, the convenient use of "♀" seemed to Whitehead to be explanatorily very useful—but only because of his plan to avoid introducing notations of relation-e of higher than dyadic *adicity*. All the same, developing mathematics of *Principia's* Volumes 1–3 using the schematic "♀" was given no explicit justification. The justification would come from comprehension of relations of higher *adicity*.

The question of justification nonetheless arises at various places. It is nicely illustrated in the following which questions how $\alpha \underset{''}{\downarrow}$ as a relation-e is defined. This is very important for definitions of *multiplication* and *exponentiation*. *Principia* has

$*113.02 \qquad \alpha \times \beta =_{df} s`(\alpha \underset{,,}{\downarrow} ``\beta)$

$*116.01 \qquad \alpha \exp \beta =_{df} \mathrm{Prod}`(\alpha \underset{,,}{\downarrow} ``\beta)$.[20]

In both, we see the expression "$\alpha \underset{,,}{\downarrow} ``\beta$". But to try to parse it, one may well feel lost. Indeed, it may feel as though a dead end is reached. Recall that *Principia* has:

$$cls{*}37.01 \qquad R``\beta =_{df} \hat{\gamma}(\exists y)(y \epsilon \beta \,.\, \gamma R x).$$

Thus, what we get in applying this definition to $\alpha \underset{,,}{\downarrow} ``\beta$ is the following:

$$\vdash \alpha \underset{,,}{\downarrow} ``\beta = \hat{\gamma}(\exists y)(y \epsilon \beta \,.\, \gamma(\alpha \underset{,,}{\downarrow})y).$$

But how are we to parse $\gamma(\alpha \underset{,,}{\downarrow})y$ as a *wff*? We seem wholly stuck, unable to get what is needed because it seems that we have never been given $\alpha \underset{,,}{\downarrow}$ as a relation-e sign! In fact, by *30.01, one cannot legitimately write $\alpha \underset{,,}{\downarrow} ``\beta$ unless $\alpha \underset{,,}{\downarrow}$ is given as a relation-e sign. Is it? Naturally, we are drawn to

$*38.03 \qquad \alpha \underset{,,}{♀} y =_{df} ♀y``\alpha.$

i.e., $\qquad \alpha \underset{,,}{\downarrow} y =_{df} \downarrow y``\alpha.$

This seems to be of no help. We need to arrive at:

$$\vdash \gamma\left(\alpha \underset{,,}{\downarrow}\right) y \equiv \gamma = \alpha \underset{,,}{\downarrow} y.$$

But it seems as though $\alpha \underset{,,}{♀}$ has no definition. In short, *38.03 does not seem to support the proof of the following part of

$$*38.3 \qquad \vdash \alpha \underset{,,}{♀} ``\beta = \hat{\gamma}(\exists y)(y \epsilon \beta \,.\, \gamma = \alpha \underset{,,}{♀} y).$$

Clearly, this requires that $\alpha \underset{,,}{♀}$ be a relation-e sign. How does one

[20] I have added brackets for clarity.

define $\alpha \underset{''}{\downarrow}$ as a relation-e sign?

In comments in *Principia*'s Volume 2, Whitehead reveals that he understood this well enough. It is clear that Whitehead wants $\alpha \times \beta$ to be what nowadays we call the "Cartesian product" (or "cross product"). Using a bit of modern set notation to illustrate, we can see that one needs to define the Cartesian product so that if $\alpha = \{x, y\}$ and $\beta = \{x, z\}$, we get:

$$\alpha \times \beta = \{x \downarrow x, x \downarrow z, y \downarrow x, y \downarrow z\}.$$

Principia makes this explicit:

> We write $\beta \times \alpha$ for the arithmetical class-product of β and α, and define it as the class of all ordinal couples of which the referent is a member of α and the relatum a member of β, *i.e.* as
> $$\hat{R}(\exists x, y)(x \in \alpha \,.\, y \in \beta \,.\, R = x \downarrow y).$$
> By $*40.7$, this class is $s`(\alpha \underset{''}{\downarrow}``\beta)$. (*PM* 2: 67)

As Whitehead noted, this requires:

$$*40.7 \qquad \vdash s`(\alpha \underset{''}{\text{♀}}``\beta) = \hat{\gamma}(\exists x, y)(x \in \alpha \,.\, y \in \beta \,.\, \gamma = x \text{♀} y).$$

But unless one can assure that there is a relation-e sign $\alpha \underset{''}{\downarrow}$ the proof of $*40.7$ stalls. To see this, notice:

$$\vdash s`(\alpha \underset{''}{\text{♀}}``\beta) = \hat{\gamma}(\exists \xi)(\xi \in \alpha \underset{''}{\text{♀}}``\beta \,.\, \gamma \in \xi)$$
$$\vdash s`(\alpha \underset{''}{\text{♀}}``\beta) = \hat{\gamma}(\exists \xi)((\exists y)(y \in \beta \,.\, \xi(\alpha \underset{''}{\text{♀}})y) \,.\, \gamma \in \xi).$$

The comments reveal that Whitehead thinks that $\alpha \underset{''}{\downarrow}$ is a relation-e sign. But where is it defined? There must be an explanation. We need a deeper understanding of what Whitehead was doing.

The use of ♀ crops up again in $*150$, and we find a definition which again seems to follow the pattern:

$$*150.03 \qquad Q \underset{\bullet\prime}{\text{♀}} y =_{df} \text{♀} y \,\dot{;}\, Q$$
$$rel*150 \qquad Q \underset{\bullet\prime}{\text{♀}} P =_{df} \text{♀} P \,\dot{;}\, Q$$

$*150.01 \qquad S;Q =_{df} S|Q|\breve{S}.$

But once again we encounter the difficulty as to what has rendered relations-e that are $Q\underset{\bullet,}{\female}$ and $\underset{\bullet,}{\female}y$ in the first place. When we come to *Principia’s* Volume 2, $*182$, the problem feels even more alarming. This concerns cases of $\hat{\female}$ and $\underset{,,}{\hat{\female}}$. *Principia* introduced $\hat{\female}$ with the following definitions:

$$
\begin{array}{ll}
*182.01 & \hat{\female} =_{df} \hat{y}\hat{x}(y = x\female x). \\
rel*182.01 & \hat{\female} =_{df} \hat{T}\hat{x}(T = x\female x). \\
relrel*182.01 & \hat{\female} =_{df} \hat{T}\hat{R}(T = R\female R).
\end{array}
$$

At long last, we reach a comment that explains what is going on. We find:

$*182.04 \qquad \vdash \underset{,,}{\hat{\downarrow}}\text{‘}\alpha = \downarrow\alpha\text{“}\alpha$

> Observe that in $\underset{,,}{\hat{\female}}$, we first take $\underset{,,}{\downarrow}$ and then put a circumflex over it. If we first took $\underset{,,}{\hat{\downarrow}}$, we could not then place two commas under it, because $\hat{\downarrow}$ is a relation, not a double descriptive function, and two commas can only significantly be placed under a double descriptive function. (*PM* 2: 474)

Whitehead says that one *cannot* put a relation-e sign in the position of $\female$ in $*182.01$. We have seen already that one can put $\downarrow$ and $|$ into the position of $\female$ in $*38.01$ and $*38.02$, respectively. That is because $\downarrow$ is not a relation-e sign; it is, e.g., $x\downarrow y$ and $P \downarrow S$ that are relation-e signs. Similarly, $|$ is not itself a relation sign; it is $P \mid S$ that is a relation-e sign for relative product. Now quite clearly, $\hat{\female}$ is different. It *is* a relation-e sign. An important instance is:

$$
\begin{array}{ll}
 & \hat{\downarrow} =_{df} \hat{T}\hat{x}(T = x\downarrow x) \\
 & \hat{\downarrow} =_{df} \hat{T}\hat{R}(T = R\downarrow R) \\
 & \vdash \hat{\downarrow}\text{‘}R = R\downarrow R \\
\text{i.e.,} & \vdash (\imath T)(T(\hat{\downarrow})R) = R\downarrow R.
\end{array}
$$

Thus we see that *Principia* has:

$*182.03 \qquad \vdash \hat{\lceil}‘R = R^2$

i.e., $\qquad \vdash (\imath T)\big(T(\hat{\lceil})R\big) = R \mid R.$

This is illuminating. By $*30.01$, we have the following:

$$\hat{\underset{,,}{\downarrow}}‘\alpha =_{df} (\imath T)(T(\hat{\underset{,,}{\downarrow}})\alpha).$$

This is gibberish unless $\hat{\underset{,,}{\downarrow}}$ is a relation-e sign. What, then, established that $\hat{\underset{,,}{\downarrow}}$ is a relation-e sign? Whitehead's comment after $*182.04$ finally tells us.

Whitehead instructed us to *just do it*—just put the sign $\underset{,,}{\downarrow}$ in the position of ♀ in $\hat{\text{♀}}$ so as to make the relation-e sign $\hat{\underset{,,}{\downarrow}}$. That is, just put the sign $\underset{,,}{\downarrow}$ in the position of ♀ in the definition $*182.01$ to get the following:

$$\hat{\underset{,,}{\downarrow}} =_{df} \hat{\mu}\hat{\alpha}\,(\mu = \alpha \underset{,,}{\downarrow} \alpha).$$

i.e., $\quad \hat{\underset{,,}{\downarrow}} =_{df} \hat{\mu}\hat{\alpha}\,(\mu = \downarrow \alpha“\alpha)$ by $*38.01$.

The plan of just doing it certainly solves all our troubles. In short, it was *not* the definition $*38.03$ that assured us that $\alpha\underset{,,}{\downarrow}$ is a relation-e sign. We were looking in the wrong place. Nothing about *cls*$*38.03$ itself assures that there is any such relation-e that is $\alpha\underset{,,}{\downarrow}$. What assures it? Whitehead just puts $\underset{,,}{\text{♀}}$ for ♀ in the definitions *cls* $*$ 38.03 and *cls*$*38.02$. He seems simply to have given himself the following instances:

$$\alpha \underset{,,}{\downarrow} =_{df} \hat{\mu}\hat{\beta}\,(\mu = \alpha \underset{,,}{\downarrow} \beta)$$
$$\underset{,,}{\downarrow}\beta =_{df} \hat{\mu}\hat{\alpha}\,(\mu = \alpha \underset{,,}{\downarrow} \beta).$$

This solves all the problems. But it leaves us with the question of what legitimates it.

There has to be a legitimation of Whitehead's sanctioning *Just do it*. My answer is that Whitehead knew that he was entitled to just do it because he knew full well how to eliminate his use of the "♀" altogether. In the case at hand, we can introduce the relation $\alpha \underset{\text{"}}{\downarrow}$ via comprehension. We have:

$$\vdash (R)(R = \hat{\mu}\,\hat{\nu}\,\hat{\sigma}\left(\mu = \nu \underset{\text{"}}{\downarrow} \sigma\right) . \supset .$$
$$(\alpha)(\exists P)(P = \hat{\mu}\hat{\beta}\,(< \mu, \alpha, \beta > \epsilon\ R)) \,.$$
$$(\beta)(\exists Q)(Q = \hat{\mu}\hat{\alpha}\,(< \mu\ \alpha, \beta > \epsilon\ R))).$$
$$\alpha \underset{\text{"}}{\downarrow} =_{df} (\imath P)(P = \hat{\mu}\hat{\beta}\,(\mu = \alpha \underset{\text{"}}{\downarrow} \beta))$$
$$\underset{\text{"}}{\downarrow} \beta =_{df} (\imath Q)(Q = \hat{\mu}\hat{\alpha}\,(\mu = \alpha \underset{\text{"}}{\downarrow} \beta))$$
$$\alpha \underset{\text{"}}{\downarrow} \beta =_{df} \underset{\text{"}}{\downarrow} \beta \text{``} \alpha. \qquad \text{by } *38.03$$

The procedure is clear and completely resolves the concern that definition *38.03 doesn't itself support the existence of a relation-e that is $\alpha \underset{\text{"}}{\downarrow}$. The curiosities of *38 are important. They reveal the central role impredicative comprehension plays in the definitions of operations throughout the work. The use of the stand-in ♀ hides this role. With comprehension and the schematic $\phi(\mu, \nu, \sigma)$, we avoid ♀. This is precisely what legitimates Whitehead's *Just do it*.

WORKS CITED

GERHARDT, CARL IMMANUEL, ed. *Leibnizens Mathematische Schriften*. 7 vols. Berlin: Asher, 1849–63.

GRATTAN-GUINNESS, IVOR. *The Search for Mathematical Roots, 1870–1940*. Princeton: Princeton U. P., 2000.

GRIFFIN, NICHOLAS. "A Sexist Joke in *Principia Mathematica*". *Russell* 39 (2020): 138–40.

—, AND SEBASTIEN GANDON, eds. *The Whitehead–Russell Correspondence*. In progress.

LANDINI, GREGORY. "Tractarian Logicism: Operations, Numbers, Induction". *The Review of Symbolic Logic* 14 (2021): 973–1010.

LEVINE, JAMES, AND BERNARD LINSKY, eds. *Bertrand Russell and His Students 1910–1916*. In progress.

LINSKY, BERNARD. *The Evolution of Principia Mathematica*. Cambridge: Cambridge U. P., 2011.

LITTLEWOOD, J. E. *Littlewood's Miscellany*. Béla Bollobás, ed. Cambridge: Cambridge U. P., 1990.

QUINE, W. V. O. *The Time of My Life: an Autobiography*. Cambridge, MA: MIT P., 1985.

RUSSELL, BERTRAND. *PoM*.

—. *OKEW*3.

—. *IMP*.

—. "Logical Atomism" (1924). In *LK*; **26** in *Papers* 9.

VIRGIL. *Aeneid*.

WHITEHEAD, A. N. *An Introduction to Mathematics*. London: Williams & Norgate, 1911.

—, AND B. RUSSELL. *PM*.